Resenhas IME-USP 1993, Vol. 1, No. 2/3, 131 - 149.

Rapid Convergence to equilibrium in Ferromagnetic Stochastic Ising Models[1]

Richard Holley

Abstract: This is a review of what is known about exponentially fast convergence to equilibrium in finite range attractive Stochastic Ising Models. The main goal of this review is to explain the necessary and sufficient condition for fast convergence, that has recently been proven by Martinelli and Olivieri. All of the necessary background is explained so that this review is self contained.

Key words: Stochastic ising model, fast convergence to equilibrium, mixing conditions.

§1 Introduction.

The stochastic Ising model was first introduced in the early 50's by N. Metropolis et. al. [12] as a method for sampling typical configurations from Gibbs states. The idea was to make a Markov process whose equilibrium states are Gibbs states and then run the process long enough to reach equilibrium, at which time the configuration of the process should be a typical configuration for the corresponding Gibbs state. Thus right from the beginning the question "How long does it take to reach equilibrium?" was an important one. About ten years later Roy Glauber [4] introduced the stochastic Ising model again; this time from the point of view of nonequilibrium statistical mechanics. The first question that Glauber asked was how fast does the system relax to equilibrium? For the next seven or eight years the work on the stochastic Ising model was mainly experimental simulations. Then around 1970 Frank Spitzer [14], and R.L. Dobrushin and I.I. Pyatetski–Shapiro [2] introduced the general subject of interacting particle systems and a small group of people began studying interacting particle systems in general from a theoretical point of view. Over the years the subject attracted a growing number of researchers. By 1985 it was becoming obvious that in the time reversable case (stochastic Ising model) one could say a lot about the rate of convergence to equilibrium if one knew enough about the equilibrium state (see R. Holley [7]). Exactly what one must know about the equilibrium state in order to conclude that the system relaxes to equilibrium exponentially fast has recently been discovered by F. Martinelli and E. Olivieri [11]. It is the purpose of this paper to review the necessary background and then prove the Martinelli–Olivieri Theorem, so that the entire proof of their condition is contained here.

This paper is organized as follows: In the first section we will give some basic facts about Gibbs states. In the second section we introduce the stochastic Ising model and prove that it exists. In the third section we derive some of the important properties of the stochastic Ising model that we will need in our proofs. The forth section contains some of the properties that derive from assuming that the

[1] This work was partially supported by N.S.F. Grant DMS-9204119.

system is ferromagnetic. Finally in the fifth section we prove the Martinelli-Olivieri necessary and sufficient condition for exponentially fast convergence to equilibrium.

§2 The Ising Model.

The goal of this section is to introduce some notation and basic facts concerning the Ising model. We will think of the Ising model as a model for the spins in a piece of iron, and for simplicity the spins are restricted to be either up or down. Thus we take $E = \{-1,1\}^{Z^d}$ to be the space of spin configurations, and denote configurations of spins by Greek letters such as $\sigma, \eta, \omega \in E$, $\sigma(k)$ the spin at $k \in Z^d$ in the configuration σ.

Give $\{-1,1\}$ the discrete topology and E the resulting product topology. Let $\mathcal{C}(E)$ denote real valued continuous functions on E with $\|f\| = \sup_{\sigma \in E} |f(\sigma)|$. $\mathcal{D} \subset \mathcal{C}(E)$ will denote the cylinder functions (local observables) on E.

We assume that we have an interaction that is given by a pair potential $J_{\{x,y\}} \in \mathbb{R}$, $x, y \in Z^d$ that is translation invariant

$$J_{\{x,y\}} = J_{\{x+k,y+k\}}, \qquad x, y, k \in Z^d,$$

and has finite range

$$J_{\{x,y\}} = 0 \text{ if } \|x-y\| > L.$$

here $\|x\| = \max\{|x_i|, i = 1, \ldots, d\}$. Any external field present will be deonted by h.

In order to describe the Gibbs states we first begin with conditional Gibbs states. The Gibbs state with the above pair interaction and external field, conditioned on the complement of the set Λ with configuration equal to σ outside of Λ is given by

$$\mu_\Lambda(\eta|\sigma) = \frac{e^{-\left(\sum_{\{x,y\}\subset\Lambda} J_{\{x,y\}}\eta(x)\eta(y) + \sum_{x\in\Lambda}\sum_{y\in\Lambda^c} J_{\{x,y\}}\eta(x)\sigma(y) + h\sum_{x\in\Lambda}\eta(x)\right)}}{Z(\Lambda,\sigma)}$$

$$\eta \in \{-1,1\}^\Lambda.$$

Here $Z(\Lambda,\sigma)$ is the normalizing constant needed to make $\mu_\Lambda(\cdot|\sigma)$ a probability measure.

Before trying to relate this to a measure on E we first set $\mathcal{F}_\Lambda = \sigma(\eta(k) : k \in \Lambda)$, the sigma algebra generated by the functions $\eta \mapsto \eta(k)$, $k \in \Lambda$. Then μ, a probability measure on $\mathcal{F}_{Z^d}$, is a Gibbs state if

$$\mu(\eta|\mathcal{F}_{\Lambda^c})(\sigma) = \mu_\Lambda(\eta|\sigma)$$

for all $\sigma \in E$ and all $\Lambda \in Z^d$. We will let $\mathcal{G}$ denote the set of all Gibbs states (for $J_{\{x,y\}}$ and h). There may be more than one element of $\mathcal{G}$; however, there is always at least one element. In this paper we will be primairly interested in the case when $\mathcal{G}$ is a singleton.

§3 The Stochastic Ising Model

The Stochastic Ising model is a Markov process, η_t, with state space E such that the measures in $\mathcal{G}$ are its stationary measures. The evolution should be "local" in the sense that each spin evolves in a way that is dependent only on the spins that are within a distance less than or equal to the range of the interaction of the given spin. Before describing the evolution in detail we need some notation. If $\eta \in E$ and $k \in Z^d$, we set

$$\eta^k(j) = \begin{cases} \eta(j) \text{ if } j \neq k \\ -\eta(k) \text{ if } j = k. \end{cases}$$

For each $k \in Z^d$ we have a positive function, which we call the flip rates, $c_k(\eta)$. Intuitively the flip rates determine the process by the following formulas:

$$P(\eta_{t+h}(k) = -\eta_t(k)|\eta_t) = c_k(\eta)h + o(h),$$

and

$$\text{If } k \neq j \text{ then } P(\eta_{t+h}(k) = -\eta_t(k) \text{ and } \eta_{t+h}(j) = -\eta_t(j)|\eta_t) = o(h).$$

We require that $c_k(\eta) = c_k(\sigma)$ if $\eta(j) = \sigma(j)$ for all $\|j-k\| \leq L$, and that they satisfy detailed balance, which intuitively says that $\mu(\eta)c_k(\eta) = \mu(\eta^k)c_k(\eta^k)$. Since this doesn't really make sense becuase $\mu(\eta) = 0$ we actually assume that

$$\mu_\Lambda(\eta|\sigma)c_k(\eta\sigma) = \mu_\Lambda(\eta^k|\sigma)c_k(\eta^k\sigma) \qquad \text{if } k \in \Lambda.$$

Here $\eta\sigma$ is the configuration of spins that equals η in Λ and equals σ in Λ^c.

For example

$$\begin{aligned} c_k(\eta) &= \frac{e^{\eta(k)(\sum_j J_{\{k,j\}}\eta(j)+h)}}{e^{-\eta(k)(\sum_j J_{\{k,j\}}\eta(j)+h)} + e^{\eta(k)(\sum_j J_{\{k,j\}}\eta(j)+h)}} \\ &= \frac{1}{1+e^{-2\eta(k)(\sum_j J_{\{k,j\}}\eta(j)+h)}}. \end{aligned}$$

This is the flip rate that Glauber used. Note that we can multiply c_k by any function of η that does not depend on $\eta(k)$ and still have detailed balance. Thus for example the Metropolis flip rates are given by $c_k(\eta)/(c_k(\eta) \vee c_k(\eta^k))$, where c_k is as above.

