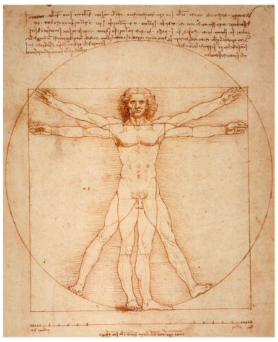

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ARTIGOS - ARTICLES

De Platão a Weil e além: genericidade através da história da matemática

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Resumo: No final do século XIX, a *genericidade* deu um grande salto no caminho da análise matemática com os desenvolvimentos promovidos pela *Escola Italiana de Geometria Algébrica*. Suas origens, no entanto, podem ser encontradas na matemática antiga em trabalhos de importantes filósofos e matemáticos, tais como Platão e Euclides. Neste artigo, tentaremos mostrar como uma noção-chave na virada estruturalista da geometria algébrica evoluiu a partir de um vago fenômeno linguístico para um conceito matemático preciso e frutífero.

Palavras-chave: genericidade; história da geometria algébrica, fundamentos da matemática, história da matemática.

From Plato to Weil and beyond: Genericity through the history of mathematics

Abstract: At the end of the 19th century, *genericity* took an important step toward mathematical analysis, due to the developments promoted by the *Italian school of algebraic geometry*. However, its origins can be traced back to ancient mathematics in the work of prominent philosophers and mathematicians, such as Plato and Euclid. In this article, we will try to show how a key notion in the structuralist turn of algebraic geometry evolved from a vague linguistic phenomenon and became a precise and fruitful mathematical concept.

Keywords: genericity, history of algebraic geometry, foundations of mathematics; history of mathematics.

Introduction

In everyday language, *genericity* acts as an important catalyst for equivalence classes. Unlike universality, genericity produces true judgments about the world even when the predication does not hold for every object that fits in the definition of the predicate. If one says, for example, that "All birds fly", one will be saying something patently false. On the other hand, if one says, "Birds fly", one will be, for sure, asserting a true proposition – even if it is false that the same does hold for every bird. In mathematics, this idea goes side by side with the use of variable terms, like x in "let x be an arbitrary element of a set" and has a long application history.

Genericity in mathematics

At face value, genericity is as old as mathematics itself. At least, since the mathematician realized that it is on the universality of mathematical proofs that the noblest characteristic of mathematics rests upon. When the first geometer proved the first theorem, or when the first algebraist solved the first equation, or even when the first computer ran the first algorithm, genericity was there. But it was only with the development of algebraic methods of mathematical analysis that the concept, as a method, began to be applied. When Weil introduced the very notion of a *generic point* on his *Foundations of Algebraic Geometry*, he showed how a linguistic trick can turn into a precise mathematical concept.

This concept, however, has an intricate root. Arguments that make use of such a tool goes back to the nineteenth century with the contributions of the famous *Italian school of algebraic geometry*. At that time, after the revolutionary work of B. Riemann (1826 – 1866), it became clear that geometry is more than human intuition is capable of grasping. Under the influence of H. Schubert (1848 – 1911), F. Klein (1849 – 1925) and M. Noether (1844 – 1921), mathematics started to become more and more *abstract* and *general*. In the words of E. Bell (BELL, 1992, p. 169)[1],

> As mathematics passed the year 1800 and entered the recent period, there was a steady trend toward increasing abstractness and generality. By the middle of the nineteenth century, the spirit of mathematics had changed so profoundly that even the leading mathematicians of the eighteenth century, could they have witnessed to outcome of half a century's progress, would scarcely have recognized it as mathematics. The older point of view of course persisted, but it was no longer that of the men who were creating new mathematics. Another quarter of a century, and it had become almost a disgrace for a first-rank mathematician to attack a special problem of the kind that would have engaged Euler in much of his work.

[1] BELL, Eric Temple. *The development of mathematics*. 1992. New York: McGraw-Hill, 1945. p. 169.

Following the progress of formal knowledge, mathematics ceased to be a science of special cases and became a broader field capable of unifying the exploration of such cases in elegant theories. As Bell remarks, in what concerns the *new mathematics*, "abstractness and generality, directed to the creation of universal methods and inclusive theories, became the order of the day." (BELL, 1992, p. 169)[2] In this context, notions that once were taken for granted, faced the need to receive new characterizations. In particular, the euclidean understanding of what is a *straight line* ceased to be sufficient.

This essay aims to give a glimpse into the evolution of the phenomena of *genericity*: from its first appearance as a vague ideal of universality, until its crystallization as a tool in the technical apparatus of the algebraist. In the course of that, we will emphasize the interplay between techniques outside mathematical thinking and the development of formal methods. To accomplish this task, we will focus on the genesis of the mathematical concept of *generic point*, which emerged in the context of the development of algebraic geometry as a kind of argument by generalization and fully embodied the older notion of genericity after the precise mathematical formulation given by André Weil.

An ancient dual

In a famous passage of a Plato's dialogue, Meno asks Socrates to elucidate his epistemological position, according to which no one is bound to learn, but everyone is a rememberer of past lives. Such a view became known as *reminiscent theory*. To accomplish the *exposée* Socrates only needed a greek speaker with no previous (formal) background in geometry. Meno then offers a born-in-the-house attendant. After pushing the boy to his limits, Socrates argues that he can make him to *recollect*, from the bottom of his soul, the things that he already knows: that is, the knowledge that the double of every square is the square of the diagonal; and, from that, the geometrical method for doubling the square by taking the square generated by its diagonal.

> [Socrates] Now you should note how, as a result of this perplexity, he will go on and discover something by joint inquiry with me, while I merely ask questions and do not teach him; and be on the watch to see if at any point you find me teaching him or expounding to him, instead of questioning him on his opinions. Tell me, boy: here we have a square of four feet,[3] have we not? You understand? – Yes.
> [Socrates] And here we add another square[4] equal to it? – Yes.
> [Socrates] And here a third,[5] equal to either of them? – Yes.

