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Random Perturbations of Dynamical Systems: Large Deviations and Averaging

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Abstract: A rewiew of the theory of random perturbations of dynamical systems is presented in this paper.Limit theorems for large deviations is an important tool in problems concerning the long time behavior of the perturbed system.But for some important classes of dynamical systems ,for example,for Hamiltonian systems,such an approach does not works. A new approach based on a developement of the averaging principle has been suggested.It turns out that for the white noise type perturbations the slow component of the perturbed motion converges,under some assum- ptions,to a diffusion process on a graph corresponding to the first integral of the nonperturbed system.Perturbations of the Hamiltonian systems in the plane and of area-preserving systems on a torus are considered.The slow component of the perturbed system converges to a jumping process on the graph in the case of impuls-like perturbations.

Key words: Large deviations, Averaging principle, Random perturbations.

Introdution

This paper is a kind of short review of solved and some unsolved problems concerning random perturbations of dynamical systems. It is written in a free style. We give here no proofs and provide just the references if available, comments, and sometimes explanations of our statements. Most attention is paid to demonstration of various effects typical for the problems under consideration.

From the probabilistic point of view, problems considered here can be roughly speaking related to the laws os large numbers, or to results of the central limit theorem type, or to the limit theorems for large deviations. It is useful to keep in mind the connections with these classical topics of probability theory.

Many problems mentioned in this paper are closely connected with asymptotic problems for partial differential equations (without randomness). A probabilistic approach turns out to be very productive for those problems.

Some results, such as, for example, large deviation theory for random perturbations of finite dimensional systems, are well known, and we mention them only to consider their generalizations. Other results like the perturbations of Hamiltonian systems are rather new, and we pay them more attention.

AMS classification: 60HXX, 58G32, 60G60, 60F17

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I would like to note, finally, that the bibliography does not pretend to cover all the papers concerning this topic.

§1. Quasideterministic approximation

By a dynamical system in $I\!R^r$ we understand a system of ordinary differential equations

$$\dot{X}_t = b(X_t), \quad X_0 = x \in I\!R^r. \tag{1.1}$$

We assume that the vector field $b(x)$ is smooth enough and the derivatives are bounded.

There are many ways to introduce random perturbations in (1.1). But to some extent various forms of noise lead to similar mathematical problems and need similar mathematical tools.

The most popular, and often most natural from the physical point of view, form of random perturbations is the additive white noise:

$$\dot{X}_t^\epsilon = b(X_t^\epsilon) + \sqrt{\epsilon}\dot{W}_t, \; X_\epsilon^0 = x \in I\!R^r. \tag{1.2}$$

Here $b(x)$ is the same as in (1.1), W_t is the Wiener process in $I\!R^r$, ϵ is a small positive parameter. As is well knowm, equation (1.2) defines a diffusion Markov process in $I\!R^r$. The differential operator

$$L^\epsilon = \frac{\epsilon}{2}\Delta + \sum_{i=1}^r b^i(x)\frac{\partial}{\partial x^i}, \; b(x) = (b^1(x), \ldots, b^r(x)),$$

is closely connected with this process. In particular, the solutions of the boundary problems connected with the operator L^ϵ and with the parabolic operator $\frac{\partial}{\partial t} - L^\epsilon$ can be written as the expectations of proper functionals of the process X_t^ϵ (see, for example, [F2]).

Of course, one can consider perturbations of the form

$$\dot{X}_t^\epsilon = b(X_t^\epsilon) + \sqrt{\epsilon}\,\sigma(X_t^\epsilon)\dot{W}_t, \; X_\epsilon^0 = x \in I\!R^r, \tag{1.3}$$

where $\sigma(x)$ is a matrix. The corresponding differential operator will be equal to

$$L^\epsilon = \frac{1}{2}\sum_{i,j=1}^r a^{ij}(x)\frac{\partial^2}{\partial X^i \partial X^j} + \sum_{i=1}^r b^i(x)\frac{\partial}{\partial X^i}, \; (a^{ij}(x)) = \sigma(x)\sigma^*(x).$$

One more form of white-noise-type perturbations: Let $\nu_t, t \geq 0$, be a continuous time Markov chain in the phase space $\{1, 2, \ldots, n\}$ with the transition intensities $c_{ij}, 1 \leq i, j \leq n, \; i \neq j$. Set

$$\dot{X}_t^\epsilon = b(X_t^\epsilon) + \sqrt{\epsilon}\sigma_{\nu_t}(X_t^\epsilon)\dot{W}_t, \ X_0^\epsilon = x \in I\!R^r. \tag{1.4}$$