The following notation will be useful to us. If $f \in \mathcal{C}(E)$, define

$$\Delta_k f(\eta) \equiv f(\eta^k) - f(\eta).$$

From our above intiutive description of the stochastic Ising model, we want a Markov process, η_t on E whose infinitesimal generator, when restricted to $\mathcal{D}$, is given by

$$\mathcal{L}f(\eta) = \sum_{k \in Z^d} c_k(\eta)\Delta_k f(\eta), \qquad f \in \mathcal{D}.$$

Our immediate goal is to show that there is a unique such process. We follow the development in [5]

In order to get some feel for the problem, suppose f is $\mathcal{F}_\Lambda$ measurable. Then

$$\mathcal{L}f(\eta) = \sum_{k \in \Lambda} c_k(\eta)\Delta_k f(\eta).$$

Thus

$$||\mathcal{L}f(\eta)|| \leq 2||c_0||\,|\Lambda|\,||f||$$

and $\mathcal{L}f$ is $\mathcal{F}_{\Lambda\cup\partial\Lambda}$ measurable. Here $\partial\Lambda = \{k \notin \Lambda : \mathrm{dist}(k,\Lambda) \leq L\}$.

Let $\Lambda \cup \partial\Lambda = \Lambda_1$, $\Lambda_1 \cup \partial\Lambda_1 = \Lambda_2$, $\Lambda_2 \cup \partial\Lambda_2 = \Lambda_3$, etc. Then

$$||\mathcal{L}^2 f(\cdot)|| \leq (2||c_0||)^2|\Lambda_1||\Lambda|||f||.$$

and by induction

$$||\mathcal{L}^n f(\cdot)|| \leq (2||c_0||)^n|\Lambda_{n-1}||\Lambda_{n-2}|\ldots|\Lambda|||f||.$$

If $\Lambda \subset [-l,l]^d$ then $\Lambda_n \subset [-l-nL, l+nL]^d$ and $|\Lambda_n| \leq (2(l+nL)+1)^d$. Thus

$$||\mathcal{L}^n f(\cdot)|| \leq (2||c_0||)^n \prod_{j=0}^{n-1}(2(l+jL)+1)^d||f||.$$

We would like to expand $e^{\mathcal{L}t}$ in a power series, so the question becomes "Does

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{L}^n f$$

converge for any $t > 0$?" We begin by setting $A_f = ||f||\prod_{j=0}^{2l}(2(l+jL)+1)^d$. Then

$$\begin{aligned}
(2||c_0||)^n &\prod_{j=0}^{n-1}(2(l+jL)+1)^d||f|| \\
&= (2||c_0||)^n A_f \left(\prod_{j=2l+1}^{n-1}(2L+\frac{2l+1}{j})^d\right)(\frac{((n-1)!)}{(2l)!})^d \\
&\leq (2||c_0||)^n((n-1)!)^d(2L+1)^{(n-1)d}A_f.
\end{aligned}$$

If $d = 1$ the series converges for all $t < \frac{1}{2||c_0||(2L+1)}$. If $d \geq 2$ we need to be more careful.

Let $N_k = \{j \in Z^d : \|k-j\| \le L\}$, and let $D(\Lambda) = \{f : f \text{ is } \mathcal{F}_\Lambda \text{ measurable}\}$. If $f \in D(\Lambda)$ then $\mathcal{L}f = \sum_{k\in\Lambda} c_k\Delta_k f$ and $c_k\Delta_k f \in D(\Lambda \cup N_k)$.

$$\mathcal{L}^2 f = \sum_j c_j\Delta_j \sum_{k\in\Lambda} c_k\Delta_k f = \sum_{k\in\Lambda}\sum_{j\in\Lambda\cup N_k} c_j\Delta_j c_k\Delta_k f.$$

and

$$\begin{aligned}\mathcal{L}^n f &= \sum_{k_n} c_{k_n}\Delta_{k_n}\sum_{k_{n-1}} c_{k_{n-1}}\Delta_{k_{n-1}}\dots\sum_{k_1} c_{k_1}\Delta_{k_1} f\\ &= \sum_{k_1\in\Lambda}\sum_{k_2\in\Lambda\cup N_{k_1}}\dots\sum_{k_n\in\Lambda\cup N_{k_1}\cup\ldots\cup N_{k_{n-1}}} c_{k_n}\Delta_{k_n}c_{k_{n-1}}\Delta_{k_{n-1}}\dots c_{k_1}\Delta_{k_1} f.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{L}^n f\| &\le \|c_0\|^n 2^n\|f\|\sum_{k_1\in\Lambda}\sum_{k_2\in\Lambda\cup N_{k_1}}\dots\sum_{k_n\in\Lambda\cup N_{k_1}\cup\ldots\cup N_{k_{n-1}}} 1\\ &\le \|c_0\|^n 2^n\|f\|(|\Lambda|+(n-1)(2L+1)^d)(|\Lambda|+(n-2)(2L+1)^d)\dots|\Lambda|\\ &\le \|c_0\|^n 2^n\|f\|(|\Lambda|+n(2L+1)^d)^n = \|c_0\|^n 2^n n!\|f\|\frac{(|\Lambda|+n(2L+1)^d)^n}{n!}\\ &\le \|c_0\|^n 2^n n!\|f\|e^{(|\Lambda|+n(2L+1)^d)} = e^{|\Lambda|}n!(2\|c_0\|e^{(2L+1)^d})^n\|f\|.\end{aligned}$$

The above inequality allows us to define $e^{\mathcal{L}t}$ for small t. We want to extend it to all $t \ge 0$ and at the same time to show that it is the limit of semi-groups which correspond to processes in which all of the spins outside of a finite set are frozen in their initial positions. To define these latter semi-groups let

$$\mathcal{L}_\Lambda f = \sum_{k\in\Lambda} c_k\Delta_k f,$$

and note that $\mathcal{L}_\Lambda$ is a bounded operator for finite Λ. Let $T_t^\Lambda = e^{t\mathcal{L}_\Lambda}$ be the a positive contraction semi-group whose generator is $\mathcal{L}_\Lambda$. Next note that if $f \in D(\Lambda_0)$ then $\mathcal{L}^n f = \mathcal{L}_\Lambda^n f$ for all $n \le \frac{\text{dist}(\Lambda_0,\Lambda^c)}{L}$. Thus

$$\lim_{\Lambda\nearrow Z^d} T_t^\Lambda f = \sum_{n=0}^\infty \frac{t^n}{n!}\mathcal{L}^n f \equiv T_t f$$

for all $0 \le t < \frac{1}{2\|c_0\|e^{(2L+1)^d}}$ and all $f \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{C}(E)$ and each T_t^Λ is a contraction semi-group, $T_t^\Lambda f \to T_t f$ for all $f \in \mathcal{C}(E)$ and all $0 \le t \le \frac{1}{2\|c_0\|e^{(2L+1)^d}}$. Now use the semi-group property of each T_t^Λ to conclude that $\lim_{\Lambda\nearrow Z^d} T_t^\Lambda f \equiv T_t f$ for all $t \ge 0$ and all $f \in \mathcal{C}(E)$.

§3 Properties of T_t (T_t^Λ).

In this section we list several of the properties of T_t and T_t^Λ that will be useful to us. The first three properites follow immediately from the choice of the flip rates and the construction of the semi-group.

