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Limits of Semigroups Depending on Parameters

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Abstract: It is reasonable to compare dissipative semigroups with a global attractor by restricting the flows to the attractor. However, if the rate of approach to the attractor is not uniform with respect to parameters, then the transient behavior near the attractor will give more information. We introduce a concept which takes into account this transient behavior. The concept also is useful when the limit system is conservative. We give the general theory with applications to parabolic and hyperbolic PDE on thin domains as well as situations where the limit problem is conservative.

Key words: Nonlinear semigroups, global attractor, dissipative system, omega-limit set.

1. Introduction.

The basic problem in dynamical systems is to compare the flows defined by different semigroups. In the study of finite dimensional semigroups restricted to a finite dimensional compact manifold $\mathcal{M}$ without boundary, this comparison is made most often through the notion of topological equivalence. Two semigroups defined on $\mathcal{M}$ are said to be *topologically equivalent* if there is a homeomorphism from $\mathcal{M}$ to $\mathcal{M}$ which preserves orbits and the sense of direction in time. If the manifold $\mathcal{M}$ is compact with boundary, the same notion has been used provided that the flow is transversal to the boundary.

If the semigroups are defined on a finite dimensional Banach space X, then extreme care must be exercised in order to discuss the behavior of orbits at "∞" and only very special cases have been considered. One way to avoid the consideration of ∞ is to assume that each semigroup $T(t)$, $t \geq 0$ has a global attractor $\mathcal{A}$. Recall that a global attractor for $T(t)$, $t \geq 0$, is a compact set $\mathcal{A}$ which is invariant (that is, $T(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$) such that, for any bounded set $B \subset X$, we have $\text{dist}_X(T(t)B, \mathcal{A}) \to 0$ as $t \to \infty$. In such a situation, if the space is finite dimensional, we often can reduce the discussion to the study of the topological equivalence of the flows on a manifold with boundary. This manifold will contain the attractor and also the boundary will be transversal to the flow.

In the infinite dimensional case, there is considerable literature devoted to the adaptation of ideas of finite dimensions to infinite dimensions. (see [1], [4], [6], [15], [20] and the references therein). If each of the semigroups has a global attractor, then it is reasonable at first to consider the topological equivalence of the flows on the attractors. Specific applications in this direction have been made to functional differential equations [6] and some classes of parabolic and hyperbolic

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partial differential equations ([9], [10], [12], [18], [19]).

If we are considering a family of semigroups depending upon a parameter and if each of the semigroups has a global attractor with the rate of approach to the attractor being uniform with respect to the parameter, then the restriction of the discussion to the attractor is reasonable. However, if the rate of approach to the attractor is not uniform in the parameter, then the transient behavior near the attractor may give more information about the limit semigroup. Such situations arise when we consider dissipative systems for which the dissipation is approaching zero. We introduce a new concept which takes into account some of this transient behavior. When this is done, we do not expect to obtain very specific information about any particular orbit of the limiting equation. We will discover only how slow movement near the attractor is reflected in the way that the orbits wander around in the phase space. For example, if the limit system is conservative, then orbits of a nearby dissipative system should wander across several of the constant energy surfaces and, therefore, across several of the orbits, of the conservative system. Our concept is an attempt to capture some of this information. In this latter situation, it sometimes is possible to obtain similar information by considering only the limits of the attractors as we show below.

Although the presentation in the text will be more general, we begin the discussion with the dissipative case. Let Y_0, E be Banach spaces. For a fixed set S in E with $0 \in \bar{S}$, the closure of S, and any $\epsilon \in S$, let $T_\epsilon(t)$, $t \geq 0$, be a C^0-semigroup on Y_0 and suppose that $\mathcal{A}_\epsilon$ is the global attractor for $T_\epsilon(t)$, $t \geq 0$. Our objective is to compare some properties of the flow defined by $T_0(t)$, $t \geq 0$, to analogous properties of the flow defined by $T_\epsilon(t)$, $t \geq 0$, for $\epsilon \in S$ with $|\epsilon|$ small.

If we suppose that $T_0(t)$ also has a global attractor $\mathcal{A}_0$, then we could compare the flows by topological equivalence on the attractors. This is the strongest type of comparison of flows that can be expected in the sense that it uses the very detailed properties of the flows. However, it can be of interest to make the comparison of flows with some weaker concept.

For any sets A, B in a Banach space X, we let

$$\delta_X(A, B) = \sup_{a \in A} \operatorname{dist}_X(a, B).$$

We say that the attractors $\mathcal{A}_\epsilon$ are *upper semicontinuous* (resp. *lower semicontinuous*) on S at $\epsilon = 0$ if

$$\lim_{\epsilon \in S \to 0} \delta_{Y_0}(\mathcal{A}_\epsilon, \mathcal{A}_0) = 0 (\text{resp.} \lim_{\epsilon \in S \to 0} \delta_{Y_0}(\mathcal{A}_0, \mathcal{A}_\epsilon) = 0).$$

We say that the attractors $\mathcal{A}_\epsilon$ are *continuous* on S at $\epsilon = 0$ in the Hausdorff sense if they are both upper and lower semicontinuous on S at $\epsilon = 0$. We say that the dissipative semigroups $T_\epsilon(t)$, $t \geq 0$, are *Hausdorff continuous* on S at $\epsilon = 0$ if the attractors $\mathcal{A}_\epsilon, \epsilon \in \bar{S}$ are continuous on S at $\epsilon = 0$ in the Hausdorff sense.

Upper semicontinuity of the attractors at $\epsilon = 0$ have been considered in several situations (see [1], [5], [6], [7], [9]). If, in addition, we want to have lower semicontinuity (and therefore Hausdorff continuity) of the attractors, we must impose some restriction on the flow. For gradient flows, for example, it is enough to suppose that the equilibrium points of the semigroup $T_0(t)$, $t \geq 0$, are hyperbolic (see [1], [8], [11]).

From these remarks, it is clear that Hausdorff continuity is much weaker than the concept of topological stability. Specific examples will be given later. We now introduce another notion which is weaker than Hausdorff continuity.

Definition 1.1. Let $\mathcal{N}_E(0, \delta)$ be the δ-neighborhood of 0 in E. The ω-limit set $\tilde{\omega}_S(\mathcal{A}_\cdot)$ of the family of sets $\mathcal{A}_\epsilon$, $\epsilon \in S \cap \mathcal{N}_E(0, \delta)$ is defined by the relation

$$\tilde{\omega}_S(\mathcal{A}_\cdot) = \cap_{\delta>0} \mathrm{Cl} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon . \tag{1.1}$$

We remark that the definition of $\tilde{\omega}_S(\mathcal{A}_\cdot)$ does not use directly the semigroup $T_0(t)$. On the other hand, if we assume that $T_0(t)$ also has a global attractor $\mathcal{A}_0$, then, in Section 2, we show that Hausdorff continuity implies that

$$\tilde{\omega}_S(\mathcal{A}_\cdot) = \mathcal{A}_0 \tag{1.2}$$

On the other hand, simple examples show that the relation (1.2) is weaker than Hausdorff continuity. If we suppose that $\mathrm{Cl} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon$ is compact, then the relation

$$\tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0 \tag{1.3}$$

is equivalent to the statement that $\mathcal{A}_\epsilon$, $0 \leq \epsilon \leq \epsilon_0$, are upper semicontinuous at $\epsilon = 0$.

The set $\tilde{\omega}_S(\mathcal{A}_\cdot)$ does not use much of the information about the semigroup $T_0(t), t \geq 0$. Therefore, it is not necessary to have $T_0(t), t \geq 0$, dissipative. This means that it is possible to consider, for example, dissispative systems of ordinary differential equations or even dissipative partial differential equations for which the dissipation approaches zero. In this way, we should be able to obtain information about those orbits of the conservative system which can be obtained from the limits of invariant sets (the attractors) of dissipative systems. In Section 7, we give a complete description of $\tilde{\omega}_S(\mathcal{A}_\cdot)$ for a second order dissipative differential equation. More general situations will be considered in later publications.

The limit $\tilde{\omega}_S(\mathcal{A}_\cdot)$ only uses information about the attractors. As a consequence, the transient behavior of the semigroups $\{T_\epsilon(t), t \geq 0\}$ for initial data not on the attractor is completely ignored. To gain some information about this transient behavior which will apply also to situations where the limit semigroup is conservative, we introduce another definition of ω-limit set (in Section 3, a more general situation is considered).

Definition 1.2. Let $T_\epsilon(t)$, $t \geq 0, \epsilon \in S \cap \mathcal{N}_E(0, \delta)$, be a family of semigroups on a Banach space Y_0. For a given set $B \subset Y_0$, the *ω-limit set of B with respect to the family of semigroups* $T_\epsilon(t)$, $t \geq 0, \epsilon \in S \cap \mathcal{N}_E(0, \delta)$, is denoted by $\hat{\omega}_S(B)$ and is defined in the following way: a point $y \in \hat{\omega}_S(B)$ if and only if there are sequences $\{\epsilon_n\} \subset S \cap \mathcal{N}_E(0, \delta)$, $\{t_n\} \subset [0, \infty)$, $\{y_n\} \subset B$ such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n)y_n \to y$ as $n \to \infty$.

Let us now suppose that B is a bounded set in Y_0 which contains

$$\cup_{S\cap\mathcal{N}_E(0,\delta)}\mathcal{A}_\epsilon\ .$$

Then it is easy to show that $\hat{\omega}_S(B) \supset \tilde{\omega}_S(\mathcal{A}_.)$. The set $\hat{\omega}_S(B)$ in the general situation can be larger than $\tilde{\omega}_S(\mathcal{A}_.)$. If the semigroups depend in a nice way upon ϵ and B is a bounded set in Y_0 such that the ω-limit set $\omega_0(B)$ of B with respect to $T_0(t)$ exists and is nonempty, then $\hat{\omega}_S(B) \supset \omega_0(B)$ (Theorem 3.2). We remark that it is not assumed that $T_0(t)$ has a global attractor.

Now suppose that B is a bounded set in Y_0 such that $\cup_{\epsilon\in S\cap\mathcal{N}_E(0,\delta)} \cup_{t\geq t_0} T_\epsilon(t)B$ is bounded. If the semigroups depend in a nice way upon ϵ, then we prove (Theorem 3.3) that the set $\hat{\omega}_S(B)$ is invariant under $T_0(t)$ if either $T_0(t)$ is asymptotically smooth or if $T_0(t)$ is a group. In addition, if we assume that $T_0(t)$ also has a global attractor $\mathcal{A}_0$, the semigroups depend in a nice way upon ϵ and $B \supset \mathcal{A}_0$, then (Theorem 3.7)

$$\hat{\omega}_S(B) = \mathcal{A}_0\,. \tag{1.4}$$

In Sections 4 and 5, we show that (1.4) is true for partial differential equations of parabolic and hyperbolic type on thin domains. This requires a more general definition than the one given above and also requires several a priori estimates from our previous work ([9], [12], [13], [19]).

As remarked earlier, the limit semigroup need not be dissipative. In Section 6, we give examples of a retarded delay equation and a nonlinear partial differential equation of Fitzhugh-Nagumo type to indicate how additional information is obtained by the consideration of the limit $\hat{\omega}_S(B)$ rather than the limit $\tilde{\omega}_S(\mathcal{A}_.)$.

2. Properties of the limit $\tilde{\omega}_S(\mathcal{A}_.)$.

Let Y_0, E be Banach spaces. For a fixed set S in E with $0 \in \bar{S}$, the closure of S, and any $\epsilon \in \bar{S}$, we introduce a Banach subspace Y_ϵ of Y_0 and a C^0-semigroup $T_\epsilon(t)$, $t \geq 0$, on Y_ϵ. We suppose that there is a continuous projection $P_\epsilon : Y_0 \to Y_\epsilon$. In the applications, we often have $\lim_{\epsilon\to 0} \mathrm{dist}_{Y_0}(v, Y_\epsilon) = 0$ for all $v \in Y_0$. We also suppose that $T_\epsilon(t)$, $t \geq 0$, $\epsilon \in S$, has a global attractor $\mathcal{A}_\epsilon$. Usually, we assume that

(2.1) There exist $\delta > 0$ and a bounded set $B_0 \subset Y_0$ such that

$$\cup_{\epsilon\in S\cap\mathcal{N}_E(0,\delta)}\mathcal{A}_\epsilon \subset B_0.$$

Frequently, we assume also that $T_\epsilon(t)P_\epsilon y \to T_0(t)y$ as $\epsilon \in S \to 0$ uniformly for (t, y) in compact sets; that is, For any $t_0 > 0$ and any compact set $U \subset [t_0, \infty) \times Y_0$ and any $\eta > 0$, there is a $\delta_0 = \delta_0(\eta, U) > 0$ such that, for $\epsilon \in S \cap \mathcal{N}_E(0, \delta_0)$ and $(t, y) \in U$, we have

$$\|T_\epsilon(t)P_\epsilon y - T_0(t)y\|_{Y_0} \leq \eta\,. \tag{2.2}$$

Sometimes we need the following stronger convergence hypothesis:

For any $t_0 > 0$ and any bounded set $U \subset [t_0, \infty) \times Y_0$ and any $\eta > 0$, there is a $\delta_0 = \delta_0(\eta, U) > 0$ such that, for $\epsilon \in S \cap \mathcal{N}_E(0, \delta_0)$ and $(t, y) \in U$, we have

$$\|T_\epsilon(t)P_\epsilon y - T_0(t)y\|_{Y_0} \leq \eta\,. \tag{2.2bis}$$

If (2.1) and (2.2bis) hold, then it is known that the sets $\mathcal{A}_\epsilon$ are upper semicontinuous at $\epsilon = 0$.

The hypotheses (2.1) and (2.2bis) can be modified slightly as follows. Let Y_1 be a Banach subspace of Y_0 (usually Y_1 is more regular than Y_0). We can assume that

There exist $\delta > 0$ and a bounded set $B_1 \subset Y_1$ such that

$$\cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon \subset B_1\,. \tag{2.1ter}$$

Then the hypothesis (2.2bis) is replaced by

For any $t_0 > 0$ and any bounded set $U_1 \subset [t_0, \infty) \times Y_1$ and any $\eta > 0$, there is a $\delta_1 = \delta_1(\eta, U_1) > 0$ such that, for $\epsilon \in S \cap \mathcal{N}_E(0, \delta_1)$ and $(t, y) \in U_1$, we have

$$\|T_\epsilon(t)P_\epsilon y - T_0(t)y\|_{Y_0} \leq \eta\,. \tag{2.2ter}$$

If (2.1ter) and (2.2ter) hold and if $\mathcal{A}_0$ is in a bounded set of Y_1, then it is known that the sets $\mathcal{A}_\epsilon$ are upper semicontinuous at $\epsilon = 0$.

At first, we present a result relating the upper semicontinuity of the attractors $\mathcal{A}_\epsilon$ at $\epsilon = 0$ to the set $\tilde{\omega}_S(\mathcal{A}_\cdot)$ defined in (1.1).

Proposition 2.1. *For $\epsilon \in \bar{S}$, suppose that the semigroup $T_\epsilon(t)$ has a global attractor $\mathcal{A}_\epsilon$. If the attractors $\mathcal{A}_\epsilon$, $\epsilon \in \bar{S}$, are upper semicontinuous at $\epsilon = 0$, then*

$$\tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0. \tag{2.3}$$

If, in addition, we suppose that $Cl_{Y_0} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon$ is compact, then (2.3) implies that the attractors $\mathcal{A}_\epsilon$, $\epsilon \in \bar{S}$, are upper semicontinuous at $\epsilon = 0$.

Proof. Assume that the attractors $\mathcal{A}_\epsilon$, $\epsilon \in \bar{S}$, are upper semicontinuous at $\epsilon = 0$. Let $\epsilon_n \to 0$, $y_n \in \mathcal{A}_{\epsilon_n}$ and $y_n \to y_0$. Assume that $y_0 \notin \mathcal{A}_0$. Since $\mathcal{A}_0$ is compact, there exists a positive constant d such that

$$\delta_{Y_0}(y_0, \mathcal{A}_0) = d\,. \tag{2.4}$$

Since the attractors $\mathcal{A}_\epsilon$ are upper semicontinuous at $\epsilon = 0$, for any $\eta > 0$, there is an integer $n_0 = n_0(\eta)$ such that, for $n \geq n_0$, we have

$$\delta_{Y_0}(y_n, \mathcal{A}_0) \leq \eta\,. \tag{2.5}$$

If we choose $\eta = d/2$, then (2.5) contradicts (2.4).

To prove the last part of the proposition, assume that $\tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0$ and that the attractors $\mathcal{A}_\epsilon$ are not upper semicontinuous at $\epsilon = 0$. Then there are a constant $d > 0$ and sequences $\epsilon_n \to 0$ as $n \to \infty$ and $y_n \in \mathcal{A}_{\epsilon_n}$ such that $\delta_Y(y_n, \mathcal{A}_0) > d$ for all n. Since the set $\{y_n,\, n \geq 1,\}$ belongs to a compact set, there is a subsequence $y_{n_k}, n_k \to \infty$ as $k \to \infty$ such that y_{n_k} converges to a point $y_0 \in \tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0$ as $k \to \infty$. This contradiction completes the proof.

We remark that the assumption that $\mathrm{Cl}_{Y_0} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\,\delta)} \mathcal{A}_\epsilon$ is compact is a rather strong one.

Proposition 2.2. *For $\epsilon \in \bar{S}$, suppose that the semigroup $T_\epsilon(t)$ has a global attractor $\mathcal{A}_\epsilon$. If the attractors $\mathcal{A}_\epsilon$, $\epsilon \in \bar{S}$, are lower semicontinuous at $\epsilon = 0$, then*

$$\tilde{\omega}_S(\mathcal{A}_\cdot) \supset \mathcal{A}_0. \tag{2.6}$$

Therefore, Hausdorff continuity of the attractors $\mathcal{A}_\epsilon$, $\epsilon \in \bar{S}$, at $\epsilon = 0$ implies that

$$\tilde{\omega}_S(\mathcal{A}_\cdot) = \mathcal{A}_0. \tag{2.7}$$

Proof. For any n, there is an $\delta_n \in S$, $\delta_n \to 0$ as $n \to \infty$, such that, for $\epsilon \in S \cap \mathcal{N}(0,\, \delta_n)$, we have

$$\sup_{y_0 \in \mathcal{A}_0} \inf_{y_\epsilon \in \mathcal{A}_\epsilon} \|y_0 - y_\epsilon\|_{Y_0} \leq \frac{1}{n}\,.$$

For each fixed $y_0 \in \mathcal{A}_0$, there exist $\epsilon_n \in S \cap \mathcal{N}(0,\, \delta_n)$ and $y_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}$, such that

$$\|y_0 - y_{\epsilon_n}\|_{Y_0} \leq \frac{1}{n}\,.$$

This implies that $y_0 \in \tilde{\omega}_S(\mathcal{A}_\cdot)$ and the proof is complete.

We remark that relation (2.6) (resp. (2.7)) does not imply lower semicontinuity (resp. Hausdorff continuity) of the attractors at $\epsilon = 0$. In fact, consider the ODE

$$\dot{x} = -x((-1)^n \epsilon_n + (x-1)^2)$$

with $\epsilon_n = 1/n$; that is, the set $S = \{1,\, 1/2,\, \ldots,\, 1/n,\, \ldots\}$. We have $\tilde{\omega}_S(\mathcal{A}_\cdot) = \mathcal{A}_0 = [0,\, 1]$, there is no continuity of the attractors at $\epsilon = 0$. Thus, we see that the notion of continuity of attractors and the relation (2.7) are distinct concepts.

We say that a set A is *positively invariant* (resp. *negatively invariant*) under the semigroup $T_0(t)$ if $T_0(t)A \subset A$ (resp. $T_0(t)A \supset A$) for $t \geq 0$.

Proposition 2.3. *Suppose that, for $\epsilon \in S$, the semigroup $T_\epsilon(t)$ has a global attractor $\mathcal{A}_\epsilon$. If (2.2) holds (resp., (2.1ter) and (2.2ter) hold), then $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is positively invariant under $T_0(t)$. If, in addition, either*

(i) $\mathrm{Cl}_{Y_0} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon$ *is compact*

or

(ii) $T_0(t)$ is a C^0-group and (2.1), (2.2bis) (resp., (2.1ter), (2.2ter)) are satisfied,

then $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is negatively invariant under $T_0(t)$ and, thus, invariant under $T_0(t)$.

Proof. If $y_0 \in \tilde{\omega}_S(\mathcal{A}_\cdot)$, then there exist sequences $\epsilon_n \in S$, $y_n \in \mathcal{A}_{\epsilon_n}$ such that $y_n \to y_0$ in Y_0 and $\epsilon_n \to 0$ in S as $n \to \infty$.. Let $t_0 > 0$ be given. Since $T_0(t_0)y$ is continuous in y and (2.2) holds, for any $\eta > 0$, there exists an integer $n_0(\eta)$ such that, for $n \geq n_0(\eta)$,

$$
\begin{aligned}
\|T_0(t_0)y_0 - &T_{\epsilon_n}(t_0)y_n\|_{Y_0} \\
&\leq \|T_0(t_0)y_0 - T_0(t_0)y_n\|_{Y_0} + \|T_0(t_0)y_n - T_{\epsilon_n}(t_0)y_n\|_{Y_0} \\
&\leq 2\eta\,,
\end{aligned} \tag{2.8}
$$

which implies that $T_{\epsilon_n}(t_0)y_n \to T_0(t_0)y_0$ as $n \to \infty$. This proves the first part of the propostion.