Here $\sigma_1(X), \ldots, \sigma_n(X)$ are some matrices. The pair (X_t^ϵ, ν_t) forms a Markov process in $I\!R^r \times \{1, \ldots, n\}$. The infinitesimal operator A of this process on a function $f(x,k)$ smooth in x is equal to

$$Hf(x,k) = \frac{1}{2}\sum_{i,j}^{r} a_k^{ij}(x)\frac{\partial^2 f(x,k)}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x)\frac{\partial f(x,k)}{\partial x^i} + \sum_{j=1}^{n} c_{kj}(f(x,j) - f(x,k)).$$

One can consider perturbations of a different kind, which are small only in the mean sense:

$$\dot{X}_t^\epsilon = b(X_t^\epsilon, \xi_{t/\epsilon}), \ X_0^\epsilon = x \in I\!R^r. \tag{1.5}$$

Here ξ_t is a stationary process with regular enough trajetories and with some ergodic properties. Suppose that $Eb(x, \xi_t) = b(x)$. Then one can prove that the process X_t^ϵ uniformly in any finite time interval $[0, T]$ converge in probability as $\epsilon \downarrow 0$ to the solution of system (1.1) ([Kh.1], [F-W.1]). It means that the process defined by (1.5) can be considered as a random perturbation of (1.1). The solutions of (1.2) and (1.4), of course, also converge uniformly on $[0, T]$ to solution of (1.1). This convergence can be looked on as a result of the law of large numbers type.

One can consider the deviations of X_t^ϵ from X_t. In the case of (1.2), if $b(x)$ is smooth enough

$$X_t^\epsilon = X_t + \sqrt{\epsilon}X_t^{(1)} + \epsilon X_t^{(2)} + \ldots \tag{1.6}$$

The process $X_t^{(1)}$ is a Gaussian Markov (non-homogeneous in time) process. Similar expansions can be written for (1.3) and (1.4) (in the last case for fixed trajectory ν_t). For the equation (1.5) it is impossible to write down such an expansion, but one can prove a central-limit-theorem-type result: Under certain assumptions concerning the mixing properties of the process ξ_t the normalized difference $\eta_t^\epsilon = \epsilon^{-1/2}(X_t^\epsilon - X_t)$ converges weakly in C_{0T} to a Markov Gaussian process ([Kh1],[FW1]).

One more old central limit-theorem-type result worth mentioning ([Kh2], [B]): Let

$$Eb(x, \xi_t) = 0. \tag{1.7}$$

Then, under some assumptions concerning the mixing properties of ξ_t, the process $\tilde{X}_t^\epsilon = X_{t/\epsilon}^\epsilon$($X_t^\epsilon$ is the solution of (1.5)) converge weakly in C_{0T} to a diffusion process. I will consider a natural generalization of this very special result later.

All the results mentioned above, besides the last one, relate to the behavior of the perturbed process on finite time interval. But many problems arising in applications concern the behavior of the perturbed system on infinite or growing together with ϵ^{-1} time intervals.

The first approximation of the long time behavior of the perturbed system is given by the (rough) theory of large deviations. This theory is, in a sense, similar to the quasiclassic approximation in the quantum mechanics. It turns out that many characteristics of the long time behavior of the system, though they are determined by the random noise, are not random. Therefore that approximation one can call quasideterministic.

A functional $S_{0T}(\varphi), \varphi \in C_{0T}$, is called the action functional for the family of process $X_t^\epsilon, 0 \le t \le T$, as $\epsilon \downarrow 0$, if

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \epsilon \ln I\!P\Big\{ \max_{0 \le t \le 1} |X_t^\epsilon - \varphi_t| < \delta \Big\} = S_{0T}(\varphi) \quad \text{(see [FW1])}.$$

For example, in the case of processes defined by (1.2)

$$S_{0T}(\varphi) = \frac{1}{2} \int_0^T |\dot\varphi_S - b(\varphi_S)|^2 ds$$

for absolutely continuous $\varphi_S \in C_{0T}$, and $S_{0T}(\varphi) = \infty$ for the rest of C_{0T} [FW1].