1. Since c_k's satisfy detailed balance

$$\begin{aligned}\int f(\eta)\mathcal{L}^\Lambda g(\eta)\mu_\Lambda(d\eta|\sigma) &= \int g(\eta)\mathcal{L}^\Lambda f(\eta)\mu_\Lambda(d\eta|\sigma)\\ &= -\frac{1}{2}\sum_{k\in\Lambda}\int c_k(\eta)\Delta_k f(\eta)\Delta_k g(\eta)\mu_\Lambda(d\eta|\sigma),\end{aligned}$$

and

$$\int f(\eta)\mathcal{L}g(\eta)\mu(d\eta) = -\frac{1}{2}\sum_{k\in Z^d}\int c_k(\eta)\Delta_k f(\eta)\Delta_k g(\eta)\mu(d\eta)$$

for all $f, g \in \mathcal{D}$ and $\mu \in \mathcal{G}$.

2. If $\int \mathcal{L}f(\eta)\mu(d\eta) = 0$ for all $f \in \mathcal{D}$, then μ is stationary for $\{T_t : t \geq 0\}$.

3. Every $\mu \in \mathcal{G}$ is stationary for $\{T_t : t \geq 0\}$.

4. Let $\mathcal{K} = 2\|c_0\|(2L+1)^d$ and set $\alpha = 2\mathcal{K} \vee 1$. Then for $f \in D(\Lambda)$,

$$\sum_{\{k:\mathrm{dist}(k,\Lambda)\geq nL\}} \|\Delta_k T_t f\| \leq (.61)^{t_0}\left(\frac{t}{t_0}\right)^n e^{\mathcal{K}(t-t_0)}|||f|||.$$

Here $t_0 = n/\alpha$ and $|||f||| = \sum_{j\in Z^d}\|\Delta_j f\|$.

Note that if $t \leq t_0$ then

$$\sum_{\{k:\mathrm{dist}(k,\Lambda)\geq nL\}} \|\Delta_k T_t f\| \leq (.61)^{t_0}|||f|||.$$

A proof of property 4 can be found in [9]. We sketch the proof here. The proof is based on the following lemma the proof of which can be found in Liggett's book [10].

Lemma. *If $\varphi(t,\eta) : [0,\infty) \times E \mapsto \mathbb{R}$ and $\frac{d\varphi}{dt}(t,\eta) = \Omega\varphi(t,\eta) + \psi(t,\eta)$ for some infinitesimal generator of a positive contraction semi-group, Ω, then $\|\varphi(t,\cdot)\| \leq \|\varphi(0,\cdot)\| + \int_0^t \|\psi(s,\cdot)\|ds$.*

To apply this lemma we let $\varphi(t,\eta) = \Delta_k T_t f(\eta)$, then

$$\begin{aligned}\frac{d}{dt}\varphi(t,\eta) =& \Delta_k \mathcal{L} T_t f(\eta)\\ =& \Delta_k \sum_j c_j \Delta_j T_t f(\eta)\\ =& \sum_{|j-k|>L} c_j(\eta)\Delta_j\Delta_k T_t f(\eta) + \sum_{|j-k|\leq L}\Delta_k c_j \Delta_j T_t f(\eta)\\ =& \sum_{|j-k|>L} c_j(\eta)\Delta_j \varphi(t,\eta) + \sum_{|j-k|\leq L}\Delta_k c_j\Delta_j T_t f(\eta).\end{aligned}$$

Thus

$$\|\Delta_k T_t f\| \leq \|\Delta_k f\| + 2\|c_0\| \int_0^t \sum_{|j-k|\leq L} \|\Delta_j T_s f\| ds.$$

Therefore,

$$\sum_{\{k:\text{dist}(k,\Lambda)\geq nL\}} \|\Delta_k T_t f\| \leq \sum_{\{k:\text{dist}(k,\Lambda)\geq nL\}} \|\Delta_k f\| + 2\|c_0\|(2L+1)^d \int_0^t \left(\sum_{\{k:\text{dist}(k,\Lambda)\geq (n-1)L\}} \|\Delta_k T_s f\| \right) ds.$$

Now set $F_n(t) = \sum_{\{k:\text{dist}(k,\Lambda)\geq nL\}} \|\Delta_k T_t f\|$. Then $F_n(t) \leq \mathcal{K} \int_0^t F_{n-1}(s) ds$ for $n > 1$, and $F_0(t) \leq |||f||| + \mathcal{K} \int_0^t F_0(s) ds$.

Thus by induction on n we get

$$F_n(t) \leq |||f||| \left(e^{\mathcal{K}t} - \sum_{j=0}^{n-1} \frac{(\mathcal{K}t)^j}{j!} \right) \leq (.61)^{t_0} \left(\frac{t}{t_0}\right)^n e^{\mathcal{K}(t-t_0)} |||f|||.$$

5. If $\text{dist}(\Lambda_0, \Lambda^c) \geq nL$ and $f \in D(\Lambda_0)$ then

$$\|T_t f(\cdot) - T_t^\Lambda f(\cdot)\| \leq |||f|||(.61)^{t_0} \left(\frac{t}{t_0}\right)^{n+1} e^{\mathcal{K}(t-t_0)}.$$

6. If $\text{dist}(\Lambda_1, \Lambda_2) \geq 2nL$ and $f \in D(\Lambda_1)$ and $g \in D(\Lambda_2)$, then $\|T_t(fg)(\cdot) - T_t f(\cdot) T_t g(\cdot)\| \leq 2|||fg|||(.61)^{t_0} \left(\frac{t}{t_0}\right)^{(n+1)} e^{\mathcal{K}(t-t_0)}$.

The proofs of 5. and 6. are similar to that of 4.

Remark: The bounds in 4. and 6. also apply to T_t^Λ for any $\Lambda \subset Z^d$.

§4 Monotonicity and coupling

For the rest of this paper we assume that all $J_{\{k,j\}} \leq 0$. This has the effect of making configurations with neighboring spins in the same direction more likely than configurations in which lots of neighbors have opposite spins. This shows up in the flip rates. Recall that one possible choice of flip rates is given by

$$c_0(\eta) = \frac{1}{1 + e^{-2\eta(0)(\sum_j J_{\{0,j\}}\eta(j)+h)}}.$$

In order to see what the effect of assuming that all of the $J_{\{k,j\}}$ are less than or equal to zero is we make E into a lattice by defining

$$(\eta \vee \sigma)(k) = \eta(k) \vee \sigma(k)$$
$$(\eta \wedge \sigma)(k) = \eta(k) \wedge \sigma(k).$$

We say that $f : E \mapsto \mathbb{R}$ is increasing if $\eta \geq \sigma$ implies that $f(\eta) \geq f(\sigma)$. Then $c_k(\eta)$ is decreasing on $\{\eta : \eta(k) = 1\}$ and increasing on $\{\eta : \eta(k) = -1\}$. Flip rates c_k, $k \in Z^d$ that satisfy these monomonicity conditions will be called attractive.

A process with attractive flip rates allows us to construct a process on $E \times E$, call it $(\eta_t^{(1)}, \eta_t^{(2)})$, with the following properties:

1) $E^{(\eta^{(1)},\eta^{(2)})}[f(\eta_t^{(1)})] = T_t f(\eta^{(1)})$.

2) $E^{(\eta^{(1)},\eta^{(2)})}[f(\eta_t^{(2)})] = T_t f(\eta^{(2)})$.

3) If $\eta^{(1)} \geq \eta^{(2)}$ then $P^{(\eta^{(1)},\eta^{(2)})}(\eta_t^{(1)} \geq \eta_t^{(2)}) = 1$ for all $t \geq 0$.