We now show that there exists $\tilde{y} \in Y_0$ such that $T_0(t_0)\tilde{y} = y_0$ if (i) holds. Since the set $\mathrm{Cl}_{Y_0} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \mathcal{A}_\epsilon$ is compact, there are subsequences $\epsilon_{n_m} \to 0$ as $m \to \infty$ and $y_{n_m} \in \mathcal{A}_{\epsilon_{n_m}}$ such that $T_{\epsilon_{n_m}}(-t_0)y_{n_m} \to \tilde{y} \in Y_0$ as $m \to \infty$. Since $T_0(t_0)y$ is continuous in y, it follows that $T_0(t_0)T_{\epsilon_{n_m}}(-t_0)y_{n_m} \to T_0(t_0)\tilde{y} \in Y_0$ as $m \to \infty$. On the other hand, relation (2.2) implies that, for any $\eta > 0$, there exists an integer $n_{m_0}(\eta)$ such that, for $n \geq n_{m_0}(\eta)$,

$$\|T_0(t_0)T_{\epsilon_{n_m}}(-t_0)y_{n_m} - T_{\epsilon_{n_m}}(t_0)T_{\epsilon_{n_m}}(-t_0)y_{n_m}\|_{Y_0} \leq \eta\,.$$

Since $T_{\epsilon_{n_m}}(t_0)T_{\epsilon_{n_m}}(-t_0)y_{n_m} = y_{n_m}$ also converges to y_0, we conclude that

$$T_0(t_0)\tilde{y} = y_0\ .$$

Assume now that (ii) holds. From (2.2 bis), we deduce that, for any $\eta > 0$, there exists an integer $n_0(\eta)$ such that, for $n \geq n_0(\eta)$,

$$\|T_{\epsilon_n}(t_0)T_{\epsilon_n}(-t_0)y_n - T_0(t_0)T_{\epsilon_n}(-t_0)y_n\|_{Y_0} \leq \eta\,, \tag{2.9}$$

which implies that $T_0(t_0)T_{\epsilon_n}(-t_0)y_n \to y_0$ as $n \to \infty$. Since $T_0(-t_0)y$ is continuous in y, we have that $T_0(-t_0)T_0(t_0)T_{\epsilon_n}(-t_0)y_n \to T_0(-t_0)y_0$ as $n \to \infty$.

We can reproduce exactly the same proof when the hypotheses (2.1ter) and (2.2ter) hold.

Proposition 2.4. *Suppose that S is open and there is a $\delta_0 > 0$ such that, for $0 < \delta \leq \delta_0$, the set $S \cap \mathcal{N}(0,\delta)$ is connected, and, for $\epsilon \in S$, the semigroup $T_\epsilon(t)$*

has a global attractor $\mathcal{A}_\epsilon$ and that $\mathrm{Cl}_{Y_0} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta_0)} \mathcal{A}_\epsilon$ is compact. Moreover, assume that the following property holds:

$$(2.10) \qquad \begin{gathered} (\textit{there is a time } t_1 > 0 \textit{ so that, for any } \epsilon_1 \in S \cap \mathcal{N}_E(0,\delta_0), \\ \textit{for any } \eta > 0 \textit{ and for any } t_2 > t_1, \textit{ there is a } \theta_0 = \theta_0(\epsilon_1, \eta, t_2) > 0 \\ \textit{such that, for } \epsilon \in S \cap \mathcal{N}_E(0,\delta_0), \|\epsilon_1 - \epsilon\|_E \le \theta_0, y_\epsilon \in \mathcal{A}_\epsilon, \\ \|T_{\epsilon_1}(t) P_{\epsilon_1} y_\epsilon - T_\epsilon(t) y_\epsilon\|_{Y_0} \le \eta \text{ for } t_1 \le t \le t_2 . \end{gathered}$$

If we assume also that $\cup_{\tilde\epsilon \in S \cap \mathcal{N}_E(0,\delta_0)} P_{\tilde\epsilon} \mathrm{Cl} \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta_0)} \mathcal{A}_\epsilon$ is bounded in Y_0, then $\tilde\omega_S(\mathcal{A}.)$ is connected.

Proof. If $\tilde\omega_S(\mathcal{A}.)$ is not connected, then, since $\tilde\omega_S(\mathcal{A}.)$ is compact, there exist two compact sets F_1, F_2 such that

$$(2.11) \qquad \begin{aligned} &(i)\ \tilde\omega_S(\mathcal{A}.) = F_1 \cup F_2 , \\ &(ii)\ \inf_{f_1 \in F_1, f_2 \in F_2} \|f_1 - f_2\|_{Y_0} \ge d_0 > 0 . \end{aligned}$$

Let $\tilde F_i = \{y \in Y_0 : \inf_{f_i \in F_i} \|y - f_i\|_{Y_0} \le \frac{d_0}{8}\}$. Obviously, there is a positive number $\delta_0^* \le \delta_0$ such that, for $\epsilon \in S \cap \mathcal{N}_E(0,\delta_0^*)$, we have $\mathcal{A}_\epsilon \subset \tilde F_1 \cup \tilde F_2$. Since $\mathcal{A}_\epsilon$ is connected, this implies that

$$(2.12) \qquad \text{either } \mathcal{A}_\epsilon \subset \tilde F_1 \text{ or } \mathcal{A}_\epsilon \subset \tilde F_2 \ \text{ for } \ \epsilon \in S \cap \mathcal{N}_E(0,\delta_0^*) .$$

Let $\epsilon_1 \in S \cap \mathcal{N}_E(0,\delta_0^*)$ be such that $\mathcal{A}_{\epsilon_1} \subset \tilde F_1$. There is a time $\tau_0 = \tau_0(\epsilon_1)$, $\tau_0 \ge t_1$, such that

$$(2.13) \qquad T_{\epsilon_1}(t) P_{\epsilon_1} \mathcal{A}_\epsilon \subset \mathcal{N}_{Y_0}(\mathcal{A}_{\epsilon_1}, \frac{d_0}{16}) \text{ for } t \ge \tau_0,\ \epsilon \in S \cap \mathcal{N}_E(0,\epsilon_0) .$$

On the other hand, by (2.10), there exists a $\theta_0 = \theta_0(\epsilon_1)$ such that, for $\|\epsilon_1 - \epsilon\|_E \le \theta_0$, $\epsilon \in S \cap \mathcal{N}_E(0,\epsilon_0)$,

$$(2.14) \qquad \|T_{\epsilon_1}(\tau_0) P_{\epsilon_1} \varphi_\epsilon - T_\epsilon(\tau_0)\varphi_\epsilon\|_{Y_0} \le \frac{d_0}{16} \text{ for } \varphi_\epsilon \in \mathcal{A}_\epsilon .$$

From (2.13) and (2.14), we deduce that $T_\epsilon(\tau_0)\mathcal{A}_\epsilon \subset \mathcal{N}_{Y_0}(\mathcal{A}_{\epsilon_1}, \frac{d_0}{8})$ and, since $\mathcal{A}_\epsilon$ is invariant under $T_\epsilon(\tau_0)$,

$$(2.15) \qquad \mathcal{A}_\epsilon \subset \mathcal{N}_{Y_0}(\mathcal{A}_{\epsilon_1}, \frac{d_0}{8}) \quad \text{for } \|\epsilon_1 - \epsilon\|_E \le \theta_0,\ \epsilon \in S \cap \mathcal{N}_E(0,\epsilon_0) .$$

The properties (2.11),(2.12) and (2.15) imply that

$$(2.16) \qquad \mathcal{A}_\epsilon \subset \tilde F_1 \quad \text{for } \|\epsilon_1 - \epsilon\|_E \le \theta_0, \epsilon \in S \cap \mathcal{N}_E(0,\epsilon_0) .$$

We now write the set $S \cap \mathcal{N}_E(0, \epsilon_0)$ as $S \cap \mathcal{N}_E(0, \epsilon_0) = S_1 \cup S_2$, where $S_i = \{\epsilon \in S : \|\epsilon\|_E < \epsilon_0,\ \mathcal{A}_\epsilon \subset \tilde{F}_i\}$. By the above proof, the sets S_1 and S_2 are open. Since $S \cap \mathcal{N}_E(0, \epsilon_0)$ is connected, either $S_1 = \emptyset$ or $S_2 = \emptyset$. This contradicts (2.11) and, therefore, $\tilde{\omega}_S(\mathcal{A}_\cdot)$ must be connected. This completes the proof.

We end this section with a few remarks.

Remark 2.1. There are situations in which, for each $\epsilon \in S$, the semigroups $T_\epsilon(t)$, $t \geq 0$, may be dissipative and possess global attractors whereas the semigroup $T_0(t)$ may possess a first integral and therefore may not be dissipative. We present such examples in Section 6.

Remark 2.2. In the numerical approximation of an evolutionary equation which defines a semigroup $T_0(t)$, we obtain approximate semigroups $T_\epsilon(t)$ where ϵ is a measure of the accuracy of the approximation. The space Y_ϵ is usually a finite dimensional subspace of Y_0 and the operator P_ϵ is a continuous projection onto this subspace.

Remark 2.3. In this section (and in the following one), we consider only the case where the spaces Y_ϵ are Banach subspaces of Y_0. The opposite situation where Y_0 is a Banach subspace of the Banach space Y_ϵ, for $\epsilon \in S$, and where each space Y_ϵ, $\epsilon \in S$, is a Banach subspace of another Banach space Y_1, also is very interesting. Unfortunately, general theorems in this situation seem to contain too many hypotheses which would not help to clarify the concepts. For this reason, we describe only a particular case of such situations in Sections 4 and 5. We will define the limit sets $\tilde{\omega}_S(\tilde{A}_\cdot)$ and $\hat{\omega}_S$ only for problems on thin domains.

As we have remarked in the introduction, we would like to be able to understand more about the attractor for $\epsilon = 0$ from the behavior of the semigroups for $\epsilon > 0$. To do this, we need a more general definition of limit as in Section 1. We also must have a more general setup in order to treat more complicated situations.

3. Definition and properties of the limit set $\hat{\omega}_S$.

We keep the same spaces E, Y_0, Y_ϵ and projections P_ϵ as in Section 2.

Definition 3.1. For a given set $B \subset Y_0$, the *ω-limit set of B with respect to the family of semigroups $T_\epsilon(t)$, $t \geq 0$, and projections P_ϵ, $\epsilon \in S \cap \mathcal{N}_E(0, \delta)$*, is denoted by $\hat{\omega}_S(B)$ and is defined in the following way: a point $y \in \hat{\omega}_S(B)$ if and only if there are sequences $\{\epsilon_n\} \subset S \cap \mathcal{N}_E(0, \delta)$, $\{t_n\} \subset [0, \infty)$, $\{y_n\} \subset B$ such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n)P_{\epsilon_n}y_n \to y$ as $n \to \infty$.

An equivalent definition of $\hat{\omega}_S(B)$ is as follows.

Definition 3.1bis. For a given set $B \subset Y_0$, the *ω-limit set of B with respect to the family of semigroups $T_\epsilon(t)$, $t \geq 0$, and projections P_ϵ, $\epsilon \in S \cap \mathcal{N}_E(0, \delta)$*, is

$$\hat{\omega}_S(B) = \cap_{\delta > 0} \operatorname{Cl} \cup_{\bar{\epsilon} \in (S \times \mathbb{R}^+) \cap \mathcal{N}_{E \times \mathbb{R}^+}(0, \delta)} T_\epsilon(t) P_\epsilon B\,,$$

where

$$\bar{\epsilon} = (\epsilon, \frac{1}{t}), \quad \mathcal{N}_{E\times\mathbb{R}^+}(0, \delta) = \{\bar{\epsilon} = (\epsilon, \frac{1}{t}) \in E \times \mathbb{R}^+ : \|\epsilon\|_E \le \delta, \frac{1}{t} \le \delta\}.$$

The set $\hat{\omega}_S(B)$ in the general situation can be larger than $\tilde{\omega}_S(\mathcal{A}_.)$ and, in some situations, coincides with $\mathcal{A}_0$. If $B \supset \cup_{\epsilon\in S\cap\mathcal{N}_E(0,\epsilon_0)}\mathcal{A}_\epsilon$ for some $\epsilon_0 > 0$, then $\hat{\omega}_S(B) \supset \tilde{\omega}_S(\mathcal{A}_.)$.

Theorem 3.1. *Suppose that $\mathcal{C}_0$ is an invariant set of the C^0-semigroup $T_0(t)$ on Y_0 which is compact in Y_0 (resp., bounded in Y_0) (resp., bounded in Y_1). If hypothesis (2.2) (resp., (2.2bis)) (resp., (2.2ter)) holds and if B is a set in Y_0 such that $B \supset \mathcal{C}_0$, then*

$$\hat{\omega}_S(B) \supset \mathcal{C}_0.$$

Proof. Fix $t_0 > 0$. If $v_0 \in \mathcal{C}_0$, there exist two sequences $v_{0n} \in \mathcal{C}_0$ and $t_n \to \infty$ as $n \to \infty$, $t_n \ge t_0 > 0$, such that $v_0 = T_0(t_n)v_{0n}$. By (2.2), for any $v_{0n} \in \mathcal{C}_0$ and t_n, we can find a positive number $\epsilon_n \in S$, $\epsilon_n \to 0$ as $n \to \infty$, such that

$$\|T_{\epsilon_n}(t_n)P_{\epsilon_n}v_{0n} - T_0(t_n)v_{0n}\|_{Y_0} < \frac{1}{n}.$$

Therefore, we have sequences $t_n \to \infty$, $\epsilon_n \to 0$ and $v_{0n} \in \mathcal{C}_0 \subset B$ such that

$$\|T_{\epsilon_n}(t_n)P_{\epsilon_n}v_{0n} - v_0\|_{Y_0} < \frac{1}{n}.$$

This completes the proof.

Theorem 3.2. *Let B be any bounded set in Y_0 and suppose that $T_0(t)$ is a C^0-semigroup on Y_0 and that the ω-limit set $\omega_0(B) \equiv \mathcal{A}_0(B)$ of B with respect to $T_0(t)$ exists and is nonempty. If (2.2bis) is satisfied, then $\hat{\omega}_S(B) \supset \mathcal{A}_0(B)$.*

Proof. If $z \in \mathcal{A}_0(B)$, then there are sequences $t_n \to \infty$, $y_n \in B$, such that $T_0(t_n)y_n \to z$ as $n \to \infty$. Let $t_0 > 0$ be fixed. For any integer m, there exists an integer n_m such that, for $n \ge n_m$, we have $\|T_0(t_n)y_n - z\|_{Y_0} \le 1/2m$ and $t_n \ge t_0$. From the property (2.2bis), there exist $\epsilon_{n_m} \in S$, $\epsilon_{n_m} \to 0$ as $m \to \infty$, such that, for any $y \in B$,

$$\|T_0(t_{n_m})y - T_{\epsilon_{n_m}}(t_{n_m})P_{\epsilon_{n_m}}y\|_{Y_0} \le \frac{1}{2m}.$$

Therefore, we have

$$\|z - T_{\epsilon_{n_m}}(t_{n_m})P_{\epsilon_{n_m}}y_{n_m}\|_{Y_0} \le \frac{1}{m}.$$

This shows that $z \in \hat{\omega}_S(B)$ and completes the proof.

Theorem 3.3. *(i) Let B be any set in Y_0. If $T_0(t)$ is a C^0-semigroup on Y_0 and the condition (2.2) is satisfied, then $T_0(t)\hat{\omega}_S(B) \subset \hat{\omega}_S(B)$.*

(ii) If $t_0 \geq 0$ is a fixed constant and we suppose in addition that

$$B_1 \equiv \cup_{\epsilon \in S \cap \mathcal{N}_E(0,\delta)} \cup_{t \geq t_0} T_\epsilon(t) P_\epsilon B$$

is bounded, that (2.2bis) holds and either that

(H_1) $\quad$ *$T_0(t)$ is asymptotically smooth and $\cup_{t\geq 0} T_0(t) B_1$ is bounded*

or that

$$(H_2) \qquad T_0(t) \text{ is a } C^0\text{-group on } \mathbb{R},$$

then $\hat{\omega}_S(B)$ is invariant under $T_0(t)$.

Proof. If $y \in \hat{\omega}_S(B)$, then there are sequences $\{\epsilon_n\} \subset S \cap \mathcal{N}_E(0,\delta)$, $\{t_n\} \subset [0,\infty)$, $\{y_n\} \subset B$ such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n) P_{\epsilon_n} y_n \to y$ as $n \to \infty$. Let $t_0 > 0$ be fixed. From (2.2) and the continuity of $T_0(t)y$ in y, for any $\eta > 0$, there is an integer $n_0(\eta)$ such that, for $n \geq n_0(\eta)$, we have

$$\|T_0(t) T_{\epsilon_n}(t_n) P_{\epsilon_n} y_n - T_{\epsilon_n}(t_n + t) P_{\epsilon_n} y_n\|_{Y_0} < \frac{\eta}{2}$$
$$\|T_0(t) y - T_0(t) T_{\epsilon_n}(t_n) P_{\epsilon_n} y_n\|_{Y_0} < \frac{\eta}{2}.$$

As a consequence of these inequalities, we have

$$\|T_0(t) y - T_{\epsilon_n}(t_n + t) P_{\epsilon_n} y_n\|_{Y_0} < \eta,$$

which implies that $T_0(t) y \in \hat{\omega}_S(B)$. This completes the proof of the first part of the theorem.

To prove the last part of the theorem, it is sufficient to show that, for y as above and any $t_1 > 0$, there is a $\tilde{y} \in \hat{\omega}_S(B)$ such that $T_0(t_1)\tilde{y} = y$. Assume that (H_1) is satisfied and let $B_0 = \mathrm{Cl} \cup_{t\geq 0} T_0(t) B_1$. The set B_0 is a closed bounded set in Y_0 and $T_0(t) B_0 \subset B_0$. Since $T_0(t)$ is asymptotically smooth, there exists a compact set $J \subset B_0$ such that J attracts B_0. Let $t_1 > 0$ be fixed. For any integer $k > 0$, there exists an integer m_k such that

$$T_0(m_k t_1) B_0 \subset \mathcal{N}_{Y_0}(J, \frac{1}{2k}) \text{ and } m_k t_1 \geq t_0. \tag{3.1}$$

If we choose n large enough so that $t_n - (m_k + 1)t_1 > t_0$, then, from (3.1), we deduce that there exists $j_{k,n} \in J$ such that

$$\|T_0(m_k t_1) T_{\epsilon_n}(t_n - (m_k+1)t_1) P_{\epsilon_n} y_n - j_{k,n}\|_{Y_0} \leq \frac{1}{2k}. \tag{3.2}$$

Since B_1 is bounded, hypothesis (2.2bis) implies that there exists a positive number ϵ_{m_k} such that, for $0 < \epsilon_n \leq \epsilon_{m_k}$,

(3.3)
$$\|T_0(m_k t_1) T_{\epsilon_n}(t_n - (m_k+1)t_1) P_{\epsilon_n} y_n - T_{\epsilon_n}(m_k t_1) T_{\epsilon_n}(t_n - (m_k+1)t_1) P_{\epsilon_n} y_n\|_{Y_0}$$

$$< \frac{1}{2k}.$$

From inequalities (3.2) and (3.3), for $0 < \epsilon_n \le \epsilon_{m_k}$, we have

$$\|T_{\epsilon_n}(t_n - t_1)P_{\epsilon_n} y_n - j_{k,n}\|_{Y_0} \le \frac{1}{k}\,. \tag{3.4}$$

In particular, there exists a subsequence $n_k \to \infty$ such that for $0 < \epsilon_{n_k} \le \epsilon_{m_k}$,

$$\|T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} - j_{k,n}\|_{Y_0} \le \frac{1}{k}\,, \tag{3.5}$$

where $j_k = j_{k,n_k}$. Since J is compact, we can extract a subsequence of the sequence j_k, which we still denote by j_k, such that $j_k \to \tilde{y}$ as $k \to \infty$. Therefore, for any $\eta > 0$, there exists k_0 such that, for $k \ge k_0$, we have, from (3.5),

$$\|T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} - \tilde{y}\|_{Y_0} \le \frac{1}{k} + \frac{\eta}{2} \le \eta\,, \tag{3.6}$$

Therefore, $T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} \to \tilde{y}$ as $n_k. \to \infty$ and $\tilde{y} \in \hat{\omega}(B)$. Also,

$$T_0(t_1)T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} \to T_0(t_1)\tilde{y} \quad \text{as} \quad n_k \to \infty.$$

On the other hand,

$$T_{\epsilon_{n_k}}(t_1)T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} = T_{\epsilon_{n_k}}(t_{n_k})P_{\epsilon_{n_k}} y_{n_k} \to y\,.$$

Since, for any $\eta > 0$, there exists n_0 such that, for $n_k \ge n_0$,

$$\|T_{\epsilon_{n_k}}(t_1)T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} - T_0(t_1)T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k}\|_{Y_0} \le \eta\,,$$

it follows that $T_0(t_1)T_{\epsilon_{n_k}}(t_{n_k} - t_1)P_{\epsilon_{n_k}} y_{n_k} \to y$ as $n_k \to \infty$. As a consequence, we have $T_0(t_1)\tilde{y} = y$ and the theorem is proved under hypothesis (H_1).