In the case of (1.5) assume that the following limit exists

$$\lim \frac{1}{T} \ln E \exp\Big\{ \alpha \int_0^T b(x, \xi_S) ds \Big\} = H(x, \alpha), \ \alpha \in I\!R^r. \tag{1.8}$$

Then one can check that $H(x, \alpha)$ is convex in α, and under certain assumptions the action functional for family X_t^ϵ defined by (1.5) is equal to

$$S_{0T}(\varphi) = \int_0^T L(\varphi_S, \dot\varphi_S) ds$$

for absolutely continuous $\varphi \in C_{0T}$, and $S_{0T}(\varphi) = \infty$ for the rest of C_{0T}. Here $L(x, \beta)$ is the Legender transformation of $H(x, \alpha) : L(x, \beta) = \sup_\alpha(\alpha\beta - H(x, \alpha))$ (see [F2], [FW1]). If, for example, ξ_t is a Feller Markov process on a compact phase space $\mathcal{E}$ with the generator A, then limit (1.8) exists, and $H(x, \alpha)$ is equal to the first eigenvalue of the problem $A\varphi(y) + \alpha b(x, y)\varphi(y) = \lambda\varphi(y), \ y \in \mathcal{E}$. I will assume that the process ξ_t in (1.5) is such a process.

The action functional for (1.4) is calculated in [EF].

Denote

$$V(x,y) = \inf_{x,y\in I\!R^r} \{S_{0T}(\varphi),\ \varphi_0 = x, \varphi_T = y, T > 0\},$$

in the case (1.2) and (1.3), (1.4) if the matrix $\sigma(x)\sigma^*(x)$ is not degenerate, $0 \le V(x,y) < \infty$. For the process defined by (1.5) $V(x,y)$ can be equal to $+\infty$. This function in the generic case contains the main information concerning the quasideterministic approximation of the long time behavior of the perturbed system.

A typical example of long time behavior problems is given by the exit problem. Consider a domain G in $I\!R^r$ (Fig. 1), and denote $\tau^\epsilon = \inf\{t : X_t^\epsilon \notin G\}$ the first time when the trajectory X_t^ϵ exits domain G. We assume that $X_0^\epsilon = x \in G$.

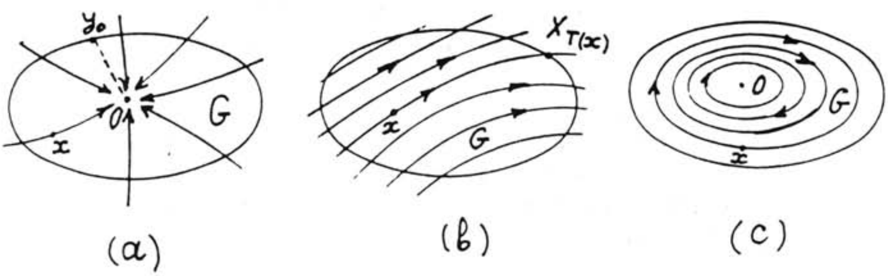

Fig. 1

If trajectories of the non-perturbed problem behave as in Fig. 1(b) (i.e., leave G in a finite time $T = T(x) < \infty$), then $\tau^\epsilon \to T(x)$ as $\epsilon \downarrow 0$ in P_x- probability (P_x means distribution in the space of trajectories starting at $x \in I\!R^r$). In this case exit of X_t^ϵ from the domain occurs due to the non-perturbed system in a finite time, and this event is not related to the long time behavior.

In the case (a) and (c) the deterministic trajectories do not exit the domain G. In these cases the exit of X_t^ϵ from G occurs due to the perturbations and $\tau^\epsilon \to \infty$ as $\epsilon \downarrow 0$.

Let us consider case (a) and introduce the quasipotential $U(y)$ of the field $b(x)$ with respect to the rest point 0 (for given perturbation):

$$U(y) := V(0,y),\ y \in I\!R^r.$$

I will explain later why $U(y)$ is called quasipotential. Let $U_0 = U(y_0) = \min_{y\in\partial G} U(y)$, and assume that y_0 is the only point of ∂G where $U(y) = U_0$.