Rather than construct a coupling with the above properties we will instead prove a lemma that we need. The proof of the lemma that we will give is by coupling, and after one sees that proof it will be clear how to make the coupling mentioned above. The following lemma is a version of a theorem that was first proved in [3]. There are numerous proofs of this version in the literature. The proof given here is from [6].

Lemma F.K.G.. *Let Λ be a finite set and let μ_1, μ_2 be probability measures on $S \equiv \{-1, 1\}^\Lambda$ such that $\mu_1(\sigma)$, $\mu_2(\sigma) > 0$ for all $\sigma \in S$. If $\mu_1(\sigma \vee \eta)\mu_2(\sigma \wedge \eta) \geq \mu_1(\sigma)\mu_2(\eta)$ for all $\sigma, \eta \in S$, then there is a measure ν on $S \times S$ such that:*

$$\sum_{\sigma \in S} \nu(\eta, \sigma) = \mu_1(\eta)$$

$$\sum_{\eta \in S} \nu(\eta, \sigma) = \mu_2(\sigma)$$

$$\nu(\eta, \sigma) = 0 \text{ unless } \eta \geq \sigma.$$

Proof:. The proof is accomplished by constructing two Markov processes that have μ_1 and μ_2 as stationary measures, and then compling those processes. We define the flip rates for our constructed processes as follows: For $x \in \Lambda$ let

$$c_x^{(i)}(\eta) = \begin{cases} 1 & \text{if } \eta(x) = -1 \\ \mu_i(\eta^x)/\mu_i(\eta) & \text{if } \eta(x) = 1. \end{cases}$$

Let Ω_i be the infinitesimal generator given by

$$\Omega_i f(\eta) = \sum_{x \in \Lambda} c_x^{(i)}(\eta)(f(\eta^x) - f(\eta)).$$

Let $T_t^{(i)} = e^{t\Omega_i}$ and set $P_t^{(i)}(\eta, A) = T_t^{(i)} I_A(\eta)$.

$\mu_i(\eta)c_x^{(i)}(\eta) = \mu_i(\eta^x)c_x^{(i)}(\eta^x)$ for all $x \in \Lambda$ and all $\eta \in S$. Therefore μ_i is stationary for $\{T_t^{(i)}, t \geq 0\}$ and Thus $\lim_{t \to \infty} P_t^{(i)}(\eta, A) = \mu_i(\eta)$ for all $\eta \in S$.

Now define the flip rates of the coupled process by

$$\bar{\Omega}(\eta,\sigma;\omega,\tau)=\begin{cases} c_x^{(1)}(\eta)\wedge c_x^{(2)}(\sigma) \\ \qquad \text{if } \eta(x)=\sigma(x) \text{ and } \omega=\eta^x, \tau=\sigma^x \\ c_x^{(1)}(\eta)-(c_x^{(1)}(\eta)\wedge c_x^{(2)}(\sigma)) \\ \qquad \text{if } \eta(x)=\sigma(x) \text{ and } \omega=\eta^x, \tau=\sigma \\ c_x^{(2)}(\eta)-(c_x^{(1)}(\eta)\wedge c_x^{(2)}(\sigma)) \\ \qquad \text{if } \eta(x)=\sigma(x) \text{ and } \omega=\eta, \tau=\sigma^x \\ c_x^{(1)}(\eta) \text{ if } \eta(x)\neq\sigma(x) \text{ and } \omega=\eta^x, \tau=\sigma \\ c_x^{(2)}(\eta) \text{ if } \eta(x)\neq\sigma(x) \text{ and } \omega=\eta, \tau=\sigma^x \\ 0 \qquad \text{otherwise.} \end{cases}$$

Define

$$\bar{\mathcal{L}}f(\eta,\sigma)=\sum_{\omega,\tau}\bar{\Omega}(\eta,\sigma;\omega,\tau)(f(\omega,\tau)-f(\eta,\sigma)).$$

Note that if $f(\eta,\sigma)=\phi(\eta)$ then $\bar{\mathcal{L}}f(\eta,\sigma)=\Omega_1\phi(\eta)$, and if $f(\eta,\sigma)=\phi(\sigma)$ then $\bar{\mathcal{L}}f(\eta,\sigma)=\Omega_2\phi(\sigma)$.

Let $\bar{T}_t=e^{t\bar{\mathcal{L}}}$ and $\bar{P}_t((\eta^{(1)},\eta^{(2)}),C)=\bar{T}_tI_C((\eta^{(1)},\eta^{(2)})$.

Then it follows from the above observations that $\bar{P}_t((\eta^{(1)},\eta^{(2)}),A\times S)=P_t^{(1)}(\eta_t^{(1)},A)$ and $\bar{P}_t((\eta^{(1)},\eta^{(2)}),S\times A)=P_t^{(2)}(\eta_t^{(2)},A)$.
Therefore, $\mu_1(A)=\lim_{t\to\infty}\bar{P}_t((\eta^{(1)},\eta^{(2)},A\times S)$, and similarly for $\mu_2(A)$.

Let $\nu(C)=\lim_{t\to\infty}\bar{P}_t((+1,-1),C)$. Again from the above observations it follows that $\nu(A\times S)=\mu_1(A)$ and $\nu(S\times A)=\mu_2(A)$.

Claim: If $\eta^{(1)}\geq\eta^{(2)}$ then $\bar{P}_t((\eta^{(1)},\eta^{(2)}),\{(\eta,\sigma):\eta\geq\sigma\})=1$ for all t. To prove this claim it is enough to show that if $\eta\geq\sigma$ and $\omega\not\geq\tau$ then $\bar{\Omega}(\eta,\sigma;\omega,\tau)=0$. This follows immediately from the definition of $\bar{\Omega}$. The only interesting case is when $\eta\geq\sigma$, $\eta(x)=\sigma(x)=1$, $\omega=\eta^x$, and $\tau=\sigma$. In this case

$$\begin{aligned}\bar{\Omega}(\eta,\sigma;\omega,\tau)&=c_x^{(1)}(\eta)-(c_x^{(1)}(\eta)\wedge c_x^{(2)}(\sigma))\\&=\frac{\mu_1(\eta^x)}{\mu_1(\eta)}-\left(\frac{\mu_1(\eta^x)}{\mu_1(\eta)}\wedge\frac{\mu_2(\sigma^x)}{\mu_2(\sigma)}\right)\\&=\frac{\mu_1(\eta^x)}{\mu_1(\eta)}\left(1-\left(1\wedge\frac{\mu_2(\sigma^x)\mu_1(\eta)}{\mu_2(\sigma)\mu_1(\eta^x)}\right)\right).\end{aligned}$$

Next note that by hypothesis, $\frac{\mu_2(\sigma^x)\mu_1(\eta)}{\mu_2(\sigma)\mu_1(\eta^x)}=\frac{\mu_2(\sigma\wedge\eta^x)\mu_1(\sigma\vee\eta^x)}{\mu_2(\sigma)\mu_1(\eta^x)}\geq 1$.

Finally $\nu(\{(\eta,\sigma):\eta\geq\sigma\}=\lim_{t\to\infty}\bar{P}_t((+1,-1),\{(\eta,\sigma):\eta\geq\sigma\})=1$.

In the future we will denote by η_t the infinite proecss with semi-group T_t. We let η_t^Λ be the process with spins outside of Λ frozen, and denote its semi-group by T_t^Λ. We also assume from now on that the flip rates are attractive.