Let us now suppose that (H_2) is satisfied. We show that, for y as in the proof of part (i) and, for any $t_1 > 0$, there exists $\tilde{y} \in \hat{\omega}_S(B)$ such that $T_0(t_1)\tilde{y} = y$. Let $t_1 > 0$ be fixed. For any t, $t - t_1 \ge t_0$, $T_\epsilon(t - t_1)P_\epsilon B$ is contained in the bounded set B_1. Therefore, from the condition (2.2bis), for any $\eta_1 > 0$, there is a positive number $\epsilon_0 = \epsilon_0(\eta_1, t_1)$, such that, for $0 < \epsilon \le \epsilon_0$, for $x \in B$,

$$\|T_\epsilon(t_1)T_\epsilon(t - t_1)P_\epsilon x - T_0(t_1)T_\epsilon(t - t_1)P_\epsilon x\|_{Y_0} \le \eta_1\,. \tag{3.7}$$

Since $T_0(-t_1)z$ is continuous in z at $z = y$, for any $\eta_2 > 0$, there exists a positive number θ_0 such that

$$\|T_0(-t_1)z - T_0(-t_1)y\|_{Y_0} < \eta_2 \text{ if } \|y - z\|_{Y_0} \le \theta_0\,. \tag{3.8}$$

Choosing $\eta_1 = \theta_0/3$ in (3.7) and using the fact that

$$T_{\epsilon_n}(t_n)P_{\epsilon_n} y_n = T_{\epsilon_n}(t_1)T_{\epsilon_n}(t_n - t_1)P_{\epsilon_n} y_n \to y \text{ as } n \to \infty,$$

we can find an integer n_0 such that, for $n \geq n_0$, we have

$$\|y - T_0(t_1)T_{\epsilon_n}(t_n - t_1)P_{\epsilon_n}y_n\|_{Y_0} \leq \frac{2\theta_0}{3}\,. \tag{3.9}$$

From (3.9) and (3.8), we deduce that, for $n \geq n_0$,

$$\|T_0(-t_1)y - T_0(-t_1)T_0(t_1)T_{\epsilon_n}(t_n - t_1)P_{\epsilon_n}y_n\|_{Y_0} \leq \eta_2\,. \tag{3.10}$$

This implies that $T_{\epsilon_n}(t_n - t_1)P_{\epsilon_n}y_n \to T_0(-t_1)y$ as $n \to \infty$.
Thus, $T_0(-t_1)y \in \hat{\omega}_S(B)$ and the theorem is proved.

Remark 3.4. The second part of the theorem also is true if the hypothesis (H_1) is replaced by the following one:

(H_3) there is a positive time t_1 such that $T_0(t)$ is compact for $t \geq t_1$.

Remark 3.5. If the hypothesis (H_1) of Theorem 3.3 holds, then, since $\hat{\omega}_S(B)$ is closed and invariant under $T_0(t)$, we conclude from the asymptotically smoothness of $T_0(t)$ that $\hat{\omega}_S(B)$ is compact.

We now want to present some situations where $\hat{\omega}_S(B) = \omega_0(B)$. As a direct consequence of Theorems 3.2 and 3.3, we have the following result.

Corollary 3.6. *Let B be a bounded set in Y_0. Suppose that $T_0(t)$ is a C^0-semigroup on Y_0 and that the ω-limit set $\omega_0(B) \equiv \mathcal{A}_0(B)$ of B with respect to $T_0(t)$ exists, is nonempty and attracts B. If the hypotheses of Theorem 3.3 are satisfied and $\hat{\omega}_S(B) \subset B$, then*

$$\hat{\omega}_S(B) = \mathcal{A}_0(B)\,.$$

Proof. From Theorem 3.2, we have $\hat{\omega}_S(B) \supset \mathcal{A}_0(B)$. From Theorem 3.3, $T_0(t)\hat{\omega}_S(B) = \hat{\omega}_S(B)$ for all $t \geq 0$. Since $\hat{\omega}_S(B) \subset B$ and $\mathcal{A}_0(B)$ attracts B, by invariance of $\hat{\omega}_S(B)$, it follows that $\hat{\omega}_S(B) \subset \mathcal{A}_0(B)$. This completes the proof.

Theorem 3.7. *Let B be a bounded set in Y_0 and suppose that $T_0(t)$ is a C^0-semigroup on Y_0 and that the ω-limit set $\omega_0(B) \equiv \mathcal{A}_0(B)$ of B with respect to $T_0(t)$ exists, is nonempty and attracts B. If, moreover, (2.2bis) holds and either $\mathcal{A}_0(B) \subset \overset{\circ}{B}$ (the interior of B) or $\mathcal{A}_0(B)$ attracts also a neighborhood of B, then*

$$\hat{\omega}_S(B) = \mathcal{A}_0(B)\,.$$

In particular, if $T_0(t)$ has a global attractor $\mathcal{A}_0$, if $\mathcal{A}_0 \subset B$ and (2.2bis) holds, then

$$\hat{\omega}_S(B) = \mathcal{A}_0\,.$$

Proof. We prove the theorem only in the case where $\mathcal{A}_0(B) \subset \overset{\circ}{B}$, since the proof in the other cases is similar. From Theorem 3.2, we know that $\hat{\omega}_S(B) \supset \mathcal{A}_0(B)$. Since

$\mathcal{A}_0(B) \subset \overset{\circ}{B}$, there exists $\eta_0 > 0$ such that, for $0 \leq \eta \leq \eta_0$, $\mathcal{N}_{Y_0}(\mathcal{A}_0(B), \eta) \subset \overset{\circ}{B}$. Let us show that, for $0 \leq \eta \leq \eta_0$,, $\hat{\omega}_S(B) \subset \mathcal{N}_{Y_0}(\mathcal{A}_0(B), \eta)$. Since $\mathcal{A}_0(B)$ attracts B, for any $0 < \eta \leq \eta_0$, there exists a positive time $\tau_0 = \tau_0(\eta, B)$ such that

$$T_0(t)B \subset \mathcal{N}_{Y_0}(\mathcal{A}_0(B), \frac{\eta}{2}) \text{ for } t \geq \tau_0 . \tag{3.11}$$

By (2.2bis), there exists a positive constant $\epsilon_0 = \epsilon_0(\eta, B, \tau_0)$ such that, for $0 \leq \epsilon \leq \epsilon_0$, for $y \in B$,

$$\|T_\epsilon(t)P_\epsilon y - T_0(t)y\|_{Y_0} \leq \frac{\eta}{2} \text{ for } \tau_0 \leq t \leq 2\tau_0 . \tag{3.12}$$

From (3.11) and (3.12), we conclude that

$$T_\epsilon(t)P_\epsilon y \in \mathcal{N}_{Y_0}(\mathcal{A}_0(B), \eta) \subset B \text{ for } \tau_0 \leq t \leq 2\tau_0, \ y \in B . \tag{3.13}$$

Since $T_\epsilon(t)P_\epsilon B \subset B$ for $\tau_0 \leq t \leq 2\tau_0$, we can apply again the above argument with $T_\epsilon(\tau_0)P_\epsilon B$. By a recursion argument, we then show that

$$T_\epsilon(t)P_\epsilon B \subset \mathcal{N}_{Y_0}(\mathcal{A}_0(B), \eta) \text{ for } t \geq \tau_0, \ 0 < \epsilon \leq \epsilon_0 , \tag{3.14}$$

which implies that

$$\hat{\omega}_S(B) \subset \mathcal{N}_{Y_0}(\mathcal{A}_0(B), \eta) . \tag{3.15}$$

Since (3.15) holds for any η, $0 < \eta \leq \eta_0$, we conclude that $\hat{\omega}_S(B) \subset \mathcal{A}_0(B)$. This completes the proof of the theorem.

In the case where each of the semigroups, including the one for $\epsilon = 0$, has a global attractor and the rate at which it attracts a bounded set is exponential and uniform in ϵ, we do not gain much additional information by considering the set $\hat{\omega}(B)$; that is, we may as well consider the limits of the attractors. This concept becomes more important when there is no global attractor for the semigroup at $\epsilon = 0$.

4. A parabolic equation on thin domains.

In this section, we obtain the analogues of Sections 2 and 3 for a parabolic equation on thin domains. At first, we define carefully the thin domains Q_ϵ over a bounded domain $\Omega \in \mathbb{R}^n$, $n = 1, 2$, with a smooth boundary. Suppose that ϵ_0 is a positive number and $g : \bar{\Omega} \times [0, \epsilon_0] \to R$ is a function of class C^3 satisfying:

$$\begin{aligned} &g(X,0) = 0, \qquad g_0(X) = \frac{\partial g}{\partial \epsilon}(X,0) > 0 \text{ for } X \in \bar{\Omega}, \\ &g(X,\epsilon) > 0 \text{ for } X \in \bar{\Omega}, \ \epsilon \in (0, \epsilon_0] . \end{aligned} \tag{4.1}$$

For $0 < \epsilon \leq \epsilon_0$, let Q_ϵ be the domain

$$(4.2) \qquad Q_\epsilon = \{(X, Y) \in R^2 : 0 < Y < g(X, \epsilon),\, X \in \Omega\}$$

and denote by ν_ϵ the outward normal to ∂Q_ϵ. Choose $\delta > 0$ so that $\tilde{Q} = \Omega \times (0, \delta)$ contains Q_ϵ for $0 < \epsilon \leq \epsilon_0$.

For α a positive constant and G^* a function belonging to $W^{1,\infty}(\tilde{Q})$, we consider the equation

$$(4.3) \qquad u_t - \Delta u + \alpha u = -f(u) - G^* \quad \text{in } Q_\epsilon$$

with the boundary conditions

$$(4.4) \qquad \frac{\partial u}{\partial \nu_\epsilon} = 0 \quad \text{in } \partial Q_\epsilon\,.$$

The function $f : R \to R$ is a C^2-function satisfying

$$(4.5) \qquad \limsup_{|x| \to +\infty} \frac{-f(x)}{x} \leq \alpha_0 < \alpha,$$

$$(4.6) \qquad |f''(x)| \leq c(1 + |x|^\gamma) \text{ for } x \in R$$

where $0 \leq \gamma < +\infty$ if $n = 1$ and $0 \leq \gamma \leq 1$ if $n = 2$.

The hypotheses on α, G^* and g could be weakened, but we avoid additional technicalities by imposing them.

In the remainder of this section, we suppose that $n = 2$. The modifications that are necessary for $n = 1$ will be clear. If we transform coordinates to the canonical domain $Q = \Omega \times (0,\, 1)$ by letting $X = x$, $Y = g(x,\, \epsilon)y$, we obtain the system

$$(4.7)_\epsilon \qquad u_t + L_\epsilon u + \alpha u = -f(u) - G^*_\epsilon \quad \text{in } Q$$

with the boundary conditions

$$(4.8)_\epsilon \qquad \frac{\partial u}{\partial \nu_{B_\epsilon}} \equiv B_\epsilon u \cdot \nu = 0 \quad \text{in } \partial Q\,,$$

where

$$(4.9) \qquad G^*_\epsilon(x, y) = G^*(x, g(x, \epsilon)y),$$

ν is the unit outward normal to ∂Q and L_ϵ is the operator:

$$(4.10) \qquad L_\epsilon = -\frac{1}{g}\, div B_\epsilon u$$

where

$$B_\epsilon u = \begin{bmatrix} gu_{x_1} - g_{x_1} y u_y \\ gu_{x_2} - g_{x_2} y u_y \\ -g_{x_1} y u_{x_1} - g_{x_2} y u_{x_2} + \frac{1}{g}(1 + (g_{x_1} y)^2 + (g_{x_2} y)^2) u_y \end{bmatrix}. \tag{4.11}$$

We also need to write the equation $(4.7)_\epsilon, (4.8)_\epsilon$ as an abstract evolutionary equation. For notation, we let $\|.\|_{0,Q}$, $\|.\|_{1,Q}$ and $\|.\|_{2,Q}$ denote respectively the classical norms in $L^2(Q)$, $H^1(Q)$ and $H^2(Q)$. For $0 < \epsilon \leq \epsilon_0$,we let $H_\epsilon = X^0_\epsilon$ be the space $L^2(Q)$ endowed with the norm $\| \cdot \|_{H_\epsilon}$ induced by the inner product

$$(u_1,\, u_2)_{H_\epsilon} = \int_Q \frac{g}{\epsilon} u_1 u_2 \, dx\, dy.$$

The hypothesis (4.1) implies that there are positive constants c_1, C_1 (independent of ϵ) such that $c_1 \|u\|_{0,Q} \leq \|u\|_{H_\epsilon} \leq C_1 \|u\|_{0,Q}$ for any $u \in L^2(Q)$. To rewrite the equation $(4.7)_\epsilon$, $(4.8)_\epsilon$, we need the bilinear form $a_\epsilon(\cdot,\cdot)$ on $(H^1(Q))^2$ (which is derived from the form:

$$(u_1,\, u_2) \mapsto \int_{Q_\epsilon} (\nabla u_1 \nabla u_2 + \alpha u_1 u_2) \, dX dY$$

by the change of variables in going from Q_ϵ to Q):

$$a_\epsilon(u_1,\, u_2) = (\mathcal{L}^{1/2}_\epsilon u_1, \mathcal{L}^{1/2}_\epsilon u_2)_{H_\epsilon} + \alpha (u_1,\, u_2)_{H_\epsilon}$$

where $\mathcal{L}^{1/2}_\epsilon$ is the gradient operator on $H^1(Q)$:

$$\mathcal{L}^{1/2}_\epsilon u = (u_{x_1} - \frac{g_{x_1}}{g} y u_y\, , u_{x_2} - \frac{g_{x_2}}{g} y u_y\, , \frac{1}{g} u_y)\, .$$

It is well known that $a_\epsilon(\cdot,\cdot)$ defines an unbounded linear operator A_ϵ on $H^1(Q)$ which is selfadjoint on H_ϵ, positive, $A_\epsilon = L_\epsilon + \alpha I$ with homogeneous Neumann boundary conditions, and $\mathcal{D}(A^{1/2}_\epsilon) \cong H^1(Q)$. By the definition of $A^{1/2}_\epsilon$, we have, for all $u \in H^1(Q)$, the following relation: $[a_\epsilon(u,u)]^{1/2} = \|A^{1/2}_\epsilon u\|_{H_\epsilon}$. Furthermore, the conditions (4.1) on g imply that there are constants c_2, C_2 such that

$$c_2(\|u\|^2_{1,Q} + \frac{1}{\epsilon^2}\|u_y\|^2_{0,Q})^{1/2} \leq \|A^{1/2}_\epsilon u\|_{H_\epsilon} \leq C_2(\|u\|^2_{1,Q} + \frac{1}{\epsilon^2}\|u_y\|^2_{0,Q})^{1/2}. \tag{4.12}$$

For $s = 0,1,2$, let X^s_ϵ be the space $D(A^{s/2}_\epsilon)$ endowed with the norm $\|u\|_{X^s_\epsilon} = \|A^{s/2}_\epsilon u\|_{H_\epsilon}$. From [9, Appendix A]), we have

$$\mathcal{D}(A_\epsilon) = \{u \in H^2(Q) : \frac{\partial u}{\partial \nu_{B_\epsilon}} = 0 \text{ in } \partial Q\}\, .$$

With this notation, the equations $(4.7)_\epsilon, (4.8)_\epsilon$ are equivalent to the abstract evolutionary equation

$$(4.13)_\epsilon \qquad u_t + A_\epsilon u = -f(u) - G^*_\epsilon\,.$$

We remark that the function $f : u \in H^1(Q) \mapsto f(u) \in L^2(Q)$ is a C^1-mapping. We compare the problem $(4.7)_\epsilon$, $(4.8)_\epsilon$ with the following problem on Ω :

$$(4.7)_0 \qquad v_t - \frac{1}{g_0}(g_0 v_{x_1})_{x_1} - \frac{1}{g_0}(g_0 v_{x_2})_{x_2} + \alpha v = -f(v) - G^*_0 \quad \text{in } \Omega\,,$$

where $G^*_0 = G^*(x,\,0)$, with the boundary conditions

$$(4.8)_0 \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{in } \partial\Omega\,.$$

As above, we let $H_0 = X^0_0$ be the space $L^2(\Omega)$ endowed with the norm $\|\cdot\|_{H_0}$ induced by the inner product

$$(v_1,\,v_2)_{H_0} = \int_\Omega g_0 v_1 v_2\, dx.$$

By hypothesis (4.1), the norm $\|\cdot\|_{H_0}$ is equivalent to the classical norm $\|\cdot\|_{0,\Omega}$ of $L^2(\Omega)$. We also introduce the bilinear form $a_0(\cdot,\,\cdot)$ on $(H^1(\Omega))^2$:

$$a_0(v_1,\,v_2) = \int_\Omega \nabla v_1 \cdot \nabla v_2 g_0\, dx + \alpha(v_1,\,v_2)_{H_0}\,.$$

The form $a_0(\cdot,\,\cdot)$ defines an unbounded linear operator A_0 on $H^1(\Omega)$ which is selfadjoint on H_0, positive. We have $H^1(\Omega) \simeq D(A_0^{1/2})$,

$$D(A_0) = \{\, v \in H^2(\Omega) : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \,\}\,,$$

and $A_0 = -\frac{1}{g_0}\frac{\partial}{\partial x_1}(g_0\frac{\partial\cdot}{\partial x_1}) - \frac{1}{g_0}\frac{\partial}{\partial x_2}(g_0\frac{\partial\cdot}{\partial x_2}) + \alpha I$ with homogeneous Neumann boundary conditions. As above, for $s = 0, 1, 2$, we let X^s_0 be the space $D(A_0^{s/2})$ endowed with the norm $\|v\|_{X^s_0} = \|A_0^{s/2} v\|_{H_0}$.

With the above notation, the abstract evolutionary equation corresponding to the problem $(4.7)_0$, $(4.8)_0$ is

$$(4.13)_0 \qquad v_t + A_0 v = -f(v) - G^*_0\,,$$

The problems $(4.13)_\epsilon$ have been studied in detail in [9]. Let $T_\epsilon(t)$ (resp. $T_0(t)$) be the semigroup generated by $(4.13)_\epsilon$ on X^1_ϵ (resp. $(4.13)_0$ on X^1_0).

The family of maps $\{ T_\epsilon(t),\, t \geq 0, \}$ is, of course, also a semigroup on $H^1(Q)$. In this section, we will define $\hat{\omega}(B)$ in the latter topology. In order to avoid confusion, we repeat the definition for this case.

Definition 4.1. For a given set $B \subset H^1(Q)$ and a given set $S \subset \mathbb{R}^+$ with $0 \in \bar{S}$, the ω-limit set of B with respect to the family of semigroups $T_\epsilon(t)$, $t > 0$, $\epsilon \in S$, is denoted by $\hat{\omega}_S(B)$ and is defined in the following way: a point $u_0 \in \hat{\omega}_S(B)$ if and only if there are sequences $\{ \epsilon_n \} \subset S$, $\{ t_n \} \subset [0, \infty)$, $\{ u_n \} \subset B$ such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n)u_n \to u_0$ in $H^1(Q)$ as $n \to \infty$.

We define the operator $M : H^1(Q) \to H^1(\Omega)$ by the relation

$$(M\varphi)(x) = \int_0^1 \varphi(x, y)\, dy \text{ for } x \in \Omega \text{ and for any } \varphi \in H^1(Q)$$

We will need the following property which is proved in [19].