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[2] BELL, *Op. Cit.*, p. 169.
[3] ABCD.
[4] DCFE
[5] CHGF.

> [Socrates] Now shall we fill up this vacant space[6] in the corner? – By all means.
> [Socrates] So here we must have four equal spaces? – Yes.
> [Socrates] Well now, how many times larger is this whole space than this other? – Four times.
> [Socrates] But it was to have been only twice, you remember? – To be sure.
> [Socrates] And does this line,[7] drawn from corner to corner, cut in two each of these spaces? – Yes.
> [Socrates] And have we here four equal lines[8] containing this space[9]? – We have.
> [Socrates] Now consider how large this space[10] is. – I do not understand.
> [Socrates] Has not each of the inside lines cut off half of each of these four spaces? – Yes.
> [Socrates] And how many spaces of that size are there in this part? – Four.
> [Socrates] And how many in this[11]? – Two.
> [Socrates] And four is how many times two? – Twice.
> [Socrates] And how many feet is this space[12]? – Eight feet.
> [Socrates] From what line do we get this figure? – From this.
> [Socrates] From the line drawn corner-wise across the four-foot figure? – Yes.
> [Socrates] The professors call it the diagonal: so if the diagonal is its name, then according to you, Meno's boy, the double space is the square of the diagonal. – Yes, certainly it is, Socrates.
> [Socrates] What do you think, Meno? Was there any opinion that he did not give as an answer of his own thought?

According to Plato, to know is to recollect innate ideas: in the framework of Plato's theory of forms, we never discover new truths. Everything that we can know was already learned by our soul in the infinite realm of eternity. In this passage (PLATO, 2005, pp. 115-121)[13], Socrates conduces the boy to *recollect*, from his innate ideas of geometrical laws, a solution for a two-dimensional version to one of the most prominent geometrical problems of ancient Greece: that is, the problem of doubling the cube; in other words, the problem of, given a cube, finding another one whose volume is the double of the first.

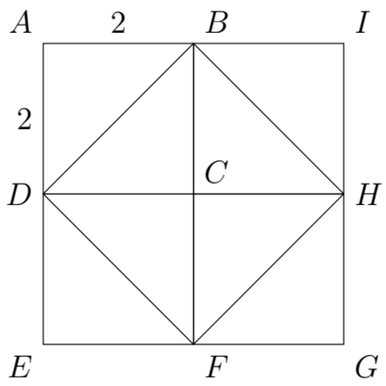

Figure 1. Plato's square.

[6] BIHC.
[7] BD.
[8] BD, DF, FH, HB.
[9] BDFH.
[10] BDFH.
[11] ABCD.
[12] BDFH.
[13] PLATO. *Meno and Other Dialogues*. 2005. Oxford University Press, 2005. pp. 115-121.

About the problem of doubling the cube, Theon of Smyrna, who quotes a work by Eratosthenes (HEATH, 1921, p. 176)[14], once said:

> Eratosthenes, in his work entitled Platonicus relates that, when the god proclaimed to the Delians through the oracle that, in order to get rid of a plague, they should construct an altar double that of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.

Whether Plato's reminiscent theory of knowledge is plausible or not, there is a pragmatic problem much more serious regarding the method exposed by Socrates for solving the problem of doubling the square. As Dieudonné observes in *History of Algebraic Geometry*, "(..) the Greeks cannot even separate algebra from geometry since algebra for them is essentially "geometric": it must be remembered that they do not calculate with numbers but with magnitudes and their relations, and that when they multiply two lengths, they obtain a magnitude of another type, namely, an area.".[15] Those limitations imposed severe restrictions on the geometrical development of the classical age. Was only with the renaissance of mathematics itself, more precisely with the work of Descartes on cubics, that a revolution was conducted into new insights and discoveries.

We can see in this example how a notion of *genericity* is deep-rooted in Socrate's remarks about the nature of our knowledge: the *idea* that leads the boy to recollect his knowledge from his soul is not something specific, being instead an idea that can be applied over and over again, with any square; but, at the same time, it is not ambiguous, being an effective method[16]. Despite the old mathematical frame, the effort of the Greek geometers, illustrated by this passage, paved the way to the study of a more abstract and generic geometry.

The Renaissance of geometry

The Renaissance was a historical period with many unfoldings in Arts, Religion, and Science. Most important are the new techniques developed in the science of graphical representation. Artists like Albrecht Dürer (1471 – 1528) were responsible for giving a new approach to

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[14] HEATH, Thomas Little. *A History of Greek Mathematics, Volume I, From Thales to Euclid* (first published in 1921 by the Clarendon Press, Oxford), 1981. Dover. New York. p. 176.
[15] DIEUDONNÉ, Jean. *History Algebraic Geometry:* An Outline of the History and Development of Algebraic Geometry. 1985. Chapman & Hall. p. 1.
[16] For several reasons, we cannot (and we are not) argue that the Ideal form of the elements is *generics* in the sense that we nowadays think about it. However, we can find some shared elements between both; in particular, in the reproducibility of a platonic Ideal in the plethora of instances that resembles it.

the artistic representation of the real world. It is not by chance that this period became known as the moment when Art discovers perspective[17].

If we understand perspective as a way of "representing objects as it appears", then it has existed in Arts since antiquity. However, the vague notion of *point of view* started its development as a precise technique only in the period that preceded the Renaissance, with the work of eminent artists like Giotto (1267 – 1337) and, posteriorly, Brunelleschi (1377 – 1446).