If $\eta_0^\Lambda(k)=1$ for all $k\notin\Lambda$ and if $\eta_0^\Lambda(k)\geq\eta(k)$ for all $k\in\Lambda$ then we can couple η_t and η_t^Λ in such a way that $\eta_t^\Lambda\geq\eta_t$ for all $t\geq 0$. Similarly, if $\eta_0^\Lambda(k)=-1$ for all $k \notin \Lambda$ and if $\eta_0^\Lambda(k) \le \eta(k)$ for all $k \in \Lambda$ then we can couple η_t and η_t^Λ in such a way that $\eta_t^\Lambda \le \eta_t$ for all $t \ge 0$. Therefore if f is increasing then for all $\eta \in E$

$$T_t^\Lambda f(-1) \le T_t f(\eta) \le T_t^\Lambda f(+1).$$

Letting t go to infinity in the above inequalities we see that

$$\int f(\sigma)\mu_\Lambda(d\sigma|-1) \le \liminf_{t\to\infty} T_t f(\eta) \le \limsup_{t\to\infty} T_t f(\eta) \le \int f(\sigma)\mu_\Lambda(d\sigma|+1)$$

for all Λ. If $|\mathcal{G}| = 1$ then

$$\lim_{\Lambda \nearrow Z^d} \int f(\sigma)\mu_\Lambda(d\sigma|-1) = \lim_{\Lambda \nearrow Z^d} \int f(\sigma)\mu_\Lambda(d\sigma|+1) = \int f(\sigma)\mu(d\sigma),$$

where $\mathcal{G} = \{\mu\}$. Therefore if $|\mathcal{G}| = 1$, $T_t f(\eta) \mapsto \int f(\sigma)\mu(d\sigma)$ for all $\eta \in E$ and all increasing f. There are a lot of increasing functions. For example define

$$\chi_A(\eta) = \prod_{j\in A} \frac{(1+\eta(j))}{2} = \begin{cases} 1 \text{ if } \eta(j) = 1 \text{ for all } j \in A \\ 0 \text{ otherwise.} \end{cases}$$

All functions in $D(\Lambda)$ are a linear combination of $\{\chi_A : A \in \Lambda\}$. If we combine these observations with the fact that $\mathcal{D}$ is dense in $\mathcal{C}(E)$, we have a proof of the following theorem.

Theorem:. *If the flip rates of the stochastic* Ising *model are attractive and, if* $|\mathcal{G}| = 1$, *then for all* $f \in \mathcal{C}(E)$ *and all* $\eta \in E$

$$\lim_{t\to\infty} T_t f(\eta) = \int f(\sigma)\mu(d\sigma).$$

For general $f \in \mathcal{C}(E)$ the convergence of $T_t f$ to $\int f\,d\mu$ may be arbitrairly slow; however, if we are interested in the rate of convergence of $T_t f$ to $\int f d\mu$ for $f \in \mathcal{D}$, it is enough to study the rate of convergence of $T_t\chi_A$ to $\int \chi_A d\mu$.

Note that if $A \subset Z^d$, is a finite set, then

$$\sum_{j\in A} \chi_{\{j\}} - \chi_A$$

is increasing. Therefore $T_t(\sum_{j\in A} \chi_{\{j\}} - \chi_A)(-1) \le T_t(\sum_{j\in A} \chi_{\{j\}} - \chi_A)(+1)$ or

$$T_t\chi_A(+1) - T_t\chi_A(-1) \le$$

$$\sum_{j\in A} \left(T_t\chi_{\{j\}}(+1) - T_t\chi_{\{j\}}(-1)\right) = |A| \left(T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1)\right).$$

Thus the overall rate of convergence is goverened by the rate that $(T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1))$ goes to zero.

We next give some examples to demonstrate the possible rates of convergence that one might obtain.

For the first example we take $J_{\{j,k\}} \equiv 0$ and $h = 0$ – i.e. no interaction, so that we may take c_k to be constant, say $c_k \equiv 1/2$.

In this case

$$T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1) = e^{-t}.$$

This example is typical of what happens at high temperatures, i.e. small $J_{\{j,k\}}$.

As a second example we take $d = 2$, $h = 0$, and

$$J_{\{j,k\}} = \begin{cases} J < -\frac{1}{2}\log(1+\sqrt{2}) & \text{for } \|j-k\| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case $|\mathcal{G}| > 1$ and $T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1)$ does not go to zero at all.

For our third example we take $d = 2$, $h = 0$, and

$$J_{\{j,k\}} = \begin{cases} -\frac{1}{2}\log(1+\sqrt{2}) & \text{for } |j-k| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_k(\eta) = \eta(k)$ then

$$\begin{aligned} T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1) &= \frac{1}{2}\left(E^{+1}[\eta_t(0)] - E^{-1}[\eta_t(0)]\right) \\ = \|T_t\phi_0\| \geq \|T_t\phi_0\|_{L^2(\mu)} &= \|T_t\phi_0\|_{L^2(\mu)}\|\phi_0\|_{L^2(\mu)} \\ \geq \int T_t\phi_0(\sigma)\phi_0(\sigma)\mu(d\sigma) &= \|T_{t/2}\phi_0\|^2_{L^2(\mu)}. \end{aligned}$$

Let $\vec{n} = ([\alpha t + 1], 0)$. Then

$$\begin{aligned} &\int \phi_0(\sigma)\phi_{\vec{n}}(\sigma)\mu(d\sigma) = \int T_{t/2}(\phi_0(\cdot)\phi_{\vec{n}}(\cdot))(\sigma)\mu(d\sigma) \\ &\leq \int T_{t/2}\phi_0(\sigma)T_{t/2}\phi_{\vec{n}}(\sigma)\mu(d\sigma) + (.61)^{t/2} \\ &\leq \sqrt{\int (T_{t/2}\phi_0(\sigma))^2\mu(d\sigma)\int (T_{t/2}\phi_{\vec{n}}(\sigma))^2\mu(d\sigma)} + (.61)^{t/2} \\ &= \|T_{t/2}\phi_0\|^2_{L^2(\mu)} + (.61)^{t/2} \end{aligned}$$

Thus $T_t\chi_0(+1) - T_t\chi_0(-1) \geq \int \phi_0(\sigma)\phi_{\vec{n}}(\sigma)\mu(d\sigma) - (.61)^{t/2}$.

We need the following fact about the Ising model, see [13].

Fact:. *For the above choice of $J_{\{j,k\}}$ and h,*

$$\lim_{k\to\infty} -\frac{\log(\int \phi_{\vec{0}}(\sigma)\phi_{(k,0)}(\sigma)\mu(d\sigma))}{\log(k)} = \frac{1}{4}$$

$$i.e. \qquad \int \phi_{\vec{0}}(\sigma)\phi_{(k,0)}\mu(d\sigma) \sim k^{-\frac{1}{4}}.$$

Combining this fact with the above inequality we conclude that

$$T_t\chi_0(+1) - T_t\chi_0(-1) \gtrsim \text{const}\,(t^{-\frac{1}{4}}).$$

Our next theorem (see [1]) shows that the above examples are essentially the only possibilities.

Theorem. *For a process with finite range attractive flip rates, if*

$$T_t\chi_0(+1) - T_t\chi_0(-1) = o(t^{-d})$$

then

$$T_t\chi_0(+1) - T_t\chi_0(-1) \to 0 \text{ exponentially fast.}$$

Before proving this theorem we prepare some lemmas.