Then under mild additional assumptions we have in the case (a):

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_x\Big\{|X^\epsilon_{\tau^\epsilon} - y_0| > \delta\Big\} = 0, \ \forall \delta > 0, x \in G. \tag{1.9}$$

$$\lim_{\epsilon \downarrow 0} \epsilon \ln E_x \tau^\epsilon = U_0, x \in G. \tag{1.10}$$

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_x\Big\{\exp\{\frac{U_0 - h}{\epsilon}\} < \tau^\epsilon < \exp\{\frac{U_0 + h}{\epsilon}\}\Big\} = 1, \ \forall h > 0. \tag{1.11}$$

So, in the generic case first exit from G occurs near not random point $y_0 \in \partial G$, and the logarithmic asymptotic of the exit time is also not random. The equalities (1.9)-(1.11) were established in [WF], [FW1].

If the set $Y_0 = \{y \in \partial G : U(y) = U_0 = \min_{y \in \partial G} U(y)\}$ consists of more than one point the situation becomes more delicate: a limit distribution of $X^\epsilon_{\tau^\epsilon}$ as $\epsilon \downarrow 0$ on Y_0 exists. One can find a number of results concerning this case on [GF], [D1], [D2].

The exit of X^ϵ_t from the domain G occurs, actually, after many returns from the periphery of G to a small neighborhood os the attracting point,and each excursion takes relatively little time. Therefore the following refinement of (1.11) can be proved [GOV], [CGOV]:

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_x\Big\{\frac{\tau^\epsilon}{E\tau^\epsilon} > t\Big\} = e^{-t}, x \in G, t \geq 0.$$

Let us consider now a dynamical system in $\mathbb{R}^r$ with many attracting sets. We say that two points $x, y \in \mathbb{R}^r$ are equivalent ($x \sim y$) if

$$V(x, y) = V(y, x) = 0.$$

For example, all the points of a periodic trajectory of system (1.1) are equivalent, since $S_{0T}(\varphi) = 0$ if $\varphi_S, 0 \leq S \leq T$, is a piece of a non-perturbed trajectory. But some points belonging to different trajectories can be also equivalent: for instance, all points in the neighborhood of the point 0 in Fig. 1(c) are equivalent.

Suppose the dynamical system (1.1) satisfies the following condition

(Condition A):

There are a finite number of compacts $K_1, K_2, \dots, K_l$ such that

1) any two points x, y belonging to the same compact are equivalent;

2) if $x \in K_i, y \notin K_i$, then $x \not\sim y$;

3) every $\omega-$limit set of system (1.1) (it means the set of limit points of X_t as $t \to \infty$) belongs to one of K_i.

Denote

$$V_{ij} = V(x, y),\ x \in K_i,\ y \in K_j$$

the value $V(x, y)$ independent of the choice of $X \in K_i$ and $y \in K_j$.

The matrix (V_{ij}) contains important information about quasideterministic approximation. Define $j(i)$ by the conditions

$$V_{ij(i)} = \min_{k:k \neq i} V_{ik}.$$

In the generic case this equality defines $j(i)$ in a unique way. Consider the sequence

$$i, j(i), j(j(i)), j(j(j(i))), \ldots.$$

At some point the numbers start to repeat and we observe a cycle. This cycle can cover all states $K_1, \ldots, K_l$. But, in general, it covers only part of them. The other states belong to different cycles. For example, in Fig. 2 we have $j(1) = 2, j(2) = 3, j(3) = 1, j(4) = 5, j(5) = 6, j(6) = 7, j(7) = 5, j(8) = 9, j(9) = 8$.

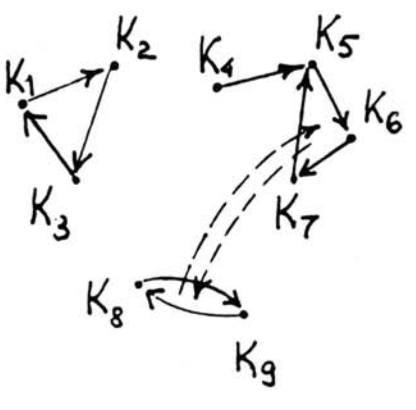

Fig. 2

Note, that the states $K_1, \ldots, K_9$ in Fig. 2 are not necessarily points in $\mathbb{R}^r$. Some of them can present compacts like periodic solutions, invariant torus, etc. It follows from the large deviation estimates that trajectories starting in the domain of attraction of K_i after first exit from this domain come to the domain of attraction of $K_{j(i)}$ with probability tending to 1 as $\epsilon \downarrow 0$. From $K_{j(i)}$ the trajectory will go to a neighborhood of $K_{j(j(i))}$ and so on according to the sequence