In mathematics, however, at that time, it was not completely transparent how to *formally* understand perspective. Giotto himself tried to depict perspective via algebraic equations by calculating the distance between lines; however, because of the inadequacy of the mathematical methods of that time, he was not able to obtain trusted representations. Despite that, it became clear that geometry should offer a model to shape this matter, being doubtless that "the great advance in perspective drawing by the artists of the Renaissance made inevitable the emergence of a geometrical theory including perspective as a special case." (BELL, 1992, p. 158)[18] Let us remember that, in the first centuries of this new era, Euclidean geometry was still the only known geometry. It was with Descartes' work that this started to change when important contributions were made in the road to the algebraization of geometrical concepts.

The further development of mathematics led to a revolution, and to the establishment of the first reliable algebraic method to analyze geometrical elements. The revolution was settled by René Descartes (1596 – 1650), a French philosopher and mathematician, famous for the so-called *Cartesian coordinate system*. It is important to stress that the very idea of coordinates was not discovered first by Descartes himself. Those systems were applied at least since Apollonius of Perugia, who set up about 200 B.C.E., a coordinate system to study conic sections[19]. However, Descartes was the first who came up with the idea of *unifying* algebra and geometry via a system of coordinates. In particular, was Descartes who conceived the now widely spread method of *graphing a function*.

In a remark about Poncelet's *Traité des propriétés projectives*, Darboux (1842 – 1917) observes "We know, moreover, by the unfortunate publication of the Saratoff notes, that it was by the aid of Cartesian analysis that the principles which serve as the base of the *Traité des propriétés projectives* were first established" (BELL, 1992, p. 339)[20]. Even if we try, we would never be able to underestimate Descartes' influence in Western thought, which was profound and can be perceived even today. The discoveries and developments led by him made it possible to refine and

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[17] GOMBRICH, Ernst Han. *The story of art*. London: Phaidon, 1995.
[18] BELL, *Op. Cit.*, p. 158.
[19] KRANTZ, Steven G. *An episodic history of mathematics*: Mathematical culture through problem solving. 2010. Maa. 2010. p. 151.
[20] BELL, *Op. Cit.*, p. 339.

expand our theories about geometrical objects, as, for instance, about what is a point and how we should comprehend the behavior of parallel straight lines in deformative spaces.

Perspective as *point at infinity*

For many centuries, the standard way to understand geometry was through the basis settled by Euclid's Elements[21]. One of the important axioms that make up the book, postulates that, in two-dimensional geometry

> (...) if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.[22]

This axiom became known as the *parallel postulate* because one of its consequences is that two parallel straight lines cannot ever meet. The problem that this imposes on the notion of perspective is that, depending on the point of view, the axiom did not hold. If we consider a space deformed by perspective, two parallel straight lines *do* meet. To see that, we can imagine the optical illusion produced by the horizon at the far end of a big road in the desert. In this view, the borderlines of the road, at a certain point, collapse into a single one. This *vanishing point*, in projective geometry, is what geometers started to call a point *at infinity*.

Figure 2. Vanishing point at the horizon. Designed by starline / Freepik. Source: <https://www.freepik.com/free-vector/perspective-road-towards-horizon_4724876.htm>

[21] HEATH, Thomas Little. *The thirteen books of Euclid's Elements*. Cambridge University Press, 1968.
[22] HEATH, 1968, *Op. Cit.*, p. 155.

With the development of new ways of thinking about the geometrical structure of reality, Euclid's geometry loses its place as the necessary way of representing the broader notion of *space*. As time went by, it became clear to geometers that in rejecting Euclid's fifth axiom, several new geometries became not only possible but, more importantly, useful. An important example of this revolution is the concept of *point at infinity* of projective geometry. If we do not restrict our space with the law that two parallels cannot coincide, then we can mathematically model the notion of *horizon* as a point where everything collapses.

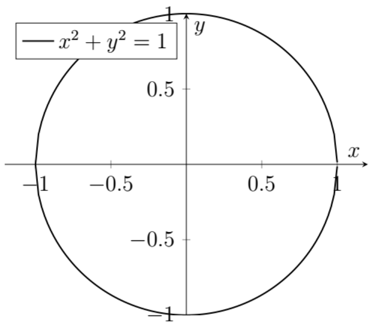

Figure 3. A circle of radius equal to 1 illustrating Descartes concept of coordinated system.