Proposition 4.1. *Let t_0 be any fixed positive number. For any bounded set $U \subset [t_0, \infty) \times H^1(Q)$ and any $\eta > 0$, there exists a positive number $\epsilon_0 = \epsilon_0(\eta, U)$ such that, for $\epsilon \in S \cap \mathcal{N}(0, \epsilon_0)$, and $(t, u_0) \in U$, we have*

$$\|T_\epsilon(t)u_0 - T_0(t)Mu_0\|_{X^1_\epsilon} < \eta\,. \tag{4.14}$$

Moreover, for any positive constant r, there is a positive constant $K(r)$ depending only on r such that, if $\sup_{\varphi \in B} \|\varphi\|_{H^1(Q)} < r$, then, for $u_0 \in B$, for $t > 0$,

$$t\|T_\epsilon(t)u_0 - T_0(t)Mu_0\|_{X^1_\epsilon} \leq \epsilon K(r) e^{K(r)t}\,. \tag{4.15}$$

Theorem 4.2. *For any bounded set B in $H^1(\Omega)$, we have:*

1. *The set $\hat{\omega}_S(B)$ is contained in $H^1(\Omega)$, is invariant under $T_0(t)$ and $\hat{\omega}_S(B) \supset \mathcal{A}_0(MB) \equiv \omega_0(MB)$.*
2. *If we assume moreover that either $\mathcal{A}_0(MB)$ is contained in $\mathring{B}$ or $\mathcal{A}_0(MB) \subset B$ attracts also a neighborhood of MB, then*

$$\hat{\omega}_S(B) = \mathcal{A}_0(MB)\,. \tag{4.16}$$

In particular, if we assume that B contains the global attractor $\mathcal{A}_0$ of $T_0(t)$, then

$$\hat{\omega}_S(B) = \mathcal{A}_0\,.$$

Proof. Let t_0 be a fixed positive number. Let us first show that $\hat{\omega}_S(B)$ is contained in $H^1(\Omega)$; in particular, the points in $\hat{\omega}_S(B)$ do not depend upon y. By [19], for any $\tau_0 > 0$, there exist a positive constant $r = r(B, f, \tau_0)$ and a bounded set $\mathcal{B}_B$ in $H^1(Q)$ such that, for $\epsilon \in S \cap \mathcal{N}(0, \epsilon_0)$ and all $u_0 \in B$, we have

$$\|T_\epsilon(t)u_0\|_{X^1_\epsilon} \leq r, \text{ for } t \geq \tau_0\,, \tag{4.17}$$

and

$$\cup_{\epsilon\in S\cap\mathcal{N}(0,\,\epsilon_0)}(\cup_{t\geq\tau_0}T_\epsilon(t)B)\subset\mathcal{B}_B\,. \tag{4.18}$$

If $u_0 \in H^1(Q)\cap\hat{\omega}_S(B)$ and $u_0 \notin H^1(\Omega)$, then $u_0 \neq Mu_0$ and $\|u_0 - Mu_0\|_{L^2(Q)} \neq 0$. We will obtain a contradiction to this statement by showing that, for any $\eta > 0$, we have

$$\|u_0 - Mu_0\|_{L^2(Q)} < \eta\,. \tag{4.19}$$

Since $u_0 \in \hat{\omega}_S(B)$, there are sequences $\{\epsilon_n\} \subset S\cap\mathcal{N}(0,\epsilon_0)$, $\{t_n\}\subset[0,\infty)$, $t_n \geq t_0$, $\{u_n\}\subset B$ such that $\epsilon_n \to 0$, $t_n\to\infty$, $T_{\epsilon_n}(t_n)u_n \to u_0$ in $H^1(Q)$ as $n\to\infty$. Thus there is an integer n_0 such that, for $n \geq n_0$, we have $\|T_{\epsilon_n}(t_n)u_n - u_0\|_{1,Q} \leq \eta/3$. Therefore,

$$\begin{aligned}\|u_0 - Mu_0\|_{L^2(Q)}&\\ &\leq \|u_0 - T_{\epsilon_n}(t_n)u_n\|_{L^2(Q)} + \|T_{\epsilon_n}(t_n)u_n - MT_{\epsilon_n}(t_n)u_n\|_{L^2(Q)}\\ &\qquad + \|M(T_{\epsilon_n}(t_n)u_n - u_0)\|_{L^2(Q)}\\ &\leq \frac{2}{3}\eta + \|T_{\epsilon_n}(t_n)u_n - MT_{\epsilon_n}(t_n)u_n\|_{L^2(Q)}\,.\end{aligned} \tag{4.20}$$

From Lemma 3.1 of [9] and from (4.17), we deduce that, for $0<\epsilon_n\leq\epsilon_0$,

$$\|T_{\epsilon_n}(t_n)u_n - MT_{\epsilon_n}(t_n)u_n\|_{L^2(Q)} \leq c\epsilon_n r\,.$$

Thus, choosing an integer $n_1 \geq n_0$ such that $c\epsilon_n r < \eta/3$ for $n \geq n_1$, we obtain

$$\|T_{\epsilon_n}(t_n)u_n - MT_{\epsilon_n}(t_n)u_n\|_{L^2(Q)} < \frac{\eta}{3}\,. \tag{4.21}$$

From (4.20) and (4.21), we infer (4.19) with the conclusion that $u_0 \in H^1(\Omega)$.

Let us now show that $\hat{\omega}_S(B)$ is invariant under $T_0(t)$. If $v_0 \in \hat{\omega}_S(B)$, there are sequences $\{\epsilon_n\}\subset S\cap\mathcal{N}(0,\epsilon_0)$, $\{t_n\}\subset[0,\infty)$, $\{u_n\}\subset B$ such that $\epsilon_n \to 0$, $t_n\to\infty$, $T_{\epsilon_n}(t_n)u_n\to v_0$ in $H^1(Q)$ as $n\to\infty$. Fix $t>0$. For any $\eta>0$, there exists a positive number $\delta>0$ such that, if $\|v_1 - v_0\|_{H^1(\Omega)} \leq \delta$, then $\|T_0(t)v_1 - T_0(t)v_0\|_{H^1(\Omega)} < \eta/2$. There exists also an integer $n_0 = n_0(\delta)$ such that, for $n\geq n_0$, we have $\|T_{\epsilon_n}(t_n)u_n - v_0\|_{H^1(Q)} < \delta$ and, since $v_0 = Mv_0$, $\|MT_{\epsilon_n}(t_n)u_n - v_0\|_{H^1(\Omega)} < \delta$. As a consequence of this inequality, we obtain

$$\|T_0(t)MT_{\epsilon_n}(t_n)u_n - T_0(t)v_0\|_{H^1(Q)} < \frac{\eta}{2}\,. \tag{4.22}$$

On the other hand, from (4.15), we have

$$\|T_0(t)MT_{\epsilon_n}(t_n)u_n - T_{\epsilon_n}(t)T_{\epsilon_n}(t_n)u_n\|_{H^1(Q)} < \frac{\epsilon_n K(r)}{t}e^{K(r)t}\,. \tag{4.23}$$

Fix $t > 0$. As a consequence of (4.22) and (4.23), we can choose $n_1 > n_0(\delta)$ such that, for $n > n_1$,

$$\|T_0(t)v_0 - T_{\epsilon_n}(t)T_{\epsilon_n}(t_n)u_n\|_{H^1(Q)} < \eta \, .$$

This shows that $T_0(t)\hat{\omega}_S(B) \subset \hat{\omega}_S(B)$.

We now need to show that, if $v_0 \in \hat{\omega}_S(B)$, then $T_0(-t)v_0 \in \hat{\omega}_S(B)$. The sequence $T_0(t)MT_{\epsilon_n}(t_n - 2t)u_n$ has a subsequence (which we label the same) which converges to a point $\tilde{v}_0 \in H^1(\Omega)$. From (4.15), we know that

$$\|T_0(t)MT_{\epsilon_n}(t_n - 2t)u_n - T_{\epsilon_n}(t_n - t)u_n\|_{X^1_\epsilon} \leq \frac{\epsilon_n K(r)}{t} e^{K(r)t} \, .$$

Therefore, $T_{\epsilon_n}(t_n - t)u_n \to \tilde{v}_0$ in $H^1(Q)$ as $n \to \infty$. Hence, $\tilde{v}_0 \in \hat{\omega}(B)$. Using the estimate (4.15), we see that $T_0(t)MT_{\epsilon_n}(t_n - t)u_n \to v_0$ in $H^1(Q)$ as $n \to \infty$. On the other hand, since $T_{\epsilon_n}(t_n - t)u_n \to \tilde{v}_0$ in $H^1(Q)$ as $n \to \infty$, we have that $T_0(t)MT_{\epsilon_n}(t_n - t)u_n \to T_0(t)M\tilde{v}_0 = T_0(t)\tilde{v}_0$. Therefore, $v_0 = T_0(t)\tilde{v}_0 \in \hat{\omega}_S(B)$ and $\hat{\omega}_S(B)$ is invariant.

One proves as in Theorem 3.7, with very minor changes, that $\hat{\omega}_S(B) \subset \mathcal{A}_0(MB)$.

It remains to show that $\hat{\omega}_S(B) \supset \mathcal{A}_0(MB)$. If $v_0 \in \mathcal{A}_0(MB)$, there are sequences $t_n \to \infty$, $v_n \in MB$ such that $T_0(t_n)v_n \to v_0$ as $n \to \infty$. Let $t_0 > 0$ be fixed. For any integer m, there exists an integer n_m such that, for $n \geq n_m$, we have

$$t_n \geq t_0 \ \text{ and } \ \|T_0(t_n)v_n - v_0\|_{H^1(\Omega)} \leq \frac{1}{2m} \, . \tag{4.24}$$

Since $v_n \in MB$, there exists $u_n = v_n + w_n$ in B such that $Mu_n = v_n$. By (4.15), there is an $\epsilon_{n_m} \in S$, $\epsilon_{n_m} \to 0$ as $m \to \infty$, such that

$$\|T_0(t_{n_m})v_{n_m} - T_{\epsilon_{n_m}}(t_{n_m})u_{n_m}\|_{H^1(Q)} \leq \frac{1}{2m} \, . \tag{4.25}$$

Therefore, we have

$$\|v_0 - T_{\epsilon_{n_m}}(t_{n_m})u_{n_m}\|_{H^1(Q)} \leq \frac{1}{m} \, , \tag{4.26}$$

which implies that $v_0 \in \hat{\omega}_S(B)$ and completes the proof.

Remark 4.3. As in Definition 1.1, we can introduce the limit

$$\tilde{\omega}_S(\mathcal{A}_\cdot) = \cap_{\delta > 0} \operatorname{Cl}_{H^1(Q)} \cup_{\epsilon \in S, |\epsilon| < \delta} \mathcal{A}_\epsilon \, .$$

In [9], we have proved that the attractors $\mathcal{A}_\epsilon$ are upper semicontinuous at $\epsilon = 0$. Therefore, as in Proposition 2.1, one shows that

$$\tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0 \, . \tag{4.27}$$

In [9], we have shown also that $\cup_{\epsilon \in S} \mathcal{A}_\epsilon$ is contained in a closed, bounded ball in $H^2(Q)$. Therefore, the hypothesis (i) of Proposition 2.3 holds and one can show, as in Proposition 2.3, that $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is invariant under $T_0(t)$.

If all of the equilibrium points are hyperbolic, then, by [19], the attractors $\mathcal{A}_\epsilon$ are lower semicontinuous at $\epsilon = 0$, which implies, by Proposition 2.2, that $\tilde{\omega}_S(\mathcal{A}_\cdot) = \mathcal{A}_0$.

If we assume that $S = (0, \delta_0)$, then it is easy to verify that all of the hypotheses of Proposition 2.4 are satisfied. Thus, $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is connected.

5. A damped wave equation on thin domains.

In this section, we obtain the analogues of Sections 2 and 3 for a hyperbolic equation on thin domains. We keep the notation of Section 4. In particular, we consider the same thin domain Q_ϵ. But now, for α and β positive constants, we consider the damped wave equation:

$$(5.1) \qquad u_{tt} + \beta u_t - \Delta u + \alpha u = -f(u) - G^* \text{ in } Q_\epsilon$$

with the boundary conditions

$$(5.2) \qquad \frac{\partial u}{\partial \nu_\epsilon} = 0 \text{ in } \partial Q_\epsilon,$$

where f and G^* are the functions introduced in Section 4.1. In particular, f satisfies the conditions (4.5), (4.6) with $\gamma < 1$ if $n = 2$. If we make the change of variables $X = x$, $Y = g(x, \epsilon)y$ which transforms Q_ϵ into the canonical domain Q, we obtain the system

$$(5.3)_\epsilon \qquad u_{tt} + \beta u_t + L_\epsilon u + \alpha u = -f(u) - G^*_\epsilon \text{ in } Q,$$

with the boundary conditions

$$(5.4)_\epsilon \qquad \frac{\partial u}{\partial \nu_{B_\epsilon}} \equiv B_\epsilon u \cdot \nu = 0 \text{ in } \partial Q,$$

where L_ϵ, B_ϵ, G^*_ϵ are given in (4.10), (4.11) and (4.9). As in Section 4, we introduce the operator A_ϵ and the spaces X^s_ϵ, $s = 0, 1, 2$. For $s = 1, 2$, we also define the space $Y^s_\epsilon = D(A^{s/2}_\epsilon) \times D(A^{(s-1)/2}_\epsilon)$ endowed with the norm $\|(\varphi, \psi)\|_{Y^s_\epsilon} = (\|\varphi\|^2_{X^s_\epsilon} + \|\psi\|^2_{X^{s-1}_\epsilon})^{1/2}$. Clearly, Y^1_ϵ is isomorphic to $H^1(Q) \times L^2(Q)$ and Y^2_ϵ is isomorphic to $\{\varphi \in H^2(Q) : \frac{\partial \varphi}{\partial \nu_{B_\epsilon}} = 0 \text{ in } \partial Q\} \times H^1(Q)$.

With this notation, the equations $(5.3)_\epsilon$, $(5.4)_\epsilon$, with initial data $(\varphi, \psi) \in Y^1_\epsilon$, are equivalent to the abstract evolutionary equation

$$(5.5)_\epsilon \qquad u_{tt} + \beta u_t + A_\epsilon u = -f(u) - G^*_\epsilon .$$

We compare the problem $(5.3)_\epsilon$, $(5.4)_\epsilon$ with the following problem on Ω:

$$(5.3)_0 \qquad v_{tt} + \beta v_t - \frac{1}{g_0}(g_0 v_{x_1})_{x_1} - \frac{1}{g_0}(g_0 v_{x_2})_{x_2} + \alpha v = -f(v) - G_0^* \text{ in } Q\,,$$

with the boundary conditions

$$(5.4)_0 \qquad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\,.$$

As in Section 4, we introduce the operator A_0 and, for $s = 0, 1, 2$, the spaces X_0^s. For $s = 1, 2$, we also define the spaces $Y_0^s = D(A_0^{s/2}) \times D(A_0^{(s-1)/2})$ endowed with the norm $\|(\varphi, \psi)\|_{Y_0^s} = (\|\varphi\|^2_{X_0^s} + \|\psi\|^2_{X_0^{s-1}})^{1/2}$. Clearly, Y_0^1 is isomorphic to $H^1(\Omega) \times L^2(\Omega)$ and Y_0^2 is isomorphic to $\{\varphi \in H^2(\Omega);\ \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega\} \times H^1(\Omega)$. With this notation, the abstract evolutionary equation corresponding to the problem $(5.3)_0$, $(5.4)_0$ is

$$(5.5)_0 \qquad v_{tt} + \beta v_t + A_0 v = -f(v) - G_0^*\,.$$

The problems $(5.5)_\epsilon$, for $0 \le \epsilon \le \epsilon_0$, have been studied in detail in [12]. Let $T_\epsilon(t)$ (resp. $T_0(t)$) be the semigroup generated by $(5.5)_\epsilon$ on $Y_\epsilon^1 = X_\epsilon^1 \times X_\epsilon^0$ (resp. by $(5.5)_0$ on $Y_0^1 = X_0^1 \times X_0^0$).

At first, we recall the following results from [12].

Proposition 5.1. *Fix $\beta_0 > 0$, $\epsilon_0 > 0$ and suppose that $\beta \ge \beta_0$ and $\gamma < 1$. For any $r_0 > 0$, there is a constant $C_0(r_0)$ such that, for $0 < \epsilon \le \epsilon_0$, the solution $U^\epsilon(t) = (u^\epsilon(t),\, u_t^\epsilon(t))$ of $(5.5)_\epsilon$ with $\|U^\epsilon(0)\|_{Y_\epsilon^1} \le r_0$, satisfies, for $t \ge 0$,*

$$(5.6) \qquad \|U^\epsilon(t)\|_{Y_\epsilon^1} \le C_0(r_0)\,.$$

If, in addition, Ω satisfies hypothesis (H) of [12], then:

(1) There exist a constant $K > 0$ and, for any $r_1 > 0$, $r_2 > 0$, two positive constants $K_1^*(r_1)$, $K_2^*(r_1, r_2)$ such that, for $0 \le \epsilon \le \epsilon_0$, any solution $U^\epsilon(t)$ of $(5.5)_\epsilon$ with $\|U^\epsilon(0)\|_{Y_\epsilon^i} \le r_i$, $i = 1, 2$. satisfies, for $t \ge 0$,

$$(5.7) \qquad \|u_{tt}^\epsilon(t)\|^2_{X_\epsilon^0} + \|U^\epsilon(t)\|_{Y_\epsilon^2} \le K_1^*(r_1) + K_2^*(r_1, r_2)e^{-Kt}\,.$$

(2) There is a constant K_3 such that, for $0 \le \epsilon \le \epsilon_0$, we have

$$(5.8) \qquad \|(\varphi, \psi)\|_{Y_\epsilon^2} \le K_3 \text{ for } (\varphi, \psi) \in \mathcal{A}_\epsilon\,,$$

where $\mathcal{A}_\epsilon$ is the global attractor of $(5.5)_\epsilon$ in Y_ϵ^1.

(3) There is a positive constant c and, for any $r > 0$, there is a positive constant $k(r)$ such that, for $0 \le \epsilon \le \epsilon_0$, any solution $U^\epsilon(t)$ of $(5.5)_\epsilon$ with $\|U^\epsilon(0)\|_{Y_\epsilon^2} \le r$ satisfies, for $t \ge 0$,

$$(5.9) \qquad \begin{aligned} \|U^\epsilon(t) - T_0(t)MU^\epsilon(0)\|^2_{Y_\epsilon^1} &\le (\epsilon + \|(I-M)U^\epsilon(0)\|^2_{Y_\epsilon^1})k(r)e^{k(r)t} \\ &\le c\epsilon k(r)e^{k(r)t} \\ \|U^\epsilon(t) - T_0(t)MU^\epsilon(0)\|^2_{H^1(Q)\times L^2(Q)} &\le (\epsilon + \|(I-M)U^\epsilon(0)\|^2_{Y_\epsilon^1})k(r)e^{k(r)t} \\ &\le c\epsilon k(r)e^{k(r)t}\,. \end{aligned}$$

Let $S_\epsilon(t)$ be the semigroup on Y^1_ϵ generated by the linear equation

$$(5.10)_\epsilon \qquad u_{tt} + \beta u_t + A_\epsilon u = 0$$

with initial data $(u(0), u_t(0)) = U_0 \in Y^1_\epsilon$. Let λ^0_1 be the first eigenvalue of A_0 and let b be any positive number satisfying

$$(5.11) \qquad 0 < b \leq \min(\frac{\beta}{8}, \frac{3\lambda^0_1}{4\beta}, \frac{\sqrt{3\lambda^0_1}}{8}).$$

In [12, Lemma 2.2], it is proved that, for $0 \leq \epsilon \leq \epsilon_0$, for $t \geq 0$,

$$(5.12) \qquad \|S_\epsilon(t)U_0\|_{Y^1_\epsilon} \leq \sqrt{3}e^{-\frac{2bt}{3}}\|U_0\|_{Y^1_\epsilon}.$$

In [12, Proof of Theorem 3.4], we also have shown that, for $0 \leq \epsilon \leq \epsilon_0$, for $t \geq 0$,

$$(5.13) \qquad \|S_\epsilon(t)U_0\|_{Y^2_\epsilon} \leq C_2 e^{-c_2 t}\|U_0\|_{Y^2_\epsilon},$$

where C_2, c_2 are positive constants.

Let θ be a real number, $0 \leq \theta \leq 1$, and let $[Y^2_\epsilon, Y^1_\epsilon]_\theta = [X^2_\epsilon, X^1_\epsilon]_\theta \times [X^1_\epsilon, X^0_\epsilon]_\theta$ be an interpolation space between Y^2_ϵ and Y^1_ϵ. Thanks to (5.12) and (5.13), there exist two positive constants C_*, c_* such that, for $0 \leq \epsilon \leq \epsilon_0$, $0 \leq \theta \leq 1$, $t \geq 0$,

$$(5.14) \qquad \|S_\epsilon(t)U_0\|_{[Y^2_\epsilon, Y^1_\epsilon]_\theta} \leq C_* e^{-c_* t}\|U_0\|_{[Y^2_\epsilon, Y^1_\epsilon]_\theta},$$

We also will need the following auxiliary result.

Lemma 5.2. *There is a positive constant c_0 and, for any $r > 0$, there is a positive constant $k_0(r)$ such that, for $0 \leq \epsilon \leq \epsilon_0$, any solution $U^\epsilon(t) \equiv (u^\epsilon(t), u^\epsilon_t(t))$ of $(5.5)_\epsilon$ with $\|U^\epsilon(0)\|_{Y^1_\epsilon} \leq r$ satisfies, for $t \geq 0$,*

$$(5.15) \qquad \|A_\epsilon^{-1/2}(u^\epsilon_t(t) - v_t(t))\|_{H_\epsilon} + \|u^\epsilon(t) - v(t)\|_{H_\epsilon}$$
$$\leq c_0(\epsilon + \|A_\epsilon^{-1/2}(u^\epsilon_t(0) - v_t(0))\|_{H_\epsilon} + \|u^\epsilon(0) - v(0)\|_{H_\epsilon})e^{k_0(r)t},$$

where $(v(t), v_t(t)) \equiv T_0(t)MU^\epsilon(0)$.

Sketch of the proof. The proof is very similar to the proof of the estimate (5.9) (see [12, Proposition 5.1]). For this reason, we are not going to give the entire proof. We only point out two arguments which are not contained in [12]. In the proof of (5.15), one needs the following two estimates. At first, by (4.12), we can write

$$(5.16) \qquad \|\frac{1}{\epsilon}\frac{\partial}{\partial y}(A_\epsilon^{-1}(u^\epsilon_t(t) - v_t(t))\|_{H_\epsilon} \leq C\|A_\epsilon^{-1/2}(u^\epsilon_t(t) - v_t(t))\|_{H_\epsilon}.$$

On the other hand, we have the following estimate:

$$\|A_\epsilon^{-1/2}(f(u^\epsilon) - f(v))\|_{H_\epsilon} = \sup_{w \in X^1_\epsilon} \frac{(\int_0^1 f'(v + s(u^\epsilon - v))(u^\epsilon - v)ds, w)_{H_\epsilon}}{\|A_\epsilon^{1/2}w\|_{H_\epsilon}}$$

$$\leq C \sup_{w \in X_\epsilon^1} \frac{\int_Q (1 + |v|^{\gamma+1} + |u^\epsilon|^{\gamma+1})|u^\epsilon - v|\,|w| dx dy}{\|A_\epsilon^{1/2} w\|_{H_\epsilon}}$$
$$\leq C\|u^\epsilon - v\|_{H_\epsilon}(1 + \|v\|^{\gamma+1}_{L^{3(\gamma+1)}(Q)} + \|u^\epsilon\|^{\gamma+1}_{L^{3(\gamma+1)}(Q)}),$$

which implies that

$$\|A_\epsilon^{-1/2}(f(u^\epsilon) - f(v))\|_{H_\epsilon} \leq Ck_1(r)\|u^\epsilon - v\|_{H_\epsilon}\,, \tag{5.17}$$

where $k_1(r)$ is a positive increasing function of r, independent of ϵ, for $0 \leq \epsilon \leq \epsilon_0$.