$i, j(i), j(j(i)), \dots$. The time $\tau^{\epsilon}_{i,j(i)}$ of the transition from K_i to a neighborhood of $K_{j(i)}$ also can be described by $V_{i,j(i)}$:

$$\lim_{\delta\downarrow 0} \epsilon \ln E_x \tau^{\epsilon}_{i,j(i)} = V_{i,j(i)}.$$

So we have stratification of all states in cycles of the first order. In Fig. 1 it is : $K_1K_2K_3, K_4, K_5K_6K_7, K_8K_9$.

The logarithmic asymptotics of the time or rotation in one cycle is defined by the matrix V_{ij}. In the times bigger than characteristic rotation time for given cycle a transition between the first rank cycles occurs. These transitions form a cycle of the second rank. The second order cycles, as well as their characteristic times, are also defined by (V_{ij}). Then the cycles third order appear etc. until all K_i will be involved. In each cycle in the case of general position one can find the main state, such that the trajectory X^{ϵ}_t spends most of the time until it leaves the cycle in the domain of attraction of this main state.

The explicit construction of the hierarchy of the cycles and the asymptotic expressions for the characteristic times through the numbers V_{ij} are given in [F1] (see also [FW1]). In particular, the notion of sublimiting distribution were introduced in this paper. If $t = t(\epsilon^{-1})$ is a growing function of ϵ^{-1} then the limiting distribution of $X^{\epsilon}_{t(\epsilon^{-1})}$ as $\epsilon \downarrow 0$ will be in general different for different functions $t(\epsilon^{-1})$. For slowly growing functions $t(\epsilon^{-1})$ this distribution is concentrated near the attractor $K_{i(x)}$ of the initial point x; then it tends to the point $j(i(x))$ if $V_{j(i(x)),j(j(i(x)))} > V_{i(x),j(i(x))}$. For $t(\epsilon^{-1})$ growing faster the limiting distribution concentrated near the main state of the first rank cycle containing the point $i(x)$. Then it is concentrated near the main state of the second order cycle containing $i(x)$ and so on until the main state of the all system will appear. All this construction is governed by the matrix (V_{ij}). Some refinements of the asymptotic behavior of the transition times (for the case of two stable equilibriums) is given in [CGOV], [GOV].

The main state of the system (under given perturbations) will be the compact K_{i_0} where the invariant measure of the process X^{ϵ}_t will be concentrated as $\epsilon \downarrow 0$.

Dynamical system (1.1) has, in general, many invariant measures: for instance, on each compact K_i at least one (normalized) invariant measure is concentrated. The process X^{ϵ}_t under mild additional conditions has only one normalized invariant measure μ^{ϵ}. It is an old question: what is the limit μ^{ϵ} as $\epsilon \downarrow 0$ [B], [K]. The theory of large deviations allows to calculate this limit for a wide class of dynamical systems. To formulate the result I will remind the notion of i-graph [FW]. Let us have a finite set $\mathcal{L} = (1, 2, \dots, l)$. A system of arrows leading from $n \in \mathcal{L}$ to $m \in \mathcal{L}$ is called i-graph if

1. exactly one arrow starts at any point $n \in \mathcal{L}\backslash\{i\}$;

2. from any $n \in \mathcal{L}\backslash\{i\}$ along the arrows one can come to the point i;

3. there are no loops in the system of arrows.

Let us denote G_i the set of all i-graphs in $\mathcal{L}$.

Let $i_0 \in \{1, \ldots, l\}$ be such that

$$\min_i \min_{g \in G_i} \sum_{(m \to n) \in g} V_{mn}$$

is achieved only for $i = i_0$. Then it is proved in [WF],[FW1], that for any neighborhood $\mathcal{E}(K_{i_0})$ or the compact K_{i_0}

$$\lim_{\epsilon \downarrow 0} \mu^\epsilon(I\!R^r \backslash \mathcal{E}(K_{i_0})) = 0.$$

It means that if the non-perturbed system has exactly one normalized, invariant measure μ_{i_0} concentrated on K_{i_0} then the invariant measure μ^ϵ of the process X_t^ϵ converges to μ_{i_0}.