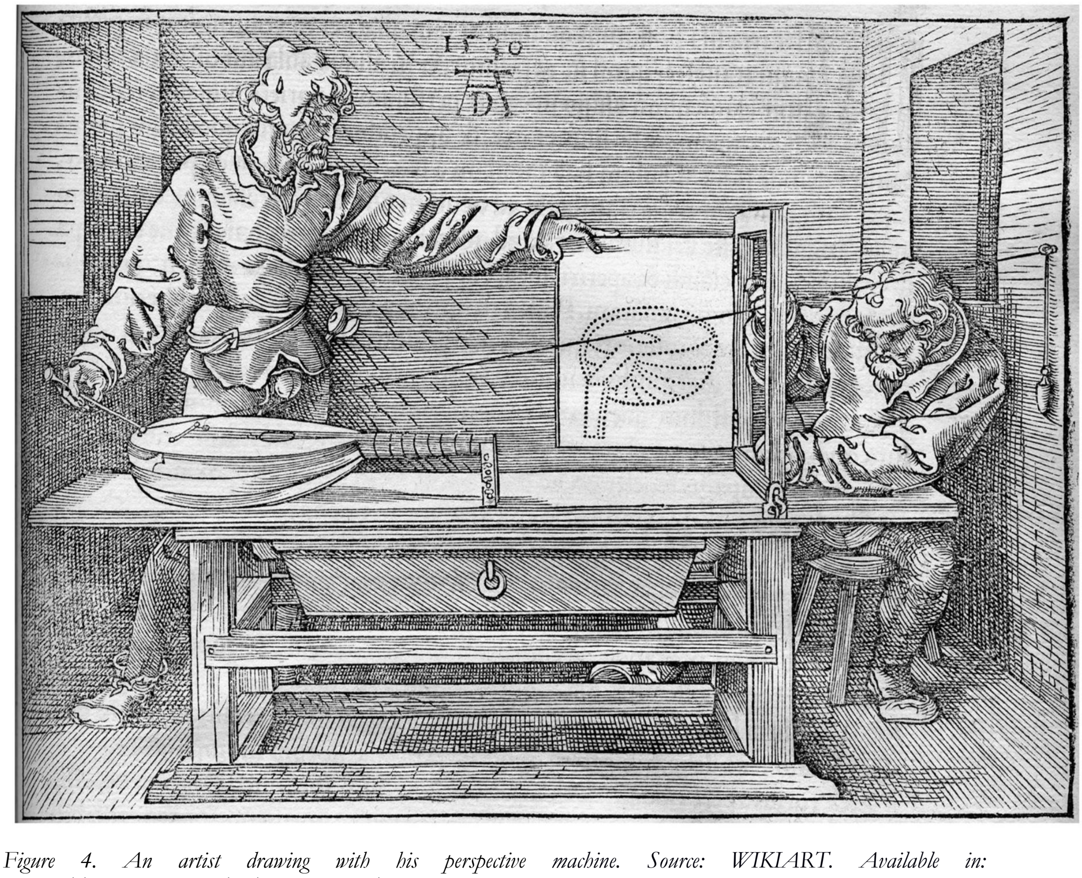

Figure 4. An artist drawing with his perspective machine. Source: WIKIART. Available in: <https://www.wikiart.org/en/albrecht-durer/man-drawing-a-lute-1523>

Italian school of algebraic geometry

In a sense, the point at infinity is fiction. Unlike fiction in Arts, however, its existence is mathematically justified. In science, fictions like this are tools to obtain models of reality. With the subsequent development of mathematics, such tools started to brand new ways of modeling the world. This is the case of algebraic geometry, which formally appeared as a branch of Field Theory and until today is the most proficuous way to understand polynomials. Thanks to advances in this area, it was indeed possible to prove *Fermat's Last Theorem*. In studying polynomials, the concept of a *point at infinity*, or a *vanishing point*, gives way to the notion of *generic point*. To comprehend how this concept emerged, we have to fly from the Renaissance to the 19th century and take a look at the group of mathematicians led by Corrado Segre: the *Italian school of algebraic geometry*.

Corrado Segre (1861 – 1924) was an Italian mathematician best known for his contributions to algebraic geometry. He was a major contributor to the early stage of development of the field, being responsible for expanding it and spreading among his students the newborn discipline. After an experience in Germany and other countries, Segre returned to Italy carrying in his luggage an intense desire to systematically develop the novelties he had learned from mathematicians at the forefront, aiming to "importing to Italy ideas that had been developing elsewhere." (LUCIANO and ROERO, 2016, p. 104)[23]

In this context, at the end of the nineteenth century, under the influence of Felix Klein's *Erlangen Program*, algebraic geometry flourished among Segre's students. What had become to be known as the *Italian school of algebraic geometry*, a movement headed by Corrado Segre, Francesco Severi, Guido Castelnuovo, and Federigo Enriques, was responsible for bringing the most important contributions to this brand new field of mathematics. Between the innovations that the fruitful work of this group introduced, lies the use of *generic points* in definitions of concepts and in theorem proving.

Regarding their influences, it is enlightening to see how Segre stressed the importance of Noether's and Brill celebrated memoir *Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie*[24]. In his famous talk at the third International Congress of Mathematicians, held in Heidelberg in 1905, "*La Geometria d'oggidì e i suoi legami coll'Analisi*"[25] (LUCIANO and ROERO, 2016, p. 135), Segre pointed out that

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[23] LUCIANO, Erika; ROERO, Clara Silvia. Corrado Segre and his disciples: the Construction of an International Identity for the Italian School of Algebraic Geometry. *From Classical to Modern Algebraic Geometry*. p. 93-241. Birkhäuser, Cham, 2016. p. 135.

[24] BRILL, Alexander; NOETHER, Max. Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie. *Mathematische Annalen*, v. 7, n. 2, p. 269-310, 1874.

[25] LUCIANO, Erika; ROERO, *Op. Cit.*, p. 135.

> A whole Italian school of geometers recognizes its starting point in the Memoir by Brill and Noether! Those concepts became even more fertile when, thanks precisely to this school, they took on a more abstract and more general character, being referred to algebraic curves, especially with the methodical introduction of the important notion of the sum of two linear series (corresponding to that of product in the field of rationality defined by an algebraic irrational). With these tools Castelnuovo obtained major new results on algebraic curves, for example regarding the issue of postulation, which I have already mentioned. More important still is the way in which it has been possible to apply that theory, or to extend it, by analogy, to surface geometry!

Thanks to the work of this school, the use of generic points (*puntos genericos*) became straightforward and unconstrained. It is not an overstatement to say that it was in the period 1880–1940 that the concept germinated more decisively. The use became so important that it drew the attention of the international community, pushing A. Weil to dedicate the firsts propositions of his *Foundations of Algebraic Geometry*[26] to explain precisely what it is to talk about generic points. Before that, however, several mathematicians built their works based on such points: Castelnuovo[27], Fano[28], Severi[29] [30], Thullen[31], to name a few. Notice how, in the following passage from an article of Severi[32], the notion is applied naturally.