Let us now consider the limit set $\hat{\omega}$. We can consider $T_\epsilon(t)$ as a semigroup on $H^1(Q) \times L^2(Q)$ and, in the definition of $\hat{\omega}(B)$, we will use the topology of $H^1(Q) \times L^2(Q)$. We may always define $\hat{\omega}(B)$ for any bounded subset of $H^1(\Omega) \times L^2(\Omega)$, but are not able to consider an arbitrary bounded set in $H^1(Q) \times L^2(Q)$. The precise definition is as follows.

Definition 5.1. Let $S \subset \mathbb{R}^+$ be a given set such that $0 \in \bar{S}$. For fixed positive constants C_1 and ϵ_0, let $\mathcal{W} = \{\mathcal{W}_\epsilon, \epsilon \in S, \epsilon \leq \epsilon_0\}$ be a collection of sets $\mathcal{W}_\epsilon$ in $(I - M)Y_\epsilon^1$ such that

$$\|(\varphi^\epsilon, \psi^\epsilon)\|_{Y_\epsilon^1} \leq C_1 \quad \text{for all } \ (\varphi^\epsilon, \psi^\epsilon) \in \mathcal{W}_\epsilon\,. \tag{5.18}$$

For a given bounded set B in $H^1(\Omega) \times L^2(\Omega)$, the *$\omega$-limit set of B with respect to the family of semigroups $T_\epsilon(t)$, $t > 0$, $\epsilon \in S$, and the family of sets $\mathcal{W}$* is denoted by $\hat{\omega}_S(B) \equiv \hat{\omega}_S(B, \mathcal{W})$ and is defined in the following way: a point $U_0 \in \hat{\omega}_S(B)$ if and only if there are sequences $\{\epsilon_n\} \subset S$, $\{t_n\} \subset [0, \infty)$, $\{U_n = V_n + W_n\}$ with $V_n \in B$, $W_n \in \mathcal{W}_{\epsilon_n}$, such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n)U_n \to U_0$ in $H^1(Q) \times L^2(Q)$ as $n \to \infty$.

Theorem 5.3. *Assume that Ω satisfies the hypothesis (H) of [12] and let $\mathcal{W}$ be a family of sets as in Definition 5.1. For any bounded set B in $H^1(\Omega) \times L^2(\Omega)$, we have*

1. For any sequences $\{\epsilon_n\} \subset S$, $\{t_n\} \subset [0, \infty)$, $\{U_n = V_n + W_n\}$ with $V_n \in B$, $W_n \in \mathcal{W}_{\epsilon_n}$, such that $\epsilon_n \to 0$, $t_n \to \infty$, the set $\cup_{n \geq 0} T_{\epsilon_n}(t_n)U_n$ is precompact in $H^1(Q) \times L^2(Q)$ and all limit points belong to $H^1(\Omega) \times L^2(\Omega)$.

2. The set $\hat{\omega}_S(B)$ is contained in Y_0^1, is invariant under $T_0(t)$ and $\hat{\omega}_S(B) \supset \mathcal{A}_0(B) \equiv \omega_0(B)$.

3. Finally, if B contains the global attractor $\mathcal{A}_0$ of $T_0(t)$, then

$$\hat{\omega}(B) = \mathcal{A}_0\,. \tag{5.19}$$

Proof. We give the proof in the case $n = 2$. The case $n = 1$ is similar and even simpler. If we set

$$\mathcal{G}_\epsilon = \begin{bmatrix} 0 \\ -G_\epsilon^* \end{bmatrix}, \quad \mathcal{F}(U) = \begin{bmatrix} 0 \\ -f(u_1) \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5.20}$$

then the variation of constants formula implies that the solution $T_\epsilon(t)U_0$ of $(5.5)_\epsilon$ can be written as

$$T_\epsilon(t)U_0 = S_\epsilon(t)U_0 + \int_0^t S_\epsilon(s)[\mathcal{F}(T_\epsilon(t-s)U_0) + \mathcal{G}_\epsilon]ds\,. \tag{5.21}$$

To prove Part 1, it is sufficient to show that, for every $\delta > 0$, there is an integer n_δ such that $\cup_{n\geq n_\delta} T_{\epsilon_n}(t_n)U_n$ is covered by a finite number of balls in $Y^1 \equiv H^1(Q)\times L^2(Q)$ of radius δ.

Let $\delta > 0$ be given. Thanks to the estimate (5.12), there is a time $T_\delta^1 > 0$ such that, for any $\epsilon \in S$, $\epsilon \leq \epsilon_0$, and any $U_0 = V_0 + W_0$, $V_0 \in B$, $W_0 \in \mathcal{W}_\epsilon$, the Y_ϵ^1 norm and the Y^1 norm of $S_\epsilon(t)U_0$ are less than $\delta/8$ for $t \geq T_\delta^1$. Therefore, if we choose an integer n_δ such that $t_n \geq T_\delta^1$ for $n \geq n_\delta$, we obtain the same bounds for $S_{\epsilon_n}(t_n)U_n$ for $n \geq n_\delta$. As a consequence, it remains to cover $\cup_{n\geq n_\delta}\int_0^{t_n} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t_n - s)U_n) + \mathcal{G}_{\epsilon_n}]ds$ by a finite number of balls in Y^1 of radius $\delta/4$.

Since, by (4.6) and (5.6), $\cup_{t\geq 0, n\geq 0}(\mathcal{F}(T_{\epsilon_n}(t-s)U_n) + \mathcal{G}_{\epsilon_n})$ is bounded in Y^1 by a positive constant which is independent of ϵ, we deduce from (5.12) that there exists a positive time T_δ^2 such that, for $t \geq T_\delta^2$,

$$\begin{aligned}&\|\int_0^t S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t-s)U_n) + \mathcal{G}_{\epsilon_n}]ds - \\ &\int_0^{T_\delta^2} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t-s)U_n) + \mathcal{G}_{\epsilon_n}]ds\|_{Y^1} \\ \leq &\int_{T_\delta^2}^{+\infty} \|S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t-s)U_n) + \mathcal{G}_{\epsilon_n}]ds\|_{Y^1} \leq \frac{\delta}{8}\,.\end{aligned}$$

If we further restrict n_δ so that $t_n \geq T_\delta^2$ for $n \geq n_\delta$, then we conclude that the proof of Part 1 will be complete if we show that

$$\cup_{n\geq n_\delta}\int_0^{T_\delta^2} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t_n-s)U_n) + \mathcal{G}_{\epsilon_n}]ds$$

can be covered by a finite number of balls in Y^1 of radius $\delta/8$.

We introduce a method of interpolation to define the spaces

$$[H^{i+1}(Q), H^i(Q)]_\theta \;=\; H^{(i+1)\theta + i(1-\theta)}(Q) \text{ for } \; i = 0,1$$

and $[L^{q_1}(Q), L^{q_2}(Q)]_\theta = L^r(Q)$ where $\frac{1}{r} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, $1 < q_2 \leq q_1 < +\infty$.

It is shown in [12] that there is a positive constant C such that, for $0 \leq \epsilon \leq \epsilon_0$, $\|U\|_{Y^i} \leq C\|U\|_{Y_\epsilon^i}$ for $i = 1,2$, where $Y^2 = H^2(Q)\times H^1(Q)$. Therefore, by interpolation, we have

$$\|U\|_{Y^{\theta+1}} \leq C\|U\|_{[Y_\epsilon^2, Y_\epsilon^1]_\theta}\,, \tag{5.22}$$

where $Y^\theta = H^{1+\theta}(Q) \times H^\theta(Q)$.

We now introduce the linear mapping

$$\mathcal{T} : (u, u_x, (1/\epsilon)u_y) \in (L^p(Q))^3 \mapsto u \in L^p(Q).$$

We know that $\mathcal{T}$ is a bounded linear mapping from $(L^2(Q))^3$ to X^1_ϵ and from $(L^{6/5}(Q))^3$ to X^0_ϵ with a norm less than some constant C independent of ϵ. Therefore, by interpolation, $\mathcal{T}$ is a bounded linear mapping from $(L^r(Q))^3$ into $[X^1_\epsilon, X^0_\epsilon]_\theta$, where $\frac{1}{r} = \frac{\theta}{2} + \frac{5(1-\theta)}{6} = \frac{5-2\theta}{6}$ and

$$\|u\|_{[X^1_\epsilon, X^0_\epsilon]_\theta} \leq C\|(u, u_x, \frac{1}{\epsilon}u_y)\|_{(L^r(Q))^3}\,. \tag{5.23}$$

If we let $U^{\epsilon_n}_n(t) = T_{\epsilon_n}(t)U_n = (u^{\epsilon_n}_n(t), u^{\epsilon_n}_{nt}(t))$, then, by the hypotheses made on the sequence U_n and by the estimate (5.6), we know that there is a positive constant C_0 such that, for $t \geq 0$, $n \geq 1$,

$$\sup(\|U^{\epsilon_n}_n(t)\|_{Y^1_{\epsilon_n}}, \|U^{\epsilon_n}_n(t)\|_{Y^1}) \leq C_0\,. \tag{5.24}$$

We set $q = 6/(\gamma + 4)$. Since $\gamma < 1$, we have $q > 6/5$. We want to show that

$$(f(u^{\epsilon_n}_n(t)), \frac{\partial}{\partial x}f(u^{\epsilon_n}_n(t)), \frac{1}{\epsilon_n}\frac{\partial}{\partial y}f(u^{\epsilon_n}_n(t)))$$

belongs to $(L^q(Q))^3$ and that there is a positive constant C_1 such that, for $t \geq 0$, $n \geq 1$,

$$\|(f(u^{\epsilon_n}_n(t)), \frac{\partial}{\partial x}f(u^{\epsilon_n}_n(t)), \frac{1}{\epsilon_n}\frac{\partial}{\partial y}f(u^{\epsilon_n}_n(t)))\|_{(L^q(Q))^3} \leq C_1\,. \tag{5.25}$$

To show this, we observe that

$$\begin{aligned}
\|\frac{1}{\epsilon_n}\frac{\partial}{\partial y}f(u^{\epsilon_n}_n(t))\|_{L^q(Q)} &\leq C\Big(\int_Q (1 + |u^{\epsilon_n}_n(t)|^{(\gamma+1)q})\frac{1}{\epsilon^q_n}|\frac{\partial u^{\epsilon_n}_n(t)}{\partial y}|^q dxdy\Big)^{1/q} \\
&\leq C\Big(\int_Q (1 + |u^{\epsilon_n}_n(t)|^{\frac{(\gamma+1)2q}{2-q}})dxdy\Big)^{\frac{2-q}{2q}}\|\frac{1}{\epsilon}\frac{\partial u^{\epsilon_n}_n}{\partial y}\|_{L^2(Q)} \\
&\leq C(1 + \|u^{\epsilon_n}_n(t)\|^{\gamma+1}_{X^1_{\epsilon_n}})\|u^{\epsilon_n}_n(t)\|_{X^1_{\epsilon_n}} \leq C_1\,,
\end{aligned}$$

by the Sobolev embedding of $H^1(Q)$ into $L^6(Q)$. In a similar way, we estimate the other terms in (5.25).

Since $q > 6/5$, there exists a real number $\theta_0, 0 < \theta_0 < 1$, such that $\frac{1}{q} = \frac{\theta_0}{2} + \frac{5(1-\theta_0)}{6}$. By (5.23), this remark and the estimate (5.25) imply that $\mathcal{F}(T_{\epsilon_n}(t)U_n)$ is uniformly bounded in $[Y^2_{\epsilon_n}, Y^1_{\epsilon_n}]_{\theta_0}$. Likewise, the hypothesis made on G^* implies that $\mathcal{G}_{\epsilon_n}$ is uniformly bounded in $[Y^2_{\epsilon_n}, Y^1_{\epsilon_n}]_{\theta_0}$ and there exists a positive constant C_2 such that, for $t \geq 0$, $n \geq 1$,

$$\|\mathcal{F}(T_{\epsilon_n}(t)U_n) + \mathcal{G}_{\epsilon_n}\|_{[Y^2_{\epsilon_n}, Y^1_{\epsilon_n}]_{\theta_0}} \leq C_2 . \tag{5.26}$$

From the properties (5.14) and (5.26), we deduce that

$$\| \int_0^{T^2_\delta} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t_n - s)U_n) + \mathcal{G}_{\epsilon_n}]ds\|_{[Y^2_{\epsilon_n}, Y^1_{\epsilon_n}]_{\theta_0}} \leq C_* C_2 T^2_\delta ,$$

which implies by (5.22) that there is a positive constant C_3 such that, for $n \geq 1$,

$$\| \int_0^{T^2_\delta} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t_n - s)U_n) + \mathcal{G}_{\epsilon_n}]ds\|_{Y^{\theta_0+1}} \leq C_3 .$$

Therefore, $\cup_{n \geq n_\delta} \int_0^{T^2_\delta} S_{\epsilon_n}(s)[\mathcal{F}(T_{\epsilon_n}(t_n - s)U_n) + \mathcal{G}_{\epsilon_n}]ds$ belongs to a bounded set of Y^{θ_0+1} and hence to a compact set of Y^1. As a consequence, this set can be covered by a finite number of balls in Y^1 of radius $\delta/8$.

We can extract subsequences $\{\epsilon_{n_k}\} \subset S$, $\{t_{n_k}\} \subset [0, \infty)$, $\{U_{n_k} = V_{n_k} + W_{n_k}\}$ such that $T_{\epsilon_{n_k}}(t_{n_k})U_{n_k} \equiv (u_{n_k}(t), u_{n_k t}(t))$ converges to an element $U_0 = (u_{01}, u_{02})$ in Y^1. Let us now show that U_0 belongs to $H^1(\Omega) \times L^2(\Omega)$.

For any $\delta > 0$ there exists an integer n_δ such that, for $n \geq n_\delta$,

$$\|u_{01} - u_{n_k}(t_{n_k})\|_{H^1(Q)} + \|u_{02} - u_{n_k t}(t_{n_k})\|_{L^2(Q)} \leq \frac{\delta}{4} . \tag{5.27}$$

However, by Lemma 3.1 of [9], we have

$$\begin{aligned} &\|u_{01} - Mu_{01}\|_{L^2(Q)} \\ &\leq 2\|u_{01} - u_{n_k}(t_{n_k})\|_{L^2(Q)} + \|u_{n_k}(t_{n_k}) - Mu_{n_k}(t_{n_k})\|_{L^2(Q)} \\ &\leq 2\|u_{01} - u_{n_k}(t_{n_k})\|_{L^2(Q)} + C\epsilon_{n_k}\|u_{n_k}\|_{X^1_{\epsilon_{n_k}}} . \end{aligned} \tag{5.28}$$

Thus, from the estimates (5.24),(5.27) and (5.28), we deduce that, for any $\delta > 0$, $\|u_{01} - Mu_{01}\|_{L^2(Q)} \leq \delta$, which implies that u_{01} belongs to $L^2(\Omega) \cap H^1(Q)$ and thus to $H^1(\Omega)$. We also have, for $n \geq n_\delta$,

$$\begin{aligned} &\|u_{02} - Mu_{02}\|_{L^2(Q)} \\ &\leq 2\|u_{02} - u_{n_k t}(t_{n_k})\|_{L^2(Q)} + \|u_{n_k t}(t_{n_k}) - Mu_{n_k t}(t_{n_k})\|_{L^2(Q)} \leq \\ &\frac{\delta}{2} + \|u_{n_k t}(t_{n_k}) - Mu_{n_k t}(t_{n_k})\|_{L^2(Q)} . \end{aligned}$$

By Lemma 3.1 of [9], for $0 \leq \epsilon \leq \epsilon_0$, for $u \in X^i_\epsilon$, $i = 0, 1$, we have $\|u - Mu\|_{L^2(Q)} \leq C\epsilon^i \|u\|_{X^i_\epsilon}$. Therefore, by interpolation, we obtain, for $u \in [X^1_\epsilon, X^0_\epsilon]_{\theta_0}$,

$\|u - Mu\|_{L^2(Q)} \leq C\epsilon^{\theta_0}\|u\|_{[X^1_\epsilon, X^0_\epsilon]_{\theta_0}}$. Using the variation of constants formula (5.21), as well as the estimates (5.12), (5.14),(5.24), (5.26), we obtain

$$\|u_{n_k t}(t_{n_k}) - Mu_{n_k t}(t_{n_k})\|_{L^2(Q)} \leq 2C_0\sqrt{3}e^{-\frac{2b}{3}t_{n_k}} + \epsilon_{n_k}^{\theta_0}CC_2\,.$$

We now choose m_δ such that $m_\delta \geq n_\delta$ and, for $n \geq m_\delta$, $2C_0\sqrt{3}e^{-\frac{2b}{3}t_n} + \epsilon_n^{\theta_0}CC_2 \leq \delta/2$. As a consequence, we have proved that, for any $\delta > 0$, $\|u_{02} - Mu_{02}\|_{L^2(Q)} \leq \delta$, which implies that u_{02} belongs to $L^2(\Omega)$. This completes the proof of Part 1 of the theorem.

We have just shown that $\hat{\omega}_S(B)$ is included in Y^1_0. Let us now show that $\hat{\omega}_S(B)$ is invariant under $T_0(t)$. If $U_0 \in \hat{\omega}_S(B)$, then there exist sequences $\{\epsilon_n\} \subset S$, $\{t_n\} \subset [0,\infty)$, $\{U_n = V_n + W_n\}$ with $V_n \in B$, $W_n \in \mathcal{W}_{\epsilon_n}$, such that $\epsilon_n \to 0$, $t_n \to \infty$, $T_{\epsilon_n}(t_n)U_n \to U_0$ in $H^1(Q)\times L^2(Q)$ as $n \to \infty$. Let $t_0 > 0$ be given. The sequence $\{T_{\epsilon_n}(t_n - t_0)U_n\}_{n\geq 1}$ is precompact and so there is a subsequence $n_k \to \infty$ such that $\epsilon_{n_k} \to 0$, $T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k} \to U_{t_0}$ in $H^1(Q) \times L^2(Q)$. Let us show that $T_0(t_0)U_{t_0} = U_0$. To this end, we are going to show that $T_0(t_0)U_{t_0} - U_0$ is as small as we want in the norm of $L^2(\Omega) \times D(A_0^{-1})$. If $U = (u_1, u_2) \in Y^1_0$, we introduce the mappings $P_1 \in \mathcal{L}(Y^1_0, H^1(Q))$, $P_2 \in \mathcal{L}(Y^1_0, L^2(Q))$ as $P_1U = u_1$, $P_2U = u_2$. If $U_{t_0} = (u_{t_01}, u_{t_02})$, we can write

$$\|P_1(T_0(t_0)U_{t_0} - U_0)\|_{L^2(Q)} \leq \tag{5.29}$$

$$\|P_1(T_0(t_0)U_{t_0} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k})\|_{L^2(Q)}$$
$$+\|P_1(T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k} - T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k})\|_{L^2(Q)}$$
$$+\|P_1(T_{\epsilon_{n_k}}(t_{n_k})U_{n_k} - U_0)\|_{L^2(Q)}\,.$$

Since $T_0(t_0)U$ is continuous in Y^1_0 at U_{t_0} and $MT_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k}$ converges to MU_{t_0} in Y^1_0, there exists an integer n_δ such that, for $n_k \geq n_\delta$, the first term in the right hand side of (5.29) is less than $\delta/3$. We also can choose n_δ such that, for $n_k \geq n_\delta$, the third term in the right hand side of (5.29) is less than $\delta/3$. By the estimate (5.15) of Lemma 5.2, we can write

$$\|A_{\epsilon_{n_k}}^{-1/2}P_2(T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k})\|_{L^2(Q)}$$
$$+\|P_1(T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k})\|_{L^2(Q)}$$
$$\leq C(\epsilon_{n_k} + \|A_{\epsilon_{n_k}}^{-1/2}P_2(I - M)T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k}\|_{H_{\epsilon_{n_k}}}$$

$$+\|P_1(I - M)T_{\epsilon_{n_k}}(t_{n_k} - t_0)U_{n_k}\|_{H_{\epsilon_{n_k}}})\,. \tag{5.30}$$

We remark that

$$\|A_\epsilon^{-1/2}(u - Mu)\|_{H_\epsilon} = \tag{5.31}$$

$$\sup_{w\in X^1_\epsilon} \frac{(u-Mu,w)_{H_\epsilon}}{\|A^{1/2}_\epsilon w\|_{H_\epsilon}} = \sup_{w\in X^1_\epsilon} \frac{(u-Mu,w-Mw)_{H_\epsilon}}{\|A^{1/2}_\epsilon w\|_{H_\epsilon}}\,.$$

From the estimates (5.30), (5.31), (5.24) as well as from Lemma 3.1 of [9], we infer that

$$\begin{aligned}\|A^{-1/2}_{\epsilon_{n_k}} P_2(T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k})\|_{L^2(Q)}\\ +\|P_1(T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k})\|_{L^2(Q)}\end{aligned}$$

$$\le C\epsilon_{n_k}\,. \tag{5.32}$$

Thus, from (5.32), it is clear that we can choose the integer n_δ such that, for $n \ge n_\delta$, the second term in the right hand side of (5.29) is less than $\delta/3$.