One can consider also the exit problem in the case when the dynamical system in G has many attractors (see [FW1], Ch.6).

As we have seen, the function $V(x, y)$ is the most important characteristic of the quasideterministic approximation. This function is defined as the solution of a variational problem for the action functional. As it often happens some geometry in the phase space is closely connected with the variational problem. The perturbation define a scalar product (or a metric) in the phase space (or, in more general situation, in the space of functions on the phase space).

For example, in the case of pure white noise perturbations (1.2) for $b(x) \equiv 0$

$$\inf(S_{0T}(\varphi) : \varphi \in C_{0T}, \varphi_0 = x, \varphi_t = y) = \frac{\rho^2(x, y)}{2t}, \tag{1.12}$$

where $\rho(.,.)$ is the Euclidian metric in $I\!R^r$. In the case (1.3) the metric $\rho(.,.)$ in (1.12) is the Riemannian metric corresponding to the form $\sum_{i,j=1}^{r} a_{ij}(x) dx^i dx^j$, where $(a_{ij}(x)) = [\sigma(x)\sigma^\star(x)]^{-1}$. In the case (1.4) (as before $b(x) \equiv 0$) $\rho(.,.)$ is a Finsler metric in $I\!R^r$. This metric is defined by the family of unit balls at each point $x \in I\!R^r$. The unit ball at point $x \in I\!R^r$ defined as the convex envelope of the Riemannian unit balls corresponding to the metrics

$$\sum_{i,j} a_{k,ij}(x) dy^i dy^j, \ k = 1, \ldots, n, \ \text{(see Fig. 3)}.$$

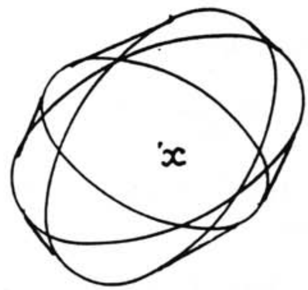

Fig. 3

Here $a_k(x) = (a_{k,ij}) = (\sigma_k(x)\sigma_k^*(x))^{-1}$. The well know S.R.S. Varadhan result [V] follows from the large deviation principle, equality, and some density bounds: If $p^\epsilon(t,x,y)$ is the transition density for the process in $I\!R^r$ corresponding to the operator

$$L^\epsilon = \frac{\epsilon}{2} \sum_{i,j=1}^{r} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j}.$$

Then

$$-\lim_{\epsilon \downarrow 0} 2t \ln p^\epsilon(t,x,y) = \rho^2(x,y), \tag{1.13}$$

where $\rho(.,.)$ is the Riemannian metric corresponding to the form

$$\sum_{i,j=1}^{2} a_{ij(x)} dx^i dx^j, \ (a_{ij}(x)) = (\sigma(x)\sigma^*(x))^{-1}.$$

In the case (1.4) one can consider transition density $p^\epsilon(t;x,l;y,k)$ for the Markov process (X_t^ϵ, ν_t) corresponding to the system

$$\begin{cases} \frac{\partial u_k^r}{\partial^t} = \frac{\epsilon}{2} \sum\limits_{i,j=1}^{r} a_k^{ij}(x) \frac{\partial^2 u_k}{\partial x^i \partial x^j} + \sum\limits_{j=1}^{n} c_{kj}(u_j^\epsilon - u_k^\epsilon) \\ x \in I\!R^r, t > 0, \ k = 1, \ldots, n; \end{cases}$$

$p^\epsilon(t;x,l;y,k)dy$ is the probability that the process (X_t^ϵ, ν_t) starting at (x,l) will be in the set (dy,k) at time t. A relation similar to (1.13) holds for $p^\epsilon(t,x,l;y,k)$:

$$-\lim_{\epsilon\downarrow 0} 2t \ln p(t; x, l; y, k) = \rho^2(x, y),$$

where $\rho(x, y)$ is the Finsler metric in $\mathbb{R}^r$ mentioned above.