> Sia P un punto doppio proprio della superficie F , che, secondo la nostra definizione, non sarà dunque una superficie generale dello $S4$. Da un punto generico *O* dello spazio ambiente esce un cono di corde della P , il quale non contiene la retta OP . Per questo fatto la OP non potrà appartenere alla varietà delle ∞+ corde di F ; nè quindi potrà la congiungente di due punti di F , che si approssimano a P con una legge qualsiasi, tendere alla OP . [33]

The use of *superficie generale* remounts to Veronese's *Foundations of Geometry*[34], where, in a very Kantian flavor, the author distinguishes between ordinary surfaces of our intuition and

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[26] WEIL, André. *Foundations of algebraic geometry*. American Mathematical Soc., 1946.
[27] CASTELNUOVO, Guido. Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica. *Rendiconti del circolo Matematico di Palermo*, v. 7, n. 1, p. 89-110, 1893.
[28] FANO, Gino. Sulle varietà algebriche dello spazio a quattro dimensioni con un gruppo continuo integrabile di trasformazioni proiettive in sé, Atti del R. Istituto Veneto di Scienze, Lettere e Arti, s. 7, 7, 1896, pp. 1069-1103.
[29] SEVERI, Francesco. Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti. *Rendiconti del Circolo Matematico di Palermo*, v. 15, n. 1, p. 33-51, 1901.
[30] SEVERI, Francesco. Sul principio della conservazione del numero. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, v. 33, n. 1, p. 313-327, 1912.
[31] THULLEN, Peter. Determinazione della serie di equivalenza individuata dal gruppo dei punti doppi impropri d'una superficie dell'S 4. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, v. 59, n. 1, p. 256-260, 1935.
[32] SEVERI, 1901, *Op. Cit.*, p. 39.
[33] "Let P be a double point of the surface F, which, according to our definition, will not therefore be a general surface of $S4$. From a generic point of the ambient space comes a cone of chords of the P, which does not contain the line OP. From this fact follows that OP cannot belong to the variety of the ∞+ chords of F; nor can the conjunction of any two points of F, which approximate P with an arbitrary law, tend to OP." (SEVERI, 1901, p. 39)
[34] VERONESE, Giuseppe. *Fondamenti di geometria a più dimensioni ea più specie di unità rettilinee esposti in forma elementare*: lezioni per la scuola di magistero in matematica. 1891. Tipografía del Seminario. Torino, 1891.

the general surface of geometrical theories. This distinction was very important to the subsequent work of the Italian geometers and lies at the core of the notion of generic point in this period. Despite that, as the foundations of mathematics became one of the main topics of interest of mathematical groups of the twentieth century, in particular to the famous *Association of Collaborators of Nicolas Bourbaki,* several notions that somehow lacked mathematical rigor were put in suspension.

Foundations reframed: the structuralist turn

Although the notion of *generic point* was central in several works that emerged from Segre's school of algebraic geometry, we cannot find any precise definition of the Italian notion of *punto generico*. This drew the attention of the international community. One of the first mathematicians to notice that problem and to investigate it was B. L. Van der Waerden (1903 – 1996), who, in the introduction of his famous notes about de genesis of the formalization of generic points (VAN DER WAERDEN, 1971, p. 171)[35], says

> What do the Italian geometers mean by a generic point on a variety ... ? Obviously, a generic point is supposed not to have certain undesirable properties. For instance, a generic plane is not tangent to a given curve, it does not pass through a given point, etc. I asked: is it possible to find, on a given variety *U*, a point ξ not having any special properties except those which hold for all points of *U*?

The problem that Van der Waerden points out is the absence of foundations regarding tacit arguments in the use of generic points by the practitioners of the Italian school. To maintain that *general* properties are valid to *every* point, the Italians geometers rest upon nothing but the thesis that the principle of continuity holds for every algebraic variety (DIEUDONNÉ, 1972, p. 850)[36], in a non-well-justified and vague application of Schubert's principle of the conservation of number. However, as Bell put it "Schubert's 'principle of the conservation of number,' ... rested on nothing that could now be recognized as a foundation." (BELL, 1992, p. 340)[37].

One could argue that we should not be too much concerned with foundational issues if our theory is well developed and exhibits interesting results. This seems to be Corrado Segre's position. In a response to Peano, Segre once said that "In my opinion if a theory is wonderful, and therefore reaches the main aim of science, the honor of the human mind, I cannot ask

35 VAN DER WAERDEN, Bartel L. The foundation of algebraic geometry from Severi to André Weil. *Archive for History of Exact Sciences*, p. 171-180, 1971. p. 171.
36 DIEUDONNÉ, 1972, Op. Cit., p. 850.
37 BELL, Op. Cit, p. 340.

anything else." (BORGA, 1992, p. 23)[38]. Needless to say, in mathematics, this is far from being the standard view. In particular, as we shall see, there was a group of mathematicians that raised the flag of another way of thinking about the foundations of mathematics.

Bourbaki and the struggle for formal refinements

From the second quarter of the twentieth century henceforward, a group of mathematicians under the pseudonym of *Nicolas Bourbaki* published a series of treatises in several areas of mathematics intending to give a solid basis for what became known as the fundamental structures of mathematics. Founded in 1934 – 1935, the *Association of Collaborators of Nicolas Bourbaki* became famous due to the extreme rigor of their results and the great power of abstractness of their work in general. Alongside those mathematicians, was André Weil.

Figure 5. Photo taken in July 1935, during the first official conference of Bourbaki, in Besse-et-Saint-Anastaise. Standing, from left to right: Henri Cartan, René de Possel, Jean Dieudonné, André Weil and Luc Olivier. Sitting, from left to right: a "guinea pig" called Mirles, Claude Chevalley and Szolem Mandelbrojt.[39]

André Weil (1906 – 1998) was a leading researcher that worked on several fields of mathematics. From basic number theory to advanced topics in algebraic varieties, his works were impactful and profound. During World War II, Weil and his family sailed to New York City,

[38] BORGA, Marco et al. Logic and foundations of mathematics in Peano's school. *Modern Logic*, v. 3, n. 1, p. 18-44, 1992. p. 23.
[39] Source: GUILBAUD, S. *Bourbaki et la fondation des maths modernes*. 2015. Available in: <https://lejournal.cnrs.fr/articles/bourbaki-et-la-fondation-des-maths-modernes>. Accessed in: Oct. 29, 2020.

where they remained with support from the Rockefeller Foundation and the Guggenheim Foundation. After this period, which lasted from 1941 to 1945, the French mathematician moved to São Paulo, where he stayed for two years. In Brazil, alongside Oscar Zariski, who was another great mathematician of the 20th century, he taught at the University of São Paulo (USP). During this period, he published his seminal work on algebraic geometry under the name of *Foundations of Algebraic Geometry*[40], which even today is a source of insights among mathematicians all around the world.