It remains to show that

$$\|A_0^{-1}P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(\Omega)} \le \delta\,. \tag{5.33}$$

We can write

(5.34)

$$\begin{aligned}\|A_0^{-1}P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(\Omega)} \le \|(A_0^{-1} - A^{-1}_{\epsilon_{n_k}})P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(Q)}\\ + \|A^{-1}_{\epsilon_{n_k}} P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(Q)}\,.\end{aligned}$$

Thanks to [9, Lemma 4.5] and to the estimate (5.24), we obtain

$$\|(A_0^{-1} - A^{-1}_{\epsilon_{n_k}})P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(Q)} \le CC_0\epsilon_{n_k}\,. \tag{5.35}$$

To estimate the last term in (5.34), we note first that

$$\|A^{-1}_{\epsilon_{n_k}} P_2(T_0(t_0)U_{t_0} - U_0)\|_{L^2(Q)} \tag{5.36}$$

$$\begin{aligned}&\le \|A^{-1}_{\epsilon_{n_k}} P_2(T_0(t_0)U_{t_0} - T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k})\|_{L^2(Q)}\\ &+\|A^{-1}_{\epsilon_{n_k}} P_2(T_0(t_0)MT_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k} - T_{\epsilon_{n_k}}(t_0)T_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k})\|_{L^2(Q)}\\ &+\|A^{-1}_{\epsilon_{n_k}} P_2(T_{\epsilon_{n_k}}(t_{n_k})U_{n_k} - U_0)\|_{L^2(Q)}\,.\end{aligned}$$

One shows easily that there exists a positive constant C such that, for $0 \le \epsilon \le \epsilon_0$, for $h \in H_\epsilon$,

$$\|A_\epsilon^{-1}h\|_{H_\epsilon} \le C\|h\|_{H_\epsilon}\,. \tag{5.37}$$

Since $MT_{\epsilon_{n_k}}(t_{n_k}-t_0)U_{n_k}$ converges to MU_{t_0} in Y^1_0 and $T_0(t_0)U$ is continuous in Y^1_0 at U_{t_0}, we conclude from (5.37) that there exists an integer n_δ such that, for $n_k \ge n_\delta$, the first term in the right hand side of (5.36) is less than $\delta/4$. Likewise, we can choose n_δ such that, for $n_k \ge n_\delta$, the third term in the right hand side of (5.36) is less than $\delta/4$. Finally, from the estimates (5.32), (5.34) to (5.37), we deduce that we can choose n_δ such that, for $n_k \geq n_\delta$, the estimate (5.33) holds. This shows that $\hat{\omega}_S(B) \subset T_0(t_0)\hat{\omega}_S(B)$. In the same way, we have $T_0(t_0)\hat{\omega}_S(B) \subset \hat{\omega}_S(B)$ and so $\hat{\omega}_S(B)$ is invariant under $T_0(t)$. Since $\hat{\omega}_S(B)$ is a bounded subset of Y_0^1 and is invariant under $T_0(t)$, it follows that $\hat{\omega}_S(B)$ is contained in the global attractor $\mathcal{A}_0$ and, in particular, is a bounded subset of Y_0^2.

It remains to prove that $\hat{\omega}_S(B)$ contains the set $\mathcal{A}_0(B) \equiv \omega_0(B)$. If $V_0 = (v_{01}, v_{02}) \in \mathcal{A}_0(B)$, then there exist sequences $t_n \to \infty$, $V_n \equiv (v_{n1}, v_{n2}) \in B$ such that $T_0(t_n)V_n \to V_0$ in Y_0^1. In particular, for any integer m, there exists an integer n_m such that, for $n \geq n_m$,

$$\|T_0(t_n)V_n - V_0\|_{Y_0^1} \leq \frac{1}{3m}\,. \tag{5.38}$$

By Lemma 5.2, the estimate (5.31) and the Lemma 3.1 of [9], we can write, for $0 < \epsilon \leq \epsilon_0$, for any W in $(I-M)Y_\epsilon^1$ with $\|W\|_{Y_\epsilon^1} \leq C_1$, that

$$\begin{aligned}&\|A_\epsilon^{-1/2}P_2(T_0(t_n)V_n - T_\epsilon(t_n)(V_n + W))\|_{H_\epsilon} + \\ &\|P_1(T_0(t_n)V_n - T_\epsilon(t_n)(V_n + W))\|_{H_\epsilon} \\ &\leq C(\epsilon + \|A_\epsilon^{-1/2}P_2W\|_{H_\epsilon} + \|P_1W\|_{H_\epsilon})e^{K_0t_n} \leq C\epsilon e^{K_0t_n}\,,\end{aligned}$$

where K_0 and C are positive constants, independent of ϵ. Therefore, we can construct a sequence $\epsilon_{n_m} \in S$ converging to 0 and choose $W_{n_m} \in \mathcal{W}_{\epsilon_{n_m}}$ such that

$$\begin{aligned}&\|A_{\epsilon_{n_m}}^{-1/2}P_2(T_0(t_{n_m})V_{n_m} - T_{\epsilon_{n_m}}(t_{n_m})(V_{n_m} + W_{n_m}))\|_{H_{\epsilon_{n_m}}} \\ &\quad + \|P_1(T_0(t_{n_m})V_{n_m} - T_{\epsilon_{n_m}}(t_{n_m})(V_{n_m} + W_{n_m}))\|_{H_{\epsilon_{n_m}}} \leq \frac{1}{3m}\,.\end{aligned} \tag{5.39}$$

From Part 1, there is a subsequence n_{m_k} of the sequence n_m such that

$$T_{\epsilon_{n_{m_k}}}(t_{n_{m_k}})(V_{n_{m_k}} + W_{n_{m_k}})$$

converges to an element V^* in Y^1 and V^* belongs to Y_0^1. We shall show that $V_0 = V^*$ by proving that $\|V_0 - V^*\|_{L^2(\Omega)\times D(A_0^{-1})}$ is as small as we wish. We can write

$$\begin{aligned}\|V_0 - V^*\|_{L^2(\Omega)\times D(A_0^{-1})} &\leq C\|V_0 - T_0(t_{n_{m_k}})V_{n_{m_k}}\|_{Y_0^1} \\ &+\|(A_0^{-1} - A_{\epsilon_{n_{m_k}}}^{-1})P_2T_0(t_{n_{m_k}})V_{n_{m_k}}\|_{L^2(Q)} \\ &+C\|P_1(T_0(t_{n_{m_k}})V_{n_{m_k}} - T_{\epsilon_{n_{m_k}}}(t_{n_{m_k}})(V_{n_{m_k}} + W_{n_{m_k}}))\|_{H_{\epsilon_{n_{m_k}}}} \\ &+C\|A_{\epsilon_{n_{m_k}}}^{-1}P_2(T_0(t_{n_{m_k}})V_{n_{m_k}} - T_{\epsilon_{n_{m_k}}}(t_{n_{m_k}})(V_{n_{m_k}} + W_{n_{m_k}}))\|_{H_{\epsilon_{n_{m_k}}}} \\ &+C\|T_{\epsilon_{n_{m_k}}}(t_{n_{m_k}})(V_{n_{m_k}} + W_{n_{m_k}}) - V^*\|_{Y^1}\,,\end{aligned}$$

which, by (5.38), (5.35), (5.39) and the convergence of $T_{\epsilon_{n_{m_k}}}(t_{n_{m_k}})(V_{n_{m_k}}+W_{n_{m_k}})$ to V^*, implies that $\|V_0 - V^*\|_{L^2(\Omega)\times D(A_0^{-1})}$ is as small as we want. This completes the proof of Part 2.

If B contains the global attractor $\mathcal{A}_0$ of $T_0(t)$, then, by Part 2, we have $\hat{\omega}_S(B) \supset \mathcal{A}_0$. Since $\hat{\omega}_S(B) \subset \mathcal{A}_0$, (5.19) holds and the theorem is proved.

Remark 5.1. We have not found any simple condition implying that $\hat{\omega}_S(B) = \mathcal{A}_0$ when the set B does not contain $\mathcal{A}_0$.

Remark 5.2. As in Definition 1.1, we can introduce the limit

$$\tilde{\omega}_S(\mathcal{A}_\cdot) = \cap_{\delta>0}\mathrm{Cl}_{H^1(Q)\times L^2(Q)} \cup_{\epsilon\in S,\,|\epsilon|<\delta} \mathcal{A}_\epsilon .$$

In [12], we have shown that the attractors $\mathcal{A}_\epsilon$ are upper semicontinuous at $\epsilon = 0$. Therefore, as in Proposition 2.1, we prove that $\tilde{\omega}_S(\mathcal{A}_\cdot) \subset \mathcal{A}_0$. If all of the equilibrium points are hyperbolic, then, by [13], the attractors $\mathcal{A}_\epsilon$ are lower semicontinuous at $\epsilon = 0$, which implies, by Proposition 2.2, that $\tilde{\omega}_S(\mathcal{A}_\cdot) = \mathcal{A}_0$.

Using the properties (5.8) and (5.9) of Proposition 5.1, we can show as in Proposition 2.3 that $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is invariant under $T_0(t)$.

Finally, if we assume that $S = (0, \delta_0)$, then it is easy to verify that, as in Proposition 2.4, the set $\tilde{\omega}_S(\mathcal{A}_\cdot)$ is connected.

6. Limits with first integrals.

In order to illustrate the difference between the set $\tilde{\omega}_S(\mathcal{A}_\cdot)$ and the set $\hat{\omega}_S(B)$, we give some simple examples of evolutionary systems for which the limit equation, as a parameter approaches zero, has a first integral.

6.1. *A retarded delay equation.* As a first example, consider the family of retarded differential difference equations

$$\dot{x}(t) = -(1+\epsilon)f(x(t)) + f(x(t-1)), \tag{6.1}$$

where $\epsilon \geq 0$ is a parameter, $f \in C^1(\mathbb{R}, \mathbb{R})$, there is positive constant δ such that $f'(x) \geq \delta$ for all x, and $f(0) = 0$.

A solution $x(t)$ of (6.1) is a continuous function on $[-1, \alpha)$, $\alpha > 0$, which is continuously differentiable on $(0, \alpha)$, has a right hand derivative at $t = 0$ and satisfies (6.1) on $[0, \alpha)$. If $Y_0 = C([-1, 0], \mathbb{R})$, then, for $\varphi \in Y_0$, a solution $x(t)$ of (6.1) with initial data φ at $t = 0$ satisfies the relations

$$\begin{aligned} x(t) &= \varphi(0) + \int_0^t [-(1+\epsilon)f(x(s)) + f(x(s-1))\,ds, \quad t \geq 0, \\ &= \varphi(t), \quad -1 \leq t \leq 0 . \end{aligned} \tag{6.2}$$

In [3], it is shown that (6.2) has a unique solution defined on some interval $[-1, \alpha)$, $\alpha > 0$. We show later that $\alpha = \infty$. If we define $(T_\epsilon(t)\varphi)(\theta) = x(t + \theta)$, $-1 \leq \theta \leq 0$, then $T_\epsilon(t), t \geq 0$, is a C^0-semigroup on Y_0 if all solutions are defined for $t \geq 0$.

Our first assertion is that, for any $\epsilon \geq 0$ and for any bounded set $B \subset Y_0$, the positive orbit $\gamma_\epsilon^+(B) = \cup_{t\geq 0} T_\epsilon(t)B$ is defined and bounded. In fact, let $\mathcal{B}(0, r)$ be the closed ball in Y_0 of radius r and center 0. If $W(x) = x^2/2$ and $x(t+\theta) = (T_\epsilon(t)\varphi)(\theta)$ for $\theta \in [-1, 0]$, then

$$\dot{W}(x(t)) = -\epsilon x(t)f(x(t)) - x(t)f(x(t)) + x(t)f(x(t-1)) \,. \tag{6.3}$$

If $\varphi \in \mathcal{B}(0, r)$ and there is a time t_0 such that $T_\epsilon(t_0)\varphi \in \partial\mathcal{B}(0, r)$, then $|x(t_0+\theta)| \leq |x(t_0)|$ for $\theta \in [-1, 0]$. Since

$$x(f(x) - f(y)) = (x^2 - xy) \int_0^1 f'(y + s(x-y))ds \tag{6.4}$$

for all $x, y \in \mathbb{R}$, it follows that $x(t_0)f(x(t_0)) \geq x(t_0)f(x(t_0-1))$. From (6.3), this implies that $\dot{W}(x(t_0)) \leq -\epsilon x(t_0)f(x(t_0)) \leq 0$. Thus, the solution cannot leave $\mathcal{B}(0, r)$; that is, $T_\epsilon(t)\mathcal{B}(0, r) \subset \mathcal{B}(0, r)$ for all $t \geq 0$. This proves that $\gamma_\epsilon^+(B)$ is defined and bounded if B is bounded. In particular, it implies that $T_\epsilon(t), t \geq 0$ is a C^0-semigroup on Y_0.

Thanks to the above assertion and to (6.1), if $\varphi \in \mathcal{B}(0, r)$, then $\dot{x}(t)$ is uniformly bounded for $t \geq 0$. This fact, together with the Arzela-Ascoli Theorem imply that $T_\epsilon(t)$ is a compact map for $t \geq 1$. As a consequence, for any $\epsilon \geq 0$ and any bounded set $B \subset Y_0$, the ω-limit set $\omega_\epsilon(B)$ of B with respect to $T_\epsilon(t)$ is a compact invariant set of $T_\epsilon(t)$.

We can even say more. If we define the maps

$$\begin{aligned}
S(t)\varphi(\theta) &= \varphi(t+\theta) - \varphi(0),\ t+\theta < 0,\\
&= 0,\ t+\theta \geq 0,\\
U_\epsilon(t)\varphi(\theta) &= \varphi(0),\ t+\theta < 0,\\
&= \varphi(0) + \int_0^{t+\theta} [-(1+\epsilon)f(x(s)) + f(x(s-1))\, ds,\ t+\theta \geq 0,
\end{aligned}$$

where $x(s) = (T_\epsilon(s)\varphi)(0)$, then $U_\epsilon(t)$ is a compact map for each $t \geq 0$ and, for any $\beta > 0$, there exists a constant $K = K(\beta)$ such that $\|S(t)\| \leq Ke^{-\beta t}$ for $t \geq 0$. Since $T_\epsilon(t) = S(t) + U_\epsilon(t)$, it follows that $T_\epsilon(t)$ is asymptotically smooth.

We have shown that (H_1) of Theorem 3.3 is satisfied. It is not difficult to show from (6.2) that condition (2.2bis) is satisfied as well. If $S = (0, 1)$, then it follows from Theorem 3.3 that $\hat{\omega}_S(B)$ is invariant under $T_0(t)$.

Since $T_\epsilon(t)\mathcal{B}(0, r) \subset \mathcal{B}(0, r)$ for $t \geq 0$, $\epsilon \geq 0$, it follows that

$$\hat{\omega}_S(\mathcal{B}(0, r)) \subset \mathcal{B}(0, r).$$

As a consequence, all of the conditions of Corollary 3.6 are satisfied. Therefore, we conclude that, for any $r > 0$, $\hat{\omega}_S(\mathcal{B}(0, r)) = \mathcal{A}_0(\mathcal{B}(0, r)) \equiv \omega_0(\mathcal{B}(0, r))$.

We next discuss $\omega_\epsilon(\varphi)$ for any $\varphi \in Y_0$ and any $\epsilon \geq 0$. For $\epsilon = 0$, it is shown in [2, Example 3.3] that, for any $\varphi \in Y_0$, $\omega_0(\varphi)$ is an equilibrium point. It is clear that any constant function satisfies the differential equation

$$\dot{x}(t) = -f(x(t)) + f(x(t-1))\,, \tag{6.5}$$

and equation (6.5) has the first integral

$$V(\varphi) = \varphi(0) + \int_{-1}^{0} f(\varphi(\theta))\, d\theta\,. \tag{6.6}$$

On each level set $V^{-1}(c)$, there is a unique equilibrium solution $e(c)$ of (6.5) given by the unique solution of the equation $e + f(e) = c$. If we linearize about any equilibrium point of (6.5), then it is easy to verify that all eigenvalues have real parts ≤ 0 and that the only eigenvalue on the imaginary axis is 0 and it is a simple eigenvalue. Since $\omega_0(\varphi)$ belongs to the set of equilibrium points and lies on $V^{-1}(c_\varphi)$, $c_\varphi = \varphi(0) + \int_{-1}^{0} f(\varphi(\theta))d\theta$, it follows that $\omega_0(\varphi) = e(c_\varphi)$, and $e(c_\varphi)$ is uniformly asymptotically stable relative to the set $V^{-1}(c_\varphi)$. From this fact, we deduce that $T_0(t)|V^{-1}(c)$ has the global attractor $\{\, e(c) \,\}$. If B is an arbitrary closed bounded set in Y_0, then, for any $\varphi \in B$, there is a unique $c(\varphi)$ such that $\varphi \in V^{-1}(c(\varphi))$. It follows that $\omega_0(B) \equiv \mathcal{A}_0(B) = I(B)$, where $I(B) = \{\, e(\varphi) : \varphi \in B \,\}$.

Now suppose that $\epsilon > 0$. In this case, (6.1) has a unique equilibrium point 0. We want to show that $\omega_\epsilon(\varphi) = \{0\}$ for each $\varphi \in Y_0$. Let $q = q_\epsilon > 1$ be a fixed constant such that $\epsilon - (q-1) > 0$. If $|x(t-1)| < q|x(t)|$, then (6.3), (6.4) imply that $\dot{W}(x(t)) \leq -\delta(\epsilon - (q-1))x^2(t)$. From [3, Theorem 4.2, Chapter 5], we see that the origin of (6.1) is uniformly asymptotically stable and $\omega_\epsilon(\varphi) = \{0\}$ for all $\varphi \in Y_0$. These properties imply that the attractor $\mathcal{A}_\epsilon = \{0\}$ for every $\epsilon > 0$. As a consequence, $\tilde{\omega}_S(\mathcal{A}_\cdot) = \{0\}$, whereas $\hat{\omega}_S(B)$ is generally much larger.

If we assume that f is analytic, then we can assume only that f is a nondecreasing function and obtain the same results as above. We outline the proof. It is based on a simple invariance principle in [2, Theorem 2.1]. It is not difficult to show that $\dot{W}(x(t)) \leq 0$ for $|x(t)| \geq |x(t+\theta)|$, $\theta \in [-1,0]$ implies that $\tilde{W}(x_t) \equiv \max_{\theta \in [-1,0]} W(x(t+\theta))$ is a nonincreasing function of t. Therefore, $\tilde{W}(x_t)$ approaches a constant as $t \to \infty$. Since $\omega_\epsilon(\varphi)$ is an invariant set, it follows that, for any $\psi \in \omega_\epsilon(\varphi)$, we have $\tilde{W}(x_t) = \tilde{W}(\psi)$, where $x(t)$ is the solution of (6.1) with initial data ψ at $t = 0$. This implies that $\max_{\theta \in [-1,0]} |x(t+\theta)|$ is a constant for all $t \in \mathbb{R}$. If f is analytic, then $x(t)$ is an analytic function [17]. If $x(t_M)$ is a point where this maximum is attained, then $\dot{x}(t_M) = 0$ and, from (6.1), for any integer $k \geq 1$, we deduce that $d^k x(t_m)/dt^k = 0$. Thus, $x(t)$ is a constant function. This shows that $\omega_\epsilon(\varphi) \in E$, the set of equilibrium points of (6.1). For $\epsilon > 0$, 0 is the only equilibrium point and it is easy to verify that it is uniformly asymptotically stable. This is enough to assert that $\mathcal{A}_\epsilon = \{0\}$. For $\epsilon = 0$, we have $\mathcal{A}_0(B) = I(B)$ as above.