Lemma. *If μ_1 and μ_2 are measures on E and there is a meausre ν on $E \times E$ such that $\mu_1(A) = \nu(A \times E)$ and $\mu_2(A) = \nu(E \times A)$ and $\nu(\{(\eta,\sigma) : \eta \le \sigma\}) = 1$ (we denote this by $\mu_1 <_s \mu_2$), then for all $f \in \mathcal{C}(E)$*

$$|\int f\,d\mu_1 - \int f\,d\mu_2| \le \sum_k \|\Delta_k f\|(\int \chi_{\{k\}}\,d\mu_2 - \int \chi_{\{k\}}\,d\mu_1).$$

Proof. $|f(\eta) - f(\sigma)| \le \sum_k \|\Delta_k f\||\chi_{\{k\}}(\eta) - \chi_{\{k\}}(\sigma)|$. Therefore

$$\begin{aligned}
|\int f\,d\mu_1 - \int f\,d\mu_2| &= |\int (f(\eta) - f(\sigma))\nu(d\eta, d\sigma)| \\
&\le \int |f(\eta) - f(\sigma)|\nu(d\eta, d\sigma) \\
&\le \int \sum_k \|\Delta_k f\||\chi_{\{k\}}(\eta) - \chi_{\{k\}}(\sigma)|\nu(d\eta, d\sigma) \\
&= \sum_k \|\Delta_k f\| \int (\chi_{\{k\}}(\sigma) - \chi_{\{k\}}(\eta))\nu(d\eta, d\sigma) \\
&= \sum_k \|\Delta_k f\|(\int \chi_{\{k\}}\,d\mu_2 - \int \chi_{\{k\}}\,d\mu_1).
\end{aligned}$$

Now let $\xi_{j,k}(t) = \|\Delta_k T_t \chi_{\{j\}}\|$.

Lemma. *For all $f \in \mathcal{C}(E)$, $\|\Delta_k T_t f\| \le \sum_j \|\Delta_j f\| \xi_{j,k}(t)$.*

Proof. For given η, t let $\mu_{t,\eta}$ be the measure such that $T_t f(\eta) = \int f(\sigma)\mu_{t,\eta}(d\sigma)$ for all $f \in \mathcal{C}(E)$. Since either $\eta \le \eta^k$ or $\eta^k \le \eta$ we have $\mu_{t,\eta} <_s \mu_{t,\eta^k}$ or $\mu_{t,\eta^k} <_s \mu_{t,\eta}$. Suppose that $\eta < \eta^k$. Then $\mu_{t,\eta} <_s \mu_{t,\eta^k}$ and

$$\begin{aligned}
|\Delta_k T_t f(\eta)| &= |\int f\, d\mu_{t,\eta^k} - \int f\, d\mu_{t,\eta}| \\
&\le \sum_j \|\Delta_j f\| (\int \chi_{\{j\}} d\mu_{t,\eta^k} - \int \chi_{\{j\}} d\mu_{t,\eta}) \\
&= \sum_j \|\Delta_j f\| |\Delta_k T_t \chi_{\{j\}}(\eta)|
\end{aligned}$$

Therefore $\|\Delta_k T_t f\| \le \sum_j \|\Delta_j f\| \xi_{j,k}(t)$.

Lemma. *For all $j, k \in Z^d$ and all $s, t \ge 0$,*

$$\xi_{j,k}(s+t) \le \sum_l \xi_{j,l}(s)\xi_{l,k}(t).$$

Proof. Apply the previous lemma to $f(\eta) = T_s \chi_{\{j\}}$.

Proof of the Theorem. Set $\delta_t = \sum_k \xi_{0,k}(t) = |||T_t\chi_{\{0\}}||| < \infty$, and note that from the previous lemma we have $\delta_{t+s} \le \delta_t \delta_s$. Therefore if $\delta_t < 1$ for some t then $\delta_t \to 0$ exponentially fast.

Next note that for $f \in \mathcal{D}$

$$\begin{aligned}
\sup_{\eta,\sigma} |T_t f(\eta) - T_t f(\sigma)| &\le \sum_k \|\Delta_k T_t f\| \\
&\le \sum_k \sum_j \|\Delta_j f\| \xi_{j,k}(t) = |||f|||\delta_t.
\end{aligned}$$

We need a condition that implies that $\delta_t < 1$ for some t. This is where the dimension comes in.

$$\begin{aligned}
\delta_t = \sum_k \xi_{0,k}(t) &\le \sum_{|k|<\alpha t} \xi_{0,k}(t) + (.61)^t \\
&\le (2\alpha t + 1)^d (T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1)) + (.61)^t.
\end{aligned}$$

Therefore if $(T_t\chi_{\{0\}}(+1) - T_t\chi_{\{0\}}(-1)) = o(t^{-d})$ then $\delta_t < 1$ for large t.

§5 Rapid Convergence Implies Rapid Mixing.

In this section we will show the equivalence of rapid convergence of the semi-group to equilibrium and certain mixing properties of the Gibbs states. We begin by investigating what can be said if the semi-group of the stochastic Ising model converges to equilibrium exponentially fast in the L^2 sense. Most of this material can be found in [9].

Theorem. *Suppose that $\mu \in \mathcal{G}$ and that there is an $\epsilon > 0$ such that $\|T_t f - \int f d\mu\|_{L^2(\mu)} \leq e^{-\epsilon t}\|f - \int f d\mu\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$. Then if $f, g \in \mathcal{D}$*

$$|\int (f\theta^\tau g\, d\mu - \int f\, d\mu \int g\, d\mu| \to 0$$

exponentailly fast as $\|\tau\| \to \infty$. Here θ_τ is shift by τ.

Proof. Let $t = \dfrac{\|\tau\|}{2L\alpha}$. Then

$$\begin{aligned}
\int f\theta_\tau g\, d\mu - \int f\, d\mu \int g\, du &= \int T_t(f\theta_\tau g)\, d\mu - \int f\, d\mu \int g\, d\mu \\
&= \int ((T_t f)(T_t(\theta_\tau g)))\, d\mu + O(.61^{\frac{\|\tau\|}{2L\alpha}}) - \int f\, d\mu \int g\, d\mu \\
&= \int \left((T_t f - \int f\, d\mu)(T_t(\theta_\tau g) - \int g\, d\mu)\right) d\mu + O(.61^{\frac{\|\tau\|}{2L\alpha}}) \\
&\leq e^{-\epsilon t}\|f - \int f\, d\mu\| e^{-\epsilon t}\|g - \int g\, d\mu\| + O(.61^{\frac{\|\tau\|}{2L\alpha}}) \\
&= \exp\left(-\frac{\epsilon\|\tau\|}{L\alpha}\right)\|f - \int f\, d\mu\|\|g - \int g\, d\mu\| + O(.61^{\frac{\|\tau\|}{2L\alpha}}).
\end{aligned}$$

Remark: If the gap in the spectrum of $\mathcal{L}$ as an operator on $L^2(\mu)$, call it $gap(2) = \epsilon$, and ϵ is sufficiently small then

$$\int \eta(0)\eta(k)\mu(d\eta) - \left(\int \eta(0)\mu(d\eta)\right)^2 \leq 2\exp\left(-\frac{\epsilon}{L\alpha}\|k\|\right)$$

and thus

$$\Xi(\beta) \equiv \sum_{k \in Z^d} \int \left(\eta(0) - \int \eta(0)\mu(d\eta)\right)\left(\eta(k) - \int \eta(k)\mu(d\eta)\right)\mu(d\eta)$$

$$\lesssim \text{constant}(d)\left(\frac{L\alpha}{\epsilon}\right)^d.$$

This gives us an upper bound on $gap(2)$ in terms of the susceptibility, Ξ. To get a better bound note that $gap(2) = \inf_{f \in L^2(\mu)} -\frac{\int f \mathcal{L} f d\mu}{\text{var}(f)}$. Thus if we let $f_\Lambda(\eta) = \sum_{k \in \Lambda} \eta(k)$, then

$$\begin{aligned} gap(2) \leq & -\frac{\int f_\Lambda \mathcal{L} f_\Lambda d\mu}{\text{var}(f_\Lambda)} = \frac{2 \sum_{k \in \Lambda} \int c_k(\eta)\mu(d\eta)}{\text{var}(f_\Lambda)} \\ = & 2 \frac{|\Lambda| \int c_0(\eta)\mu(d\eta)}{\sum_{k \in \Lambda} \sum_{j \in \Lambda} \int \left(\eta(k) - \int \sigma(k)\mu(d\sigma)\right)\left(\eta(j) - \int \sigma(j)\mu(d\sigma)\right)\mu(d\eta)}. \end{aligned}$$

Letting $\Lambda \nearrow Z^d$ we get

$$gap(2) \leq 2 \frac{\int c_0(\eta)\mu(d\eta)}{\Xi(\beta)}.$$

We mention the next two theorems to help put things in perspective. They are not needed for the proof of the result for which we are aiming. Recall that the rate of convergence of $T_t \chi_{\{0\}}$ to its equilibrium value determines the rate of convergence of the semi-group to equilibrium in the uniform sense. The next theorem shows that the analogous statement is true for convergence in the L^2 sense.