Famously, in the Introduction of his *Foundations of Algebraic Geometry*[41] A. Weil discussed two different points of view concerning mathematical practice: one that takes mathematics as being a critical work, and another holding that mathematics is, above all, a creative enterprise. As he remarks, "Algebraic geometry, despite its beauty and importance, has long been held in disrepute by many mathematicians as lacking proper foundations." (WEIL, 1946, p. 7). One of the central discomforts of Weil was the lack of foundation of the theory employed by Severi and his colleagues. As he says "To take only one instance, a personal one, this book has arisen from the necessity of giving a firm basis to Severi's theory of correspondences on algebraic curves (...)" (WEIL, 1946, p. 8).

This stance is in agreement with the wider mathematical framework of the twentieth century, as we can see in another remark of Bell, "If the mathematics of the twentieth century differs significantly from that of nineteenth, possibly the most important distinctions are a marked increase of abstractness with a consequent gain in generality, and a growing preoccupation with the morphology and comparative anatomy of mathematical structures" (BELL, 1992, p. 18)[42] . The worries that moved Bourbaki A. Weil were based on the conviction that the edifice of mathematics could not be erected above weak foundations.

These worries lead to fundamental changes in the work of the mathematician in the street, to take an expression from A. Turing[43]. Mathematics ceased to be a science of the magnitudes, as it once was with the Greeks – it is not merely a means of relations between measures anymore. With the modernist revolution that made mathematics a formal science, and because of the discovery of the paradoxes, the edifice of mathematics had to be reframed.

> [Mathematics] is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their

[40] WEIL, André. *Foundations of algebraic geometry*. American Mathematical Soc., 1946.
[41] WEIL, Op. Cit., pp. 7-12.
[42] BELL, Op. Cit, p. 18.
[43] TURING, Alan M. The reform of mathematical notation and phraseology. *The collected works of AM Turing: Mathematical logic*, pp. 211-222, 1944. p. 211.

> labyrinths and alleys, and projecting towards the periphery new avenues, more direct, broader, and more commodious.[44]

As this passage of Bourbaki suggests, in this new way of dealing with the field, mathematical science needed to be reconstructed as a whole and to be redesigned from time to time. In the new century, mathematical theories developed into mathematical structures. From that moment on, the notions of algebraic structures (such as ring, group, module, field) became "the fundamental ones". In Dieuddoné's words, "It was therefore quite natural to think of an 'abstract' extension of algebraic geometry, in which the coefficients of the equations and the coordinates of the points would belong to an arbitrary field." (DIEUDONNÉ, 1972, p. 848)[45]

The generic point find its definition

In the context of such changes for mathematics, the old idea of the Italian geometers of using a *punto generico* as a mark for the application of the concept of *general position* could then, with the foundational work of Van der Waerden and André Weil, finally receive a stable foundation.

> The Italians (not to speak of their predecessors) used these notions with a freedom which, to their critics of the orthodox algebraic school, bordered on recklessness. As long as the underlying field was C, the notion of "elements in general position" could be easily justified by an appeal to continuity (although the Italians seldom bothered to prove that these elements formed open sets in the spaces they considered). (...) Van der Waerden calls this point a generic point of *V*, for it is immediate to check that for any extension K' of *k*, any point of $V_{K'}$ is a specialization of $(1, \xi, \ldots, \xi_n)$. Such points can then be used in the same way as the "general points"' of the Italians, despite their apparently tautological character: any theorem proved for generic points (and of course expressible by algebraic equations (not inequalities!) between their coordinates) is valid for arbitrary points of corresponding varieties.[46]

With the advance of mathematical methods, genericity gained a new guise. It became a structure. If your space is not completely separable, then one can always add a generic point to it, being such a point, by definition, a dense set. In this context, *generic points* were an important step forward in the formalization of the vague idea of generality. As Dieudonné remarks "The most conspicuous progress realized during that period is the successful definition, in algebraic geometry over an arbitrary field, of the concepts of *generic point* and of *intersection multiplicity*, due

[44] FERREIRÓS, José. *The architecture of modern mathematics*. Oxford University Press, 2006. p. 4.
[45] DIEUDONNÉ, Jean. The historical development of algebraic geometry. *The American Mathematical Monthly*, v. 79, n. 8, p. 827-866, 1972. p. 848.
[46] DIEUDONNÉ, 1972, *Op. Cit.*, p. 849.

to the combined efforts of van der Waerden and A. Weil." (DIEUDONNÉ, 1972 , p. 849).

The precise definition of *generic point* can be found in page 69 of Weil's book. However, his definition presupposes that the acquaintance of the reader with other notions. In the following three steps, we try to reconstruct Proposition *1* via the definitions of its central concepts, which are *specialization* and *generic specialization*.