6.2. *An ordinary differential equation.* Suppose that $\epsilon > 0, \gamma > 0, 0 < a < 1/2$, are constants, $f(u) = u(1-u)(u-a)$, and consider the system of ordinary differential equations

$$\begin{aligned} \dot{\xi} &= f(\xi) - \eta \\ \dot{\eta} &= \epsilon(\gamma\xi - \eta)\,. \end{aligned} \tag{6.7}$$

We fix the parameters a and γ so that (6.7) has exactly three hyperbolic equilibrium points $P_j = (\xi_j^0, \eta_j^0), j = 1,2,3$, with $0 = \xi_1^0 < \xi_2^0 < \xi_3^0$, $0 = \eta_1^0 < \eta_2^0 < \eta_3^0$. We fix $\epsilon_0 > 0$ small enough so that, for $0 < \epsilon \le \epsilon_0$, the points P_1, P_3 are stable and P_2 is a saddle point with a one dimensional unstable manifold. To see that this can be done, let λ_{ij}^0 and $(\xi_{ij}^0, \eta_{ij}^0)$, $i = 1,2$, be the eigenvalues and corresponding eigenfunctions of the linearization of (6.7) near the equlilibrium point P_j, $j = 1,2,3$. A few computations reveal that, as $\epsilon \to 0$,

$$\lambda_{12}^0 = f'(\xi_2^0) + O(\epsilon), \quad \eta_{12}^0 = (\epsilon \frac{\gamma}{f'(\xi_2^0)} + O(\epsilon^2))\xi_{12}^0 ,$$

$$\lambda_{22}^0 = \epsilon \frac{\gamma - f'(\xi_2^0)}{f'(\xi_2^0)} + O(\epsilon^2), \quad \eta_{22}^0 = (f'(\xi_2^0) + O(\epsilon))\xi_{22}^0 ,$$

where $f'(\xi_2^0)$ and $f'(\xi_2^0) - \gamma$ are positive. For $j = 1, 3$ (the stable equilibrium points), we have

$$\lambda_{1j}^0 = \epsilon \frac{\gamma - f'(\xi_j^0)}{f'(\xi_j^0)}, \quad \eta_{1j}^0 = (f'(\xi_j^0) + O(\epsilon))\xi_{1j}^0$$

$$\lambda_{2j}^0 = f'(\xi_j^0) + O(\epsilon), \quad \eta_{2j}^0 = (\epsilon \frac{\gamma}{f'(\xi_j^0)} + O(\epsilon^2))\xi_{2j}^0 ,$$

where $f'(\xi_j^0)$ is negative.

We now show that equation (6.7) has a global attractor $\mathcal{A}_\epsilon$ for each $\epsilon > 0$. If we define $V(\xi, \eta) = \frac{1}{2}(|\xi|^2 + |\eta|^2)$, then it is not difficult to show that the derivative of $V(\xi, \eta)$ along the solutions of (6.7) is $\le -a\xi^2 - \frac{\epsilon}{4}\eta^2$ outside the ball of radius r_ϵ and center zero if r_ϵ is sufficiently large. This implies that the system (6.7) is bounded dissipative and thus has a global attractor $\mathcal{A}_\epsilon$. We also remark that the curve $\eta = f(\xi)$ divides the plane $\mathbb{R}^2$ into two regions R_+ (resp. R_-) where $f(\xi) - \eta$ has constant sign $+$ (resp. $-$). Let δ now be a small positive number and let U_δ be the δ-neighborhood of the curve $\eta = f(\xi)$. If $(\xi, \eta) \in \mathbb{R}^2 \setminus U_\delta$, then $\dot{\xi} > \delta$ (resp. $\dot{\xi} < \delta$) if $(\xi, \eta) \in R_+$ (resp. R_-). From this observation, we deduce that each solution $(\xi(t), \eta(t))$ of (6.7) must enter U_δ and stay in U_δ after a finite time, depending, of course, on the initial data $(\xi(0), \eta(0))$. Notice that, for some data $(\xi(0), \eta(0))$ in U_δ, $(\xi(t), \eta(t))$ will leave U_δ for some $t_0 > 0$, but will enter again in U_δ and stay there after some $t_1 > t_0$. From all of the above considerations, we conclude that every solution $(\xi(t), \eta(t))$ of (6.7) stays in $U_\delta \cap B_\epsilon$ after some positive time, where B_ϵ is a bounded set. As a consequence, for $0 < \epsilon \le \epsilon_0$, the solution must either lie on the stable manifold of P_2 or belong to the basin of attraction of either P_1 or P_3. This shows that the system is gradient-like and thus $\mathcal{A}_\epsilon$ is the union of the unstable manifolds of the equilibrium points; more specifically, it is the closure of the unstable manifold of the saddle point P_2.

If $\epsilon > 0$, it is easy to build positively invariant rectangles for (6.7). Let $\xi_m < \xi_M$ be the points such that $f(\xi_m)$ (resp. $f(\xi_M)$) is a local minimum (resp. maximum). Let a_0, b_0, c_0, d_0 be real numbers such that $a_0 < \xi_1^0$, $b_0 > \xi_3^0$, $c_0 \le f(\xi_m)$, $c_0 < \gamma a_0$, $d_0 \ge f(\xi_M)$, $d_0 > \gamma b_0$, and $f(a_0) - d_0 \ge 0$, $f(b_0) - c_0 \le 0$. We consider the rectangle $B_0 \equiv B(a_0, b_0, c_0, d_0,) = \{(\xi, \eta) : a_0 \leq \xi \leq b_0, c_0 \leq \eta \leq d_0\}$. A simple phase plane analysis shows that, for every $\epsilon \geq 0$, the vector field (6.7) points into B_0 on the boundary ∂B_0, which implies that B_0 is positively invariant under the flow defined by (6.7) (see Figure 1). Furthermore, for any $\epsilon > 0$, the ω-limit set of B is a proper subset of B_0.

We next discuss the behavior of the unstable manifold $W^u_\epsilon(P_2)$ of P_2. Fix $\nu_0 > 0$ and $\nu > 0$ small and consider the line segment near P_2 defined by $I^0_{\nu_0,\nu} = \{(\xi, \eta) : \xi = \xi^0_2 - \nu_0, \eta \in [\eta^0_2 - \nu, \eta^0_2 + \nu]\}$. Let $\tilde{\xi}^m_2$ be such that $\tilde{\xi}^m_2 < \xi^0_1$ and $f(\tilde{\xi}^m_2) = \eta^0_2$; fix $\delta > 0$ smaller than $\xi^0_2 - \nu_0 - \tilde{\xi}^m_2$ and choose ν so small that the line segment $I^1_{\delta,\nu} = \{(\xi, \eta) : \xi = \tilde{\xi}^m_2 + \delta, \eta \in [\eta^0_2 - 2\nu, \eta^0_2 + 2\nu]\}$ does not intersect the curve $\eta = f(\xi)$. A simple analysis of the vector field of (6.7) shows that there are ϵ_0 and a time $\tau_0 > 0$ such that, for $0 < \epsilon \leq \epsilon_0$, each solution with initial data in $I^0_{\nu_0,\nu}$ must intersect $I^1_{\delta,\nu}$ in time less than τ_0. As a consequence, $W^u_\epsilon(P_2)$ must intersect $I^1_{\delta,\nu}$ for $0 < \epsilon \leq \epsilon_0$ if ϵ_0 is sufficiently small. Since δ and ν are arbitrary (as well as δ_0), we conclude that the limit of $W^u_\epsilon(P_2)$ as $\epsilon \to 0$ must contain the line segment $\{ (\xi, \eta) = (\xi, \eta^0_2) : \tilde{\xi}^m_2 \leq \xi \leq \xi^0_2 \}$. Continuing with similar analysis, we conclude that, as $\epsilon \to 0$, the set $W^u_\epsilon(P_2)$ approaches the curve consisting of the union of the three curves

$$
\begin{aligned}
C_1 &\equiv \{ (\xi, \eta) = (\xi, f(\xi)) : \tilde{\xi}^m_2 \leq \xi \leq \xi^0_1 \}, \\
C_2 &\equiv \{ (\xi, \eta) = (\xi, \eta^0_2) : \tilde{\xi}^m_2 \leq \xi \leq \tilde{\xi}^M_2 \}, \\
C_3 &\equiv \{ (\xi, \eta) = (\xi, f(\xi)) : \xi^0_3 \leq \xi \leq \tilde{\xi}^M_2 \},
\end{aligned}
$$

where $\tilde{\xi}^m_2 < \xi^0_1$ (resp. $\tilde{\xi}^M_2 > \xi^0_3$) is such that $\eta^0_2 = f(\tilde{\xi}^m_2)$ (resp. $\eta^0_2 = f(\tilde{\xi}^M_2)$). Thus, if $S_0 = (0, \epsilon_0]$, then $\tilde{\omega}_{S_0}(\mathcal{A}.)$ is the union of these curves.

Let us now consider what happens when $\epsilon = 0$. In this case, the system (6.7) has a first integral given by $W(\xi, \eta) = \eta$. For any fixed constant c, the flow on the surface $W^{-1}(c)$ is given by the scalar equation

$$\dot{\xi} = f(\xi) - c. \tag{6.8}$$

For each fixed c, there is a global attractor $\mathcal{A}^c_0$ of (6.8) which consists of one point if $|c|$ is large and is a line segment otherwise (see Figure 2). For any bounded set $B \subset \mathbb{R}^2$, the attractor $\mathcal{A}_0(B)$ for (6.7) for $\epsilon = 0$ is given by

$$\mathcal{A}_0(B) = \{ \mathcal{A}^c_0 : B \cap W^{-1}(c) \neq \emptyset \}. \tag{6.9}$$

It is obvious that condition (2.2 bis) is satisfied. Theorem 3.2 implies that $\hat{\omega}_{S_0}(B) \supset \mathcal{A}_0(B)$ for every bounded set B in $\mathbb{R}^2$. Without some further restrictions on B, this is the most that we can say. As noted above, for every $\epsilon \geq 0$, the rectangle $B_0 = B(a_0, b_0, c_0, d_0)$ is positively invariant under the flow defined by (6.7). Therefore, $\hat{\omega}_{S_0}(B_0) \subset B_0$. Since the conditions of Corollary 3.6 are satisfied, it follows that $\mathcal{A}_0(B_0) = \hat{\omega}_{S_0}(B_0)$. (See Figure 3 for the comparison of the sets $\hat{\omega}_{S_0}(B_0)$ and $\tilde{\omega}_{S_0}(\mathcal{A}.)$). We see that $\hat{\omega}_{S_0}(B_0)$ captures enough of the transient dynamics for $\epsilon > 0$ in order to reproduce the corresponding attractor for $\epsilon = 0$.

Let $\tilde{\xi}_M < \xi_1^0$ (resp. $\tilde{\xi}_m > \xi_3^0$) be the point such that $f(\tilde{\xi}_M) = f(\xi_M)$ (resp. $f(\tilde{\xi}_m) = f(\xi_m)$) and let $\mathcal{B}_0$ be the closed set delimited by the four curves

$$\begin{aligned}
\mathcal{C}_0^1 &= \{(\xi,\eta) = (\xi, f(\xi)), \tilde{\xi}_M \le \xi \le \xi_m\} \\
\mathcal{C}_0^2 &= \{(\xi,\eta) = (\xi, f(\xi_m)), \xi_m \le \xi \le \tilde{\xi}_m\} \\
\mathcal{C}_0^3 &= \{(\xi,\eta) = (\xi, f(\xi)), \tilde{\xi}_m \le \xi \le \xi_M\} \\
\mathcal{C}_0^4 &= \{(\xi,\eta) = (\xi, f(\xi_M)), \xi_M \le \xi \le \tilde{\xi}_M\}
\end{aligned}$$

By a simple phase plane analysis, we can show more generally that, if B is a closed bounded set containing $\mathcal{B}_0$ and satisfying the property that $B \cap W^{-1}(c) \supset \mathcal{A}_0^c$, for any c for which $B \cap W^{-1}(c) \neq \emptyset$, then $\hat{\omega}_{S_0}(B) = \mathcal{A}_0(B) \subset B$. Since the conditions of Corollary 3.6 are satisfied, it follows that $\hat{\omega}_{S_0}(B) = \mathcal{A}_0(B)$ (see Figure 3).

A nonlinear PDE. With the same restrictions as imposed on (6.7) in the previous example and for a given constant $\delta > 0$, we consider the system of equations, called the FitzHugh-Nagumo equations

$$\begin{aligned}
u_t &= \delta u_{xx} + f(u) - v \\
v_t &= \epsilon(\gamma u - v)\,, \quad 0 < x < 1\,,
\end{aligned} \tag{6.10}$$

with the boundary condition

$$u_x = 0 \quad \text{for } x = 0, x = 1\,. \tag{6.11}$$

The initial data is taken in the space $X = H^1(0,1) \times L^2(0,1)$ (or $L^2(0,1) \times L^2(0,1)$).

Let $T_{\epsilon,\delta}(t)$ be the semigroup generated by (6.10), (6.11). It is possible to show that $T_{\epsilon,\delta}(t)$ is asymptotically smooth (it is actually an α-contraction) and has a global attractor $\mathcal{A}_{\epsilon,\delta}$ (see, for example, [16]). We remark that a solution $(u(t), v(t)) \in \mathcal{A}_{\epsilon,\delta}$ is defined and bounded for all $t \in \mathbb{R}$. Therefore, the function $v(t)$ must satisfy the equation

$$v(t) = \int_{-\infty}^{t} e^{-\epsilon(t-s)} \epsilon\gamma u(s)\, ds\,.$$

As a consequence, the function $v \in H^1(0,1)$ and $\mathcal{A}_{\epsilon,\delta} \subset H^1(0,1) \times H^1(0,1)$. Using the regularity theory of parabolic equations and the above integral expression for $v(t)$, we deduce that $\mathcal{A}_{\epsilon,\delta} \subset H^2(0,1) \times H^2(0,1)$. This implies that $\mathcal{A}_{\epsilon,\delta}$ belongs to a bounded subset of $C^1(0,1) \times C^1(0,1)$. Let us now consider (6.10), (6.11) in the space $C^1(0,1) \times C^1(0,1)$. The rectangle $B_0 = B(a_0, b_0, c_0, d_0)$ introduced in the previous example is an invariant rectangle under the flow defined by (6.10), (6.11); that is, if the initial data belongs to B_0, then the solution remains in B_0 for all $t \geq 0$. Furthermore, no invariant set can lie on the boundary of B_0. This implies that, for any $\delta_0 > 0$, there exists a positive number r_0 such that $\cup_{\epsilon>0,\delta\geq\delta_0} \mathcal{A}_{\epsilon,\delta} \subset \mathcal{B}_{r_0}$, the ball in $C^1(0,1) \times C^1(0,1)$ of center 0 and radius r_0. From this fact, we deduce that, for any $\epsilon > 0$, $\delta \geq \delta_0$, $\|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0$ if $(u_0, v_0) \in \mathcal{A}_{\epsilon,\delta}$.

Let us now consider what happens when $\epsilon = 0$. In this case, the system (6.10), (6.11), has a first integral given by $V(u, v) = v$; that is, for any given initial data $(\varphi, \psi) \in X$,the solution $(u(t, x), v(t, x))$ of (6.10), (6.11), for $\epsilon = 0$, satisfies $v(t, x) = \psi$ for all $t \geq 0$. On the set $V^{-1}(\psi)$, the dynamics of the flow is determined by the scalar parabolic equation

$$u_t = \delta u_{xx} + f(u) - \psi \tag{6.12}$$

with the boundary condition (6.11). Equation (6.12) has a global attractor for each $\delta > 0$, $\psi \in L^2(0, 1)$.

Let

$$\tilde{B}_0 = \{ (u_0, v_0) \in H^1(0, 1) \times H^1(0, 1) : (u_0(x), v_0(x)) \in$$
$$B(a_0, b_0, c_0, d_0),\ x \in [0, 1], \|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0 \}.$$

We now can apply Corollary 3.6 to see, for example, that $\mathcal{A}_{0,\delta}(\tilde{B}_0) = \hat{\omega}_{S_0,\delta}(\tilde{B}_0)$. For any bounded set $\tilde{B} \in H^1(0, 1)\times L^2(0, 1)$, we always have $\mathcal{A}_{0,\delta}(\tilde{B}) \subset \hat{\omega}_{S_0,\delta}(\tilde{B})$.

Let us describe the attractor in more detail for the diffusion coefficient δ very large. This is the same as taking a thin domain $Q_\epsilon$around a point which has the special shape obtained by rescaling x in Q_ϵ. For δ very large, we claim that the attractor $\mathcal{A}_{\epsilon,\delta} = \mathcal{A}_\epsilon$, the attractor for (6.7). To prove this claim, we introduce the notation

$$u = \xi + \tilde{u}, \quad \xi = \int_0^1 u\,dx$$
$$v = \eta + \tilde{v}, \quad \eta = \int_0^1 v\,dx\,.$$

The system (6.10) can be written as

$$\begin{aligned}
\dot{\xi} &= f(\xi) - \eta + \int_0^1 F(\xi, \tilde{u})\,dx, \\
\dot{\eta} &= \epsilon(\gamma\xi - \eta)\,, \\
\tilde{u}_t &= \delta\tilde{u}_{xx} + F(\xi, \tilde{u}) - \int_0^1 F(\xi, \tilde{u})\,dx - \tilde{v}, \\
\tilde{v}_t &= -\epsilon\tilde{v} + \epsilon\gamma\tilde{u}\,,
\end{aligned} \tag{6.13}$$

where $F(\xi, \tilde{u}) = f(\xi+\tilde{u}) - f(\xi)$. Let r_0 be the constant given above which bounds uniformly in ϵ, δ the attractors $\mathcal{A}_{\epsilon,\delta}$ in $H^1(0, 1) \times H^1(0, 1)$. Using arguments similar to the ones in [14], we can show that there is a constant $\delta_1 > 0$, such that, for $\delta \geq \delta_1$, there are constants $\beta(\delta) > 0$, $k_0 > 0$, $\beta(\delta) \to \infty$ as $\delta \to \infty$ such that, for $0 \leq \epsilon \leq \epsilon_0$ and initial data (u_0, v_0) satisfying $\|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0$, we have the following estimates for the solutions of (6.10), (6.11):

$$\|\tilde{u}(t)\|_{H^1(0,1)} \leq k_0(e^{-\frac{\delta}{2}t}\|\tilde{u}_0\|_{H^1(0,1)} + \frac{1}{\beta(d)}e^{-\frac{\delta}{2}t}\|\tilde{v}_0\|_{H^1(0,1)}) \tag{6.14}$$

$$(6.15) \qquad \|\tilde{v}(t)\|_{H^1(0,1)} \le k_0(e^{-\epsilon t}\|\tilde{v}_0\|_{H^1(0,1)} + e^{-\frac{\delta}{2}t}\|\tilde{u}_0\|_{H^1(0,1)})$$

for all $t \geq 0$.

Relations (6.14), (6.15) imply that, for any $\epsilon > 0$ and for initial data (u_0, v_0) satisfying $\|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \le r_0$, the ω-limit set of the solution must lie on the attractors $\mathcal{A}_\epsilon$ of the ODE (6.7) provided that $\delta \geq \sup(\delta_0, \delta_1)$, which proves our claim.

For $\epsilon = 0$ and $\delta \geq \delta_0$, sufficiently large, the attractor for (6.10), (6.11) is the same as the attractor $\mathcal{A}_0^{\bar{\psi}}$ of the equation

$$(6.16) \qquad \dot{u} = f(u) - \bar{\psi}, \quad \bar{\psi} = \int_0^1 \psi(x)\,dx\,.$$

For any bounded set $B \subset X$, the attractor $\mathcal{A}_{0,\delta}(B)$ for (6.10), (6.11) for $\epsilon = 0$ is given by

$$\mathcal{A}_{0,\delta}(B) = \{\, \mathcal{A}_0^{\bar{\psi}} : \bar{\psi} = \int_0^1 \psi(x)\,dx,\ \psi \in B \,\}$$

for $\delta \geq \delta_0(B)$ sufficiently large.

As remarked above, for the set $\tilde{B}_0$, Corollary 3.6 implies that, for $\delta \geq \delta_0$, $\mathcal{A}_{0,\delta}(\tilde{B}_0) = \hat{\omega}_{S_{0,\delta}}(\tilde{B}_0)$.

6.2. *An ordinary differential equation.* Suppose that $\epsilon > 0, \gamma > 0, 0 < a < 1/2$, are constants, $f(u) = u(1-u)(u-a)$, and consider the system of ordinary differential equations

$$(6.7) \qquad \begin{aligned} \dot{\xi} &= f(\xi) - \eta \\ \dot{\eta} &= \epsilon(\gamma\xi - \eta)\,. \end{aligned}$$

We show first that equation (6.7) has a global attractor $\mathcal{A}_\epsilon$ for each $\epsilon > 0$. We could do this by showing that the derivative of the function $V(\xi, \eta) = \frac{1}{2}(|\xi|^2 + |\eta|^2)$ is negative on a ball of radius r_ϵ and center zero if r_ϵ is sufficiently large. However, we prefer to use a different argument since it will be needed later.

Let $\xi_m < \xi_M$ be the points such that $f(\xi_m)$ (resp. $f(\xi_M)$) is a local minimum (resp. maximum). Let $B \equiv B(a, b, c, d,)$ be the rectangle $[a, b] \times [c, d]$ with $c \le \min\{\, f(\xi_m), \gamma a \,\}$, $d \geq \max\{\, f(\xi_M), \gamma b \,\}$, $a \le \xi_1^0$, $b \ge \xi_3^0$. A simple phase plane analysis shows that, for every $\epsilon \ge 0$, the set B is positively invariant under the flow defined by (6.7). Furthermore, for any $\epsilon > 0$, the ω-limit set of B is a proper subset of B. Therefore, (6.7) has a global attractor $\mathcal{A}_\epsilon$ for each $\epsilon > 0$. The same type of phase plane analysis shows that, for every $\epsilon \ge 0$, the system (6.7) is gradient.