Theorem.

$$gap(2) = \lim_{t \to \infty} -\frac{1}{t} \left[\int \left(\chi_{\{0\}}(\eta) T_t \chi_{\{0\}}(\eta)\right) \mu(d\eta) - \left(\int \chi_{\{0\}}(\eta)\mu(d\eta) \right)^2 \right]$$

For the proof see [8]

The following theorem is the strongest statement that I know how to prove concerning mixing of the Gibbs state as a consequense of a gap in the spectrum in the sense of L^2. See [9] for the proof.

Theorem. *If $gap(2) > 0$ then there is an $\alpha > 0$ such that for all $\phi \in \mathcal{D}$,*

$$\liminf_{n \to \infty} \inf\{-\frac{1}{n} \log(|\int \phi\psi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu|) : ||\psi||_{L^2(\mu)} \leq 1 \text{ and } \psi \text{ is } \mathcal{F}_{\Lambda_n^c} \text{ measurable } \} \geq \alpha.$$

Here $\Lambda_n = \{k \in Z^d : ||k|| \leq n\}$.

The mixing condition in the last theorem almost says that the influence of the boundary on sites deep in the interior of a cube diminishes exponentially fast in the distance of the site from the boundary; however, it is not quite that strong. In order to obtain a mixing condition that strong, we need to replace the assumption of exponentially fast convergence in L^2 with the assumption of exponentially fast convergence in the uniform sense. The following theorem ([7]) gives us the stronger mixing condition under the hypothesis of uniform convergence.

Theorem. *Assume that he proces has attractive flip rates and suppose that there is an $\epsilon > 0$ such that for all $f \in \mathcal{D}$, there is an $A_f < \infty$ such that for all $t \geq 0$, $\|T_t f - \int f\, d\mu\| \leq A_f e^{-\epsilon t}$. Then if $f \in D(\Lambda_0)$ is increasing and $\Lambda \supset \Lambda_0$,*

$$0 \leq \int f(\eta)\mu_\Lambda(d\eta| + 1) - \int f(\eta)\mu(d\eta| - 1)$$
$$\leq 2|||f|||(.61)^{\frac{\operatorname{dist}(\Lambda_0,\Lambda)}{\alpha L}} + 2A_f e^{-\epsilon\frac{\operatorname{dist}(\Lambda_0,\Lambda)}{\alpha L}}.$$

Proof. Let $t = \dfrac{\operatorname{dist}(\Lambda_0, \Lambda)}{\alpha L}$. Then

$$\begin{aligned}
0 &\leq \int f(\eta)\mu_\Lambda(d\eta| + 1) - \int f(\eta)\mu_\Lambda(d\eta| - 1)\\
&= \int f(\eta)\mu_\Lambda(d\eta| + 1) - T_t^\Lambda f(+1) + T_t^\Lambda f(+1) - T_t f(+1)\\
&\quad + T_t f(+1) - T_t f(-1) + T_t f(-1) - T_t^\Lambda f(-1)\\
&\quad + T_t^\Lambda f(-1) - \int f(\eta)\mu_\Lambda(d\eta| - 1)\\
&\leq 0 + |||f|||(.61)^{\frac{\operatorname{dist}(\Lambda_0,\Lambda^c)}{\alpha L}} + 2A_f e^{\epsilon\frac{\operatorname{dist}(\Lambda_0,\Lambda^c)}{\alpha L}} + |||f|||(.61)^{\frac{\operatorname{dist}(\Lambda_0,\Lambda^c)}{\alpha L}} + 0.
\end{aligned}$$

That the mixing condition in the last theorem is sufficient for exponentially fast convergence to equilibrium in the uniform sense is a recent result of F. Martinelli and E. Olivieri ([11]). It completes the program (in the attractive case) of finding conditions on the Gibbs states that are necessary and sufficient to guarantee exponentially fast convergence of the stochastic ISING model to its equilibrium.

Theorem (F. Martinelli and E. Olivieri). *Assume that the process has attractive flip rates. Let $\Lambda_n = \{k \in Z^d : \|k\| \leq n\}$. If there is an $\epsilon > 0$ and a $C < \infty$ such that for all n*

$$\int \chi_{\{0\}}(\sigma)\mu_{\Lambda_n}(d\sigma| + 1) - \int \chi_{\{0\}}(\sigma)\mu_{\Lambda_n}(d\sigma| - 1) \leq Ce^{-\epsilon n},$$

then $T_t\chi_{\{0\}}(1) - T_t\chi_{\{0\}}(-1) \to 0$ exponentially fast as $t \to \infty$.

Again we prepair some lemmas before proving the theorem.

For notaional convenience, define $\rho(t) = T_t\chi_{\{0\}}(1) - T_t\chi_{\{0\}}(-1)$.

Lemma. *Under the hypotheses of the previous Theorem, for all $t \geq 0$ and all $n \in Z^+$,*

$$\rho(2t) \leq 2(2n+1)^d\rho(t)^2 + 2Ce^{-\epsilon n}.$$

Proof.

$$\rho(2t) = T_{2t}\chi_{\{0\}}(1) - \int T_t\chi_{\{0\}}(\eta)\mu(d\eta)$$
$$+ \int T_t\chi_{\{0\}}(\eta)\mu(d\eta) - T_{2t}\chi_{\{0\}}(-1).$$

We show that $T_{2t}\chi_{\{0\}}(1) - \int T_t\chi_{\{0\}}(\eta)\mu(d\eta) \leq (2N+1)^d\rho(t)^2 + Ce^{-\epsilon t}$. A similar proof shows that $\int T_t\chi_{\{0\}}(\eta)\mu(d\eta) - T_{2t}\chi_{\{0\}}(-1) \leq (2N+1)^d\rho(t)^2 + Ce^{-\epsilon t}$.

$$
\begin{aligned}
&T_{2t}\chi_{\{0\}}(1) - \int T_t\chi_{\{0\}}(\eta)\mu(d\eta) = E^1[T_t\chi_{\{0\}}(\eta_t)] - E^\mu[T_t\chi_{\{0\}}(\eta_t)] \\
&= E^{(1,\mu)}[T_t\chi_{\{0\}}(\eta_t^{(1)}) - T_t\chi_{\{0\}}(\eta_t^{(2)})] \\
&= E^{(1,\mu)}[T_t\chi_{\{0\}}(\eta_t^{(1)}) - T_t\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) = \eta_t^{(2)}(j) \text{ for all } j \in \Lambda_n] \\
&+ E^{(1,\mu)}[T_t\chi_{\{0\}}(\eta_t^{(1)}) - T_t\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) > \eta_t^{(2)}(j) \text{ for some } j \in \Lambda_n].
\end{aligned}
$$

Note that $|T_t\chi_{\{0\}}(\eta) - T_t\chi_{\{0\}}(\sigma)| \leq \rho(t)$ for all η, σ. Therefore

$$
\begin{aligned}
E^{(1,\mu)}[T_t\chi_{\{0\}}(\eta_t^{(1)}) - T_t\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) > \eta_t^{(2)}(j) \text{ for some } j \in \Lambda_n] \\
\leq \rho(t)(2n+1)^d\rho(t).
\end{aligned}
$$

Now consider the first term. In what follows we use $T_t^{\Lambda_n,+1}$ to denote the semigroup in which all of the spins outside of Λ_n are frozen at $+1$, and similarly for -1.