1. "Let (*x*) and (*x'*) be two sets of *n* quantities, and $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$ the ideals they determine over a field *k*; if $\boldsymbol{\beta} \subset \boldsymbol{\beta}'$, i.e. if we have $F(x') = 0$ whenever $F(X)$ is in $k[X]$ and such that $F(x) = 0$, then we say that (*x'*) is a specialization of (*x*) over *k*." (WEIL, 1946, p. 26)[47]
2. "If two sets of generalized quantities, (*x*) and (*x'*), are specializations of each other over a field k, we say that they are generic specializations of each other over k." (WEIL, 1946, p. 27)[48]
3. "Let *V* be the locus of a point *P* over a field k. Then a point P' is a generic point of V over k if and only if it is a generic specialization of *P* over k." (WEIL, 1946, p. 69)[49]

The mathematicians of the twentieth century realized that every property of such a generic point is a property of every other point of your space, just like the Italians already did with their *punto generico*, but now with a precise formulation. In a sense, this is exactly what one expected when proving a *general* result about a mathematical structure. In a brief comment about generic points in a talk about the historical development of algebraic geometry, Dieudonné remarks (DIEUDONNÉ, 1985, p. 67)[50]

> Since Poncelet's time, it has been customary in algebraic geometry to restrict the proofs of most of the general theorems to the case where the points or the algebraic varieties under consideration are "in general position," (...) Stated a little more precisely, if the point of affine or projective space having these parameters for coordinates is considered, the problem studied implies that this point belongs to an algebraic variety *V*, and the points of *V* for which the data are not "in general position" are the points of algebraic subvarieties WW_j of *V*, distinct from *V*. If *V* is irreducible, it follows that the complement of the W_j is an open, everywhere dense subset of *V*.

Which became clear after the work of Van der Waerden in realizing the gap between the use of the concept and its mathematical definition and the foundational worries of André Weil, is that the *punto generico* that once was applied by the Italians geometers as a mean to obtain

[47] WEIL, Op. Cit, p. 26.
[48] WEIL, Op. Cit, p. 27.
[49] WEIL, Op. Cit, p. 69.
[50] DIEUDONNÉ, Jean. The historical development of algebraic geometry. *The American Mathematical Monthly*, v. 79, n. 8, p. 827-866, 1972. p. 67.

general arguments in their proofs now can be understood in a more precise manner as a dense subset of an irreducible variety.

Beyond foundations

As we saw, according to Dieudonné, generic points were introduced into mathematical practice as a matter to tighten the notion of a general position. After that, with the work of Van der Waerden and Weil, the notion became more precise and could be understood as a dense subset of an algebraically closed space. Despite its importance to the field of algebraic geometry, however, Weil's book from 1946 did not put an end to the problem of defining generic points. One of its central problems is that with the definition given by Weil, in a unique variety we do not have a single generic point, but an infinity family of identical points with the property of being generic. This happens as a consequence of the definition of generic specialization and mainly because over the same field we always have infinite quantities that are specializations of each other. To see that, consider the class of equivalence given by the set of all quantities that is equal to itself. If one's domain is infinite, then one has infinite elements that are equal to itself.

It was necessary a new refinement of the notion, which didn't take long to happen. In 1960, Alexander Grothendieck (1928 – 2014), assisted by Jean Dieudonné (1906 – 1992), started publishing a series of fascicles that later composed the *Éléments de géométrie algébrique* (EGA) and became known as a cornerstone of the field of algebraic geometry. In this book, Grothendieck established a systematic foundation of algebraic geometry via *schemes*, which he rigorously defined. In the framework of this new theory, he was able to define *a generic point* that does not use specializations, but schemes.

In doing so, the mathematicians foresaw the method of change of basis, which allowed to bring back some important features of mathematical analysis. As a consequence, the intersection between the tools of Zariski topology and the continuity argument was made possible. As Dieudonné pointed out[51]

> The notion of generic point, which had disappeared from the Serre-Chevalley theory, is now reintroduced in a natural way: for instance, if A is an integral domain, its (unique) generic point is the prime ideal (0) in Spec(A); its "generic" property is expressed by the fact that its closure is the whole space Spec(A), and thus continuity arguments in the Italian style (but in the Zariski topology!) are now again available.

[51] DIEUDONNÉ, 1972, Op. Cit., p. 863.

With this new definition, it became clear that genericity could be applied in several ways in algebraic geometry without the need of sacrifice rigor, clarity or any other mathematical desideratum. Besides its importance in formalizing the notion of genericity, the generic points also made it possible to create a general method of change of basis[52].

> Most of the time this fundamental process is applied to study the morphism $f: X \rightarrow S$ by replacing the "base" S by another one Y, in such a way that the new morphism p_2 , which is now written $f_{(Y)} : X_{(Y)} \rightarrow Y$... can be more easily handled. This "change of base" is probably the most powerful tool in the theory of schemes, generalizing in a bewildering variety of ways the old idea of "extending the scalars."

Remember how important was the introduction of the *point at infinity* to the Renaissance of geometry. During that period, work was done towards the end of *extending the scalars*: the notion of point ceased to be only "that which has no part" and started to become a more embracing concept. And now, with the *generic point*, the family of the scalars received a brand new member that allowed the mat to naturally model the notion of change of base. To understand it better, it may be useful to have a look at how generic points look like.

On how generic points looks like

After the work of Grothendieck, it became clear to the mathematical community that generic points could be rigorously applied to the analysis of algebraic varieties. One of the researchers that worked on the notion was the Fields Medalist David Mumford (1937 –). In 1988, Springer-Verlag published some of his notes under the name of *The Red Book of Varieties and Schemes*[53], which is a compilation of his annotations on lectures in algebraic geometry. Rapidly, Mumford's book became a classic in the field.

In 2007, the author received the AMS Leroy P. Steele Prize for Mathematical Exposition, with honors for "his beautiful expository accounts of a host of aspects of algebraic geometry"[54]. On the occasion of the prize, the AMS jury describes the importance of *The Red Book of Varieties and Schemes* by saying that "This is one of the few books that attempt to convey in pictures some of the highly abstract notions that arise in the field of algebraic geometry.". One of those pictures offers a way to depict generic points graphically.