We now fix the parameters a and γ so that (6.7) has exactly three hyperbolic equilibrium points $P_j = (\xi_j^0, \eta_j^0), j = 1, 2, 3$, with $0 = \xi_1^0 < \xi_2^0 < \xi_3^0$, $0 = \eta_1^0 < \eta_2^0 < \eta_3^0$. The points P_1, P_3 are stable and P_2 is a saddle point with a one dimensional unstable manifold. The global attractor $\mathcal{A}_\epsilon$ of (6.7), for each $\epsilon > 0$, is the union of the unstable manifolds of the equilibrium points; more specifically, it is the closure of the unstable manifold of the saddle point P_2.

We need to know more about the behavior of the unstable manifold for P_2. To do this, we must first analyze the flow near the equilibrium points. Let λ_{ij}^0 and $(\xi_{ij}^0, \eta_{ij}^0)$, $i = 1, 2$, be the eigenvalues and corresponding eigenfunctions of the linearization of (6.7) near the equlilibrium point P_j, $j = 1, 2, 3$. A few computations reveal that

$$\lambda_{12}^0 = f'(\xi_2^0) + O(\epsilon), \quad \eta_{12}^0 = O(\epsilon)\xi_{12}^0,$$
$$\lambda_{22}^0 = O(\epsilon), \quad \eta_{22}^0 = f'(\xi_2^0)\xi_{22}^0 + O(\epsilon).$$

as $\epsilon \to 0$. For $j = 1, 3$ (the stable equilibrium points), we have

$$\lambda_{1j}^0 = O(\epsilon), \quad \eta_{1j}^0 = f'(\xi_j^0)\xi_{1j}^0 + O(\epsilon)$$
$$\lambda_{2j}^0 = f'(\xi_2^0) + O(\epsilon), \quad \eta_{2j}^0 = O(\epsilon)\xi_{2j}^0 \;\; j = 1, 3,$$

as $\epsilon \to 0$.

If we let $W_\epsilon^u(P_2)$ be the unstable manifold of P_2, then a simple phase plane analysis and the above local properties near the equilibrium points imply that, as $\epsilon \to 0$, the set $W_\epsilon^u(P_2)$ approaches the curve consisting of the union of the three curves

$$C_1 \equiv \{ (\xi, \eta) = (\xi_1^0, \eta) : \eta = f(\xi), \, \tilde{\xi}_2^m \le \xi \le \xi_1^0 \},$$
$$C_2 \equiv \{ (\xi, \eta) = (\xi, \eta_2^0) : \tilde{\xi}_2^m \le \xi \le \tilde{\xi}_2^M \},$$
$$C_3 \equiv \{ (\xi, \eta) = (\xi_3^0, \eta) : \eta = f(\xi), \, \xi_3^0 \le \xi \le \tilde{\xi}_2^M \},$$

where $\tilde{\xi}_2^m < \xi_1^0$ (resp. $\tilde{\xi}_2^M > \xi_3^0$) is such that $\eta_2^0 = f(\tilde{\xi}_2^m)$ (resp. $\eta_2^0 = f(\tilde{\xi}_2^M)$). Thus, if $S_0 = (0, \epsilon_0]$, then $\tilde{\omega}_{S_0}(\mathcal{A}_\cdot)$ is the union of these curves.

Let us now consider what happens when $\epsilon = 0$. In this case, the system (6.7) has a first integral given by $W(\xi, \eta) = \eta$. For any fixed constant c, the flow on the surface $W^{-1}(c)$ is given by the scalar equation

$$\dot{\xi} = f(\xi) - c. \tag{6.8}$$

For each fixed c, there is a global attractor $\mathcal{A}_0^c$ of (6.8) which consists of one point if $|c|$ is large and is a line segment otherwise (see Figure 1). For any bounded set $B \subset \mathbb{R}^2$, the attractor $\mathcal{A}_0(B)$ for (6.7) for $\epsilon = 0$ is given by

$$\mathcal{A}_0(B) = \{ \mathcal{A}_0^c : B \cap W^{-1}(c) \neq \emptyset \}. \tag{6.9}$$

It is obvious that condition (2.2) is satisfied. Theorem 3.2 implies that $\hat{\omega}_{S_0}(B) \supset \mathcal{A}_0(B)$ for every bounded set B in $\mathbb{R}^2$. Without some further restrictions on B, this is the most that we can say. As noted above, for every $\epsilon \ge 0$, the rectangle $B = B(a, b, c, d)$ is positively invariant under the flow defined by (6.7). Therefore, $\hat{\omega}_{S_0}(B) \subset B$. Since the conditions of Theorem 3.3 are satisfied, it follows that $\mathcal{A}_0(B) = \hat{\omega}_{S_0}(B)$. See Figure 2 for the comparison of the sets $\hat{\omega}_{S_0}(B)$ and $\tilde{\omega}_{S_0}(\mathcal{A}_\cdot)$. We see that $\hat{\omega}_{S_0}(B)$ captures enough of the transient dynamics for $\epsilon > 0$ in order to reproduce the corresponding attractor for $\epsilon = 0$.

A nonlinear PDE. With the same restrictions as imposed on (6.7) in the previous example and for a given constant $d > 0$, we consider the system of equations

$$u_t = du_{xx} + f(u) - v \qquad (6.10)$$
$$v_t = \epsilon(\gamma u - v), \quad 0 < x < 1,$$

with the boundary condition

$$u_x = 0 \quad \text{for } x = 0, x = 1. \qquad (6.11)$$

The initial data is taken in the space $X = H^1(0, 1) \times L^2(0, 1)$.

Let $T_{\epsilon,d}(t)$ be the semigroup generated by (6.10), (6.11). It is possible to show that $T_{\epsilon,d}(t)$ is asymptotically smooth (it is actually an α-contraction) and has a global attractor $\mathcal{A}_{\epsilon,\delta}$ (see, for example, [16]). We remark that a solution $(u(t), v(t)) \in \mathcal{A}_{\epsilon,\delta}$ is defined and bounded for all $t \in \mathbb{R}$. Therefore, the function $v(t)$ must satisfy the equation

$$v(t) = \int_{-\infty}^{t} e^{-\epsilon(t-s)}\epsilon\gamma u(s)\, ds.$$

As a consequence, the function $v \in H^1(0, 1)$ and $\mathcal{A}_{\epsilon,\delta} \subset H^1(0, 1) \times H^1(0, 1)$. Using the regularity theory of parabolic equations and the above integral expression for $v(t)$, we deduce that $\mathcal{A}_{\epsilon,\delta} \subset H^2(0, 1) \times H^2(0, 1)$. This implies that $\mathcal{A}_{\epsilon,\delta}$ belongs to a bounded subset of $C^1(0, 1) \times C^1(0, 1)$. Let us now consider (6.10), (6.11) in the space $C^1(0, 1) \times C^1(0, 1)$. The rectangle $B = B(a, b, c, d)$ introduced in the previous example is an invariant rectangle under the flow defined by (6.10), (6.11); that is, if the initial data belongs to B, then the solution remains in B for all $t \geq 0$. Furthermore, no invariant set can lie on the boundary of B. This implies that, for any $d_0 > 0$, there exists a positive number r_0 such that $\cup_{\epsilon>0, d\geq d_0>0}\mathcal{A}_{\epsilon,\delta} \subset \mathcal{B}_{r_0}$, the ball in $C^1(0, 1) \times C^1(0, 1)$ of center 0 and radius r_0. From this fact, we deduce that, for any $\epsilon > 0$, $d \geq d_0$, $\|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0$ if $(u_0, v_0) \in \mathcal{A}_{\epsilon,\delta}$.

Let us now consider what happens when $\epsilon = 0$. In this case, the system (6.10), (6.11), has a first integral given by $V(u, v) = v$; that is, for any given initial data $(\varphi, \psi) \in X$,the solution $(u(t, x), v(t, x))$ of (6.10), (6.11), for $\epsilon = 0$, satisfies $v(t, x) = \psi$ for all $t \geq 0$. On the set $V^{-1}(\psi)$, the dynamics of the flow is determined by the scalar parabolic equation

$$u_t = du_{xx} + f(u) - \psi \qquad (6.12)$$

with the boundary condition (6.11). Equation (6.12) has a global attractor for each $d > 0$, $\psi \in L^2(0, 1)$.

Let

$$\tilde{B} = \{ (u_0, v_0) \in H^1(0, 1) \times H^1(0, 1) : (u_0(x), v_0(x)) \in$$
$$B(a, b, c, d),\ x \in [0, 1], \|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0 \}.$$

We now can apply Corollary 3.6 to see, for example, that $\mathcal{A}_0(\tilde{B}) = \hat{\omega}_{S_0}(\mathcal{B})$. For any bounded set $B \in H^1(0, 1) \times L^2(0, 1)$, we always have $\mathcal{A}_0(\tilde{B}) \subset \hat{\omega}_{S_0}(\mathcal{B})$.

Let us describe the attractor in more detail for the diffusion coefficient d very large. This is the same as taking a thin domain around a point which has the special shape obtained by rescaling x in Q_ϵ. For d very large, we claim that the attractor $\mathcal{A}_{\epsilon,\delta} = \mathcal{A}_\epsilon$, the attractor for (6.7). To prove this claim, we introduce the notation

$$\begin{aligned} u &= \xi + \tilde{u}, \quad \xi = \int_0^1 u\,dx \\ v &= \eta + \tilde{v}, \quad \eta = \int_0^1 v\,dx\,. \end{aligned}$$

The system (6.10) can be written as

$$\begin{aligned} \dot{\xi} &= f(\xi) - \eta + \int_0^1 F(\xi, \tilde{u})\,dx \\ \dot{\eta} &= \epsilon(\gamma\xi - \eta)\,. \\ \tilde{u}_t &= d\tilde{u}_{xx} + F(\xi, \tilde{u}) - \int_0^1 F(\xi, \tilde{u})\,dx - \tilde{v} \\ \tilde{v}_t &= -\epsilon\tilde{v} + \epsilon\gamma\tilde{u}\,, \end{aligned} \tag{6.13}$$

where $F(\xi, \tilde{u}) = f(\xi + \tilde{u}) - f(\xi)$. Let r_0 be a constant which bounds uniformly in ϵ, d the attractors $\mathcal{A}_{\epsilon,\delta}$ in $H^1(0, 1) \times H^1(0, 1)$. Using arguments similar to the ones in [14], we can show that there is a constant $d_1 > 0$, such that, for $d \geq d_1$, there are constants $\beta(d) > 0$, $k_0 > 0$, $\beta(d) \to \infty$ as $d \to \infty$ such that, for $0 \leq \epsilon \leq \epsilon_0$ and initial data (u_0, v_0) satisfying $\|(u_0, v_0\|_{H^1(0,1)\times H^1(0,1)} \leq r_0$, we have the following estimates for the solutions of (6.10), (6.11):

$$\|\tilde{u}(t)\|_{H^1(0,1)} \leq k_0(e^{-\frac{\epsilon}{2}t}\|\tilde{u}_0\|_{H^1(0,1)} + \frac{1}{\beta(d)}e^{-\frac{\epsilon}{2}t}\|\tilde{v}_0\|_{H^1(0,1)}) \tag{6.14}$$

$$\|\tilde{v}(t)\|_{H^1(0,1)} \leq k_0(e^{-\epsilon t}\|\tilde{v}_0\|_{H^1(0,1)} + e^{-\frac{\epsilon}{2}t}\|\tilde{u}_0\|_{H^1(0,1)}) \tag{6.15}$$

for all $t \geq 0$.

Relations (6.14), (6.15) imply that, for any $\epsilon > 0$ and for initial data (u_0, v_0) satisfying $\|(u_0, v_0)\|_{H^1(0,1)\times H^1(0,1)} \leq r_0$, the ω-limit set of the solution must lie on the attractors $\mathcal{A}_\epsilon$ of the ODE (6.7) provided that $d \geq d_1$. Since we know that all of the attractors must have initial data with this bound, we have proved our claim.

For $\epsilon = 0$ and $d \geq d_0$, sufficiently large, the attractor for (6.15) is the same as the attractor $\mathcal{A}_0^{\bar{\psi}}$ of the equation

$$\dot{u} = f(u) - \bar{\psi}\,, \quad \bar{\psi} = \int_0^1 \psi(x)\,dx\,. \tag{6.16}$$

For any bounded set $B \subset X$, the attractor $\mathcal{A}_0(B)$ for (6.10), (6.11) for $\epsilon = 0$ is given by

$$\mathcal{A}_0(B) = \{ \mathcal{A}_0^{\bar{\psi}} : \bar{\psi} = \int_0^1 \psi(x)\,dx,\ \psi \in B \}.$$

As remarked above, for the set $\tilde{B}$, Corollary 3.6 implies that $\mathcal{A}_0(\tilde{B}) = \hat{\omega}_{S_0}(\mathcal{B})$.

7. Linearly damped second order ODE.

In this section, we consider the second order ordinary differential equation

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -f(u) - \beta v, \end{aligned} \tag{7.1}$$

where $\beta \geq 0$ is constant, $f \in C^2(\mathbb{R}, \mathbb{R})$ has a finite number of simple zeros and $-f(u)$ is dissipative; that is,

$$\limsup_{|u| \to \infty} -\frac{f(u)}{u} \leq -2\alpha < 0. \tag{7.2}$$

For $\beta = 0$, we have the conservative system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -f(u), \end{aligned} \tag{7.3}$$

with the first integral

$$V(u, v) = \frac{1}{2}v^2 + F(u), \quad F(u) = \int_0^u f(s)\,ds. \tag{7.4}$$

For $\beta > 0$, (7.1) is a gradient system and has a global attractor $\mathcal{A}_\beta$. The equilibrium set E of (7.1) consists of the points $(u_0, 0) \in \mathbb{R}^2$ such that $f(u_0) = 0$. Since $-f$ is dissipative, the set E is bounded. Since the zeros of f are simple, the set E is finite, each point in E is hyperbolic, each unstable point corresponds to a local maximum of the function F and has a one dimensional unstable manifold $W_\beta^u(u_0, 0)$. Therefore, the attractor is one dimensional and is given by

$$\mathcal{A}_\beta = \cup_{(u_0,\,0) \in E} W_\beta^u(u_0, 0). \tag{7.5}$$

We denote the set of saddle points by $\{ s_j \in E, j = 1, 2, \ldots, M \}$. We say that the potential function F is *generic* if $V(s_j) \neq V(s_k)$ for $j \neq k$, $j, k = 1, 2, \ldots, M$.

To state our result, we need some notation. Let $\tilde{V}_j$ be the connected component of $\{ (u, v) \in \mathbb{R}^2 : V(u, v) \leq V(s_j) \}$ which contains s_j and let

$$V_j = \tilde{V}_j \cap \{ (u, v) \in \mathbb{R}^2 : V(u, v) \geq V(s_k),\ s_k \in \tilde{V}_j,\ k \neq j \}. \tag{7.6}$$

We remark that, if F is generic and if there is no $s_k \in \tilde{V}_j$ except s_j, then V_j is a region in the (u, v)-plane consisting of all points inside and on the figure eight defined by $W_0^u(s_j)$.

Proposition 7.1. *If f is a C^2-function with simple zeros, satisfies (7.2), $F = \int_0^u f(s)\,ds$ is generic, then, for $S = (0, \beta_0]$,*

$$\tilde{\omega}_S(\mathcal{A}) = \{ (u, v) \in \mathbb{R}^2 : V(u, v) \le c_M \}, \tag{7.7}$$

where $c_M = \max\{ V(s_j),\ j = 1, 2, \ldots, M \}$. More specifically,

$$\tilde{V}_j \supset \tilde{\omega}_S(W^u(s_j)) \supset V_j,\ \ j = 1, 2, \ldots, M. \tag{7.8}$$

Proof. For $\beta = 0$, the set $W_0^u(s_j)$ is called a *separatrix* for the flow defined by (7.3). If F is generic, then the curve defined by $W_0^u(s_j)$ is a figure eight. Furthermore, each orbit of $T_0(t)$ in the set $\tilde{V}_j \setminus \cup_{s_k \in \tilde{V}_j} W_0^u(s_k)$ is periodic.

The first relation in (7.8) follows from the fact that $V(\varphi) \le V(s_j)$ for $\varphi \in W_\beta^u(s_j)$, $\beta \ge 0$.

To prove the second relation in (7.8), we first suppose that there are no saddle points in $\tilde{V}_j \setminus W_0^u(s_j)$; that is, $\tilde{V}_j = V_j$. Without loss of generality, we may assume that $s_j = (0, 0)$ and that the two other equilibrium points in V_j are $(\xi_1, 0)$, $(\xi_2, 0)$ with $\xi_1 < 0 < \xi_2$. The equilibrium points $(\xi_1, 0)$, $(\xi_2, 0)$ of (7.1) are stable foci if β is small. If we let $V_j^0 = V_j \setminus W_0^u(0)$, then the set $V_j^0 = U_1 \cup U_2$, where U_1, U_2 are disjoint connected open sets containing respectively the points $(\xi_1, 0)$, $(\xi_2, 0)$, and the boundary of V_j^0 is a figure eight consisting of the set $W_0^u(0)$, the unstable manifold of the saddle point $(0, 0)$ of (7.3).

For $\beta \ge 0$, let $T_\beta(t)(u_0, v_0)$ be the solution of (7.1) passing through (u_0, v_0) at $t = 0$. Suppose that $(u_0, v_0) \in U_1$ and suppose that the periodic orbit of (7.3) through (u_0, v_0) is Γ_0 of least period τ_0. For every $\beta > 0$, the closure of $W_\beta^u(0)$ contains the point $(\xi_1, 0)$. Since the function $V(u, v)$ is continuous, there is a $(u_1, v_1) \in W_\beta^u(0)$ such that $V(u_1, v_1) = V(u_0, v_0)$ and there is a $0 \le t_1 \le \tau_0$ such that $(u_0, v_0) = T_0(t_1)(u_1, v_1)$. For any $\eta > 0$, there is a $\beta_0 > 0$, depending only on τ_0, such that, for $0 < \beta \le \beta_0$, we have

$$\|T_\beta(t_1)(u_1, v_1) - T_0(t_1)(u_1, v_1)\| = \|T_\beta(t_1)(u_1, v_1) - (u_0, v_0)\| < \eta.$$

This shows that $\tilde{\omega}_S(W^u(0))$ contains the point (u_0, v_0). Using the same argument for the set U_2, we see that $\tilde{\omega}_S(W^u(0))$ contains each point of U_2. Thus, $\tilde{\omega}_S(W^u(s_j)) \supset V_j$.

Now, we consider the case where there is exactly one other saddle point s_k in $\tilde{V}_j$. If we let $\tilde{V}_j^0 = \tilde{V}_j \setminus W_0^u(s_j)$, then $\tilde{V}_j^0 = \tilde{U}_1 \cup \tilde{U}_2$, where $\tilde{U}_1$, $\tilde{U}_2$ are nonempty open connected sets. For definiteness, suppose that $s_k \in \tilde{U}_1$. Then $W_0^u(s_k)$ is a figure eight in $\tilde{U}_1$ which contains no other saddle points. From the first part of the proof, we know that $\tilde{\omega}_S(W^u(s_k)) \supset V_k$. Now suppose that $(u_0, v_0) \in (V_j \cap \tilde{U}_1) \setminus W_0^u(s_k)$ and Γ_0 is the periodic orbit of $T_0(t)$ through (u_0, v_0). For every $\beta > 0$, the closure of $W_\beta^u(s_j)$ contains one of the equilibrium points in V_k. Now we may proceed exactly as in the first part of the proof to conclude that $\tilde{\omega}_S(W^u(s_j))$ contains (u_0, v_0). If $(u_0, v_0) \in (V_j \cap \tilde{U}_2)$, then we proceed as in the first part of the proof to obtain the same conclusion.

It is clear that this type of argument can be continued to complete the proof of (7.8).

We remark that in general we may not have $\tilde{\omega}_S(W^u_\cdot(s_j)) = V_j$. If $\tilde{V}_j \setminus W^u_0(s_j)$ contains no saddle point of (7.3), then (7.8) implies that $\tilde{\omega}_S(W^u_\cdot(s_j)) = V_j$. If $\tilde{V}_j \setminus W^u_0(s_j)$ contains a saddle point s_k of (7.3) and M denotes the interior of the loop of $W^u_0(s_j)$ that does not contain s_k and $\min\{V(\varphi) : \varphi \in M\} < V(s_k)$, then the above proof shows that $\tilde{\omega}_S(W^u_\cdot(s_j)) \supset M$ and thus $\tilde{\omega}_S(W^u_\cdot(s_j)) \neq V_j$. Also, if L is the other loop of $W^u_0(s_j)$, then L contains $W^u_0(s_k)$. Since the interior of $W^u_0(s_k)$ must contain at least two equilibrium points of (7.3) for which each orbit in a neighborhood is periodic, it follows that L contains at least two stable foci for $\beta > 0$. If the ω-limit set of $W^u_\beta(s_j) \cap L$ is one of the stable foci, then $\tilde{\omega}_S(W^u_\cdot(s_j)) \neq V_j$.

The conclusion in Proposition 7.1 is the most that can be obtained by using the limit set $\tilde{\omega}_S(\mathcal{A}_\cdot)$. If we use Corollary 3.6, we obtain information about the conservative system in a bounded set B by considering the set $\hat{\omega}_S(B)$. In fact, if $B(c) = \{(u, v) \in \mathbb{R}^2 : V(u, v) \leq c\}$, then $\hat{\omega}_S(B(c)) = B(c)$ since the set $B(c)$ is invariant under the group $T_0(t)$ defined by the solutions of (7.3).

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