We will use the following facts below in the order listed.

1. $\chi_{\{0\}}$ is an increasing function, and by a coupling argument we see that $T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta) \geq T_t\chi_{\{0\}}(\eta)$. A similar inequality, with the sense reversed, holds if we raplace $+1$ with -1.
2. $T_t^{\Lambda_n,+1}\chi_{\{0\}}(\cdot)$ is in $D(\Lambda_n)$.
3. By the F.K.G. inequality, $\mu <_s \mu_{\Lambda_n}(\cdot|+1)$ and $\mu_{\Lambda_n}(\cdot|-1) <_s \mu$.
4. $\mu_{\Lambda_n}(\cdot|+1)$ is stationary for $T_t^{\Lambda_n,+1}$, and $\mu_{\Lambda_n}(\cdot|-1)$ is stationary for $T_t^{\Lambda_n,-1}$.

Applying these obersvations we now compute as follows:

$$
\begin{aligned}
&E^{(1,\mu)}[T_t\chi_{\{0\}}(\eta_t^{(1)}) - T_t\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) = \eta_t^{(2)}(j) \text{ for all } j \in \Lambda_n] \\
&\leq E^{(1,\mu)}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(1)}) - T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) = \eta_t^{(2)}(j) \text{ for all } j \in \Lambda_n] \\
&= E^{(1,\mu)}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(2)}) - T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)}), \eta_t^{(1)}(j) = \eta_t^{(2)}(j) \text{ for all } j \in \Lambda_n] \\
&\leq E^{(1,\mu)}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(2)}) - T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)})] \\
&= E^{\mu}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(2)}) - T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)})] \\
&= E^{\mu}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(2)})] - E^{\mu}[T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)})] \\
&\leq E^{\mu_{\Lambda_n}(\cdot|+1)}[T_t^{\Lambda_n,+1}\chi_{\{0\}}(\eta_t^{(2)})] - E^{\mu_{\Lambda_n}(\cdot|-1)}[T_t^{\Lambda_n,-1}\chi_{\{0\}}(\eta_t^{(2)})] \\
&= \int \chi_{\{0\}}(\eta)\mu_{\Lambda_n}(d\eta|+1) - \int \chi_{\{0\}}(\eta)\mu_{\Lambda_n}(d\eta|-1) \leq Ce^{-\epsilon n}.
\end{aligned}
$$

By using the F.K.G. inequality it is easy to see that the hypotheses of the Martinelli–Olivieri Theorem imply that there is only one Gibbs state, and hence that $\rho(t) \to 0$ and $t \to \infty$. Thus the previous lemma tells us that under the hypotheses of the Martinelli–Olivieri Theorem, the hypotheses of the next two lemmas are satisfied.

Lemma. *If $\rho(2t) \le 2(2n+1)^d \rho(t)^2 + 2Ce^{-\epsilon n}$ for all $t \ge 0$ and all $n \in Z^+$, and if $\rho(t) \searrow 0$ as $t \to \infty$ then for all large t*

$$\rho(2t) \le \rho(t)^{\frac{3}{2}}.$$

Proof. Assume that t is large enough that $\rho(t) \le 1$ and set $n(t) = [-\frac{2}{\epsilon}\log(\rho(t)]$. Then

$$\begin{aligned}\rho(2t) \le& 2\left(2\frac{2}{\epsilon}\log(\frac{1}{\rho(t)}) + 1\right)^d \rho(t)^2 + 2Ce^{\epsilon(\frac{2}{\epsilon}\log(\rho(t)) + 1)} \\ =& 2\left(\left(2\frac{2}{\epsilon}\log(\frac{1}{\rho(t)}) + 1\right)^d \sqrt{\rho(t)} + Ce^{\epsilon}\sqrt{\rho(t)}\right)\rho(t)^{\frac{3}{2}} \\ <& \rho(t)^{\frac{3}{2}}\end{aligned}$$

if $\rho(t)$ is small enough (i.e. t is large enough).

Lemma:. *If $\rho(t) \searrow 0$ and $\rho(2t) \le \rho(t)^{\frac{3}{2}}$ for all large t then there is a $C < \infty$ such that*

$$\rho(t) \le e^{-C(\frac{3}{2})^{\log_2(t)}} = e^{-Ct^{\log_2(\frac{3}{2})}} = o\left(\frac{1}{t^d}\right).$$

This last lemma is just a messy calculus exercise that we leave to the reader.

In view of the theorem concerning the possible rates of convergence of $\rho(t)$ to 0, we see that the Martinelli–Olivieri Theorem follows immediately from our last lemma.

References

1. Aizenman, M. and Holley, R., *Percolation Theory and Ergodic Theory of Infinite Particle Systems*, IMA Vols in Math. and its Appl. ed by Harry Kesten, vol. 8, Springer Verlag, 1987, pp. 1-11.
2. Dobrushin, R.L., Pyatetski–Shapiro, I.I., and Vasilev, N.B., *Markov Processes in an Infinite Product of Discrete Spaces*, Soviet–Japanese Symposium in Probability Theory, Khavarovsk (1969).
3. Fortuin, C.M., Kasteleyn, P.W., and Ginibre, J., *who knows*, Commun. Math. Phys. **22** (1971), 89–103.
4. Glauber, R.J., *Time Dependent Statistics of the Ising Model*, J. Math. Phys. **4** (1963), 294–307.
5. Holley, R., *Markovian Interaction Processes with Finite Range Interactions*, Ann. Math. Stat. **43** (1972), 1961–1967.
6. Holley, R., *Remarks on the FKG Inequalities*, Commun. Math. Phys. **36** (1974), 227–231.
7. Holley, R., *Possible Rates of Convergence in Finite Range, Attractive Spin Systems*, Contemporary Mathematics **41** (1985), 215–234.
8. Holley, R., *Spatial Stochastic Processes*, Progress in Probability ed by Kenneth S. Alexander and Joseph C. Watkins, vol. 19, Birkhauser, 1991, pp. 89–104.
9. Holley, R. and Stroock, D.W., L_2 *Theory for the Stochastic Ising Model*, A. Wahr. Verw. Geb. **35** (1976), 87–101.
10. Liggett, T.M., *Interacting Particle Systems*, Springer Verlag, 1985.

11. Martinelli, F. and Olivieri, E., *Approach to Equilibrium of Glauber Dynamics in the One Phase Region. I: The Attractive Case*, preprint (1993).
12. Metropolis, N., Rosenbluth, A., Rosenbluth, M., Teller, A., and Teller, E., *Equations of State by Fast Computing Machines*, J.Chem. Phys. **21** (1953), 1087–1091.
13. McCoy, B.M. and Wu, T.T., *The Two-Dimensional Ising Model*, Harvard University Press, Cambridge, Massachusets, 1973.
14. Spitzer F., *Interaction of Markov Processes*, Adv. in Math. **5** (1970), 246–290.

Richard Holley
Department of Mathematics
University of Colorado
Campus Box 426
Boulder - CO 80309
U S A
e-mail : holley@euclid.colorado.edu