52 DIEUDONNÉ, 1972, Op. Cit., p. 863.
53 MUMFORD, David. The red book of varieties and schemes. *Lecture notes in mathematics*, v. 1358, p. 14-01, 1996.
54 LEBRUYN, L. *Mumford's treasure map*. 2008. Available in: <http://www.neverendingbooks.org/mumfords-treasure-map>. Accessed in: Oct. 29, 2020.

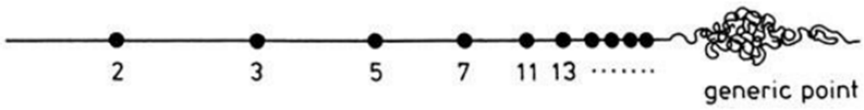

Figure 6. Spec(ℤ)[55].

In this figure, Mumford depicts the space generated by the spectrum of the ring of the integers ($Spec(\mathbb{Z})$). If we read this from left to right, then we have, in the first place, the set of all integers that are divisible by 2. And then, the set of all numbers that are divisible by 3, and then by 5, by 7, and so on. At the very end of this progression is the set of all numbers that are divisible by zero. Now, it is important to notice that, with the only exception of zero itself, *every* integer number is divisible by zero. Mumford chooses to represent this *density* inherent to the generic points as a "hairy ball". To understand why he does that, we have to realize that the generic points are the subsets of the space that is a subset of every other open set of the space, which makes them dense sets. And this is why the representation is so clever: with this "hairy ball" Mumford aims to convey, with an apparent mess, the idea that a *generic point* is a special point where all the other points collide.

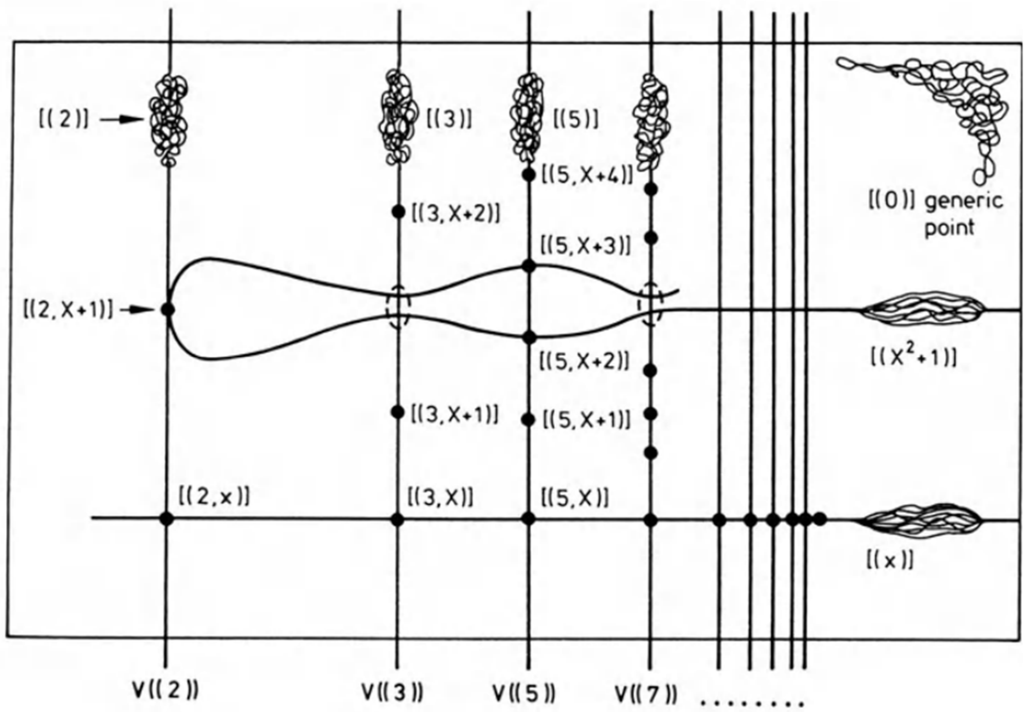

Figure 7. Mumford's "arithmetic surface"[56]*. A graphical representation of* $Spec(\mathbb{Z}[x])$.

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[55] MUMFORD, Op. Cit., p. 72.
[56] MUMFORD, *Op. Cit.*, p. 75.

In the above figure, we have what Mumford calls *the arithmetic surface*. To understand what is depicted there, we have to keep in mind that the *hairy balls* represent the density of the generic points: in each $V((x))$, Mumford represents the family of varieties that has the quotient equal to zero up to *x*. For each such family, there is a generic point. This is why the spectrum of the polynomial rings over the field of the integer has not only one, but infinite (different) generic points. It is beyond the scope of this work to explain the mathematical details of this space, but even from a layman's point of view, Mumford's graphical representation is comprehensive. In any case, it is enlightening for those who want to have an idea of how are the behaviors of generic points in the case of $Spec(\mathbb{Z}[x])$.

It is worth noticing that the [(0)], which is the set of every polynomial that is divisible by zero, is depicted outside of the grid. Once again, this is because the ideal generated by the zero-ring is such that it includes every other ring of the entire space. In this sense, it suits greatly the operations of change of base, just like Dieudonné pointed out.

Conclusion

In this essay, we tried to show how *genericity*, which first appears as a kind of smooth universality, turned into the precise concept of *generic point*. Along that path, we saw how the contributions made by Plato, Descartes, Segre, Weil, Grothendieck and, finally, Mumford, were central to the development of the generics in mathematics.

As we saw, one of the central features introduced by those points were the general arguments, which make it possible for the mathematician to shorten their proofs and increase their clarity. In the course of this development, genericity became more and more a tool for the working algebraist. Besides that, the contributions made by Weil and Grothendieck are an example of how a concept can be refined in order to be enhanced and receive better uses.

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