Consider now problems concerning the long time behavior of the perturbed system. We restrict ourselves to the case (1.2) of pure white noise perturbations. Suppose the field $b(x)$ has the representation

$$b(x) = -\nabla U(x) + l(x), \tag{1.14}$$

and $\nabla U(x)$ is orthogonal to the field $l(x)$. We assume that $U(x)$ is a smooth function (say, of the class c^3) and $l(x)$ is a smooth vector field. We call $-\nabla U(x)$ the potential part of $b(x)$, and $l(x)$ is called the rotating part. Of course, one can replace $U(x)$ in (1.14) by $U(x)$ + const. But the representation (1.14) has more deep non-uniqueness: for example, the field $l(x)$ can itself have a potential part that is orthogonal to $\nabla U(x)$. We will impose some additional assumptions that make the representation (1.14) unique at least locally.

Set 0 be an asymptotically stable equilibrium point of the field $b(x)$ (see Fig. 1a), and let $b(x)$ be directed inside G for all $x \in \partial G : b(x)n(x) > 0$, where $n(x)$ is the interior normal to ∂G. Assume that $U(0) = 0$ and $U(x) > 0$ for $x \in G \cup \partial G$. Let $\nabla U(x) \neq 0$ for $x \neq 0, x \in G \cup \partial G$. Then one can check (see [FW1], Ch. 6) that

$$V(0, y) = V(y) = 4U(y) \tag{1.15}$$

at least for $y \in G \cap \{y : U(y) < \min_{z \in \partial G} U(z)\}$. If the field $l(x)$ in (1.14) identically equal to zero the field $b(x)$ is potential: $b(x) = -\nabla U$. In this case (1.15) shows that the quasipotential $V(y)$ up to the factor 4 coincides with the potential. It is why $V(y)$ is called quasipotential. The uniqueness of the representation (1.14) with listed above assumption on $U(x)$ (at least locally) follows from (1.15). The existence of the representation (1.14), at least in some generalized sense, also follows from (1.15): one can check that if $V(y)$ is smooth then $U(x) = \frac{1}{4}V(x)$, $l(x) = b(x) + \frac{1}{4}\nabla V(x)$ give the representation (1.14) in a neighborhood of the point 0. In general the function $V(x)$ is only Lipschitz continuous. When (1.14) is true the numbers V_{ij} also can be expressed through the values of $U(x)$ at its critical points.

Some importants characteristics of the perturbed system in the case of potential vector field $b(x) = -\nabla U(x)$ can be written explicity. For example, the invariant measure μ^ϵ of the process X_t^ϵ defined by (1.2) in this case has a density $M^\epsilon(x)$:

$$M^\epsilon(x) = C \exp\left\{\frac{2U(x)}{\epsilon}\right\}. \tag{1.16}$$

The constant C is the norm-factor:

$$C^{-1} = \int_{\mathbb{R}^r} \exp\left\{\frac{-2U(x)}{\epsilon}\right\} dx.$$

The finiteness of the integral in the right side of the last equality is the condition of existence of the finite invariant measure. Using the explicit representation for the density one can easily check that as $\epsilon \downarrow 0$ the $\lim_{\epsilon \downarrow 0} \mu^\epsilon$ concentrated at the point(s) of absolute minimum of the potential.

I would like now to mention the limit theorems for not very large deviations. Let, for example, the origin 0 be an asymptotically stable equilibrium point for (1.1). Suppose we are interested in the exit problem from a neighborhood G^ϵ of the origin and $G^\epsilon = \epsilon^\alpha G$, where α is a positive parameter and G is a bounded domain in $\mathbb{R}^r, 0 \in G$. Consider, say, system (1.5). Then if $\alpha > \frac{1}{2}$ the exit time τ_α^ϵ tends to zero as $\epsilon \downarrow 0$ (some mild assumptions of non-degeneration should be made). If $\alpha = \frac{1}{2}$ the exit time $\tau_{1/2}^\epsilon$ is on order 1 and its characteristics can be calculated using the central-limit-theorem approximation mentioned above. If $\alpha = 0$ we have large deviations of order 1 considered earlier. If $\alpha \in (0, \frac{1}{2})$ we have case not very large deviations. The asymptotic behavior of the probabilities of such deviations will be the same as for approximating Gaussian process (see [FW1], Ch.7).

Finally, I would like to mention shortly some large deviation problems connected with the perturbations of infinite dimensional semiflows. Consider the following system of reaction-diffusion equations (RDE):

$$\frac{\partial u_k(t,x)}{\partial t} = D_k \Delta u_k + f_k(x, u_1, \ldots, u_n),\ t > 0,\ x \in G \subseteq \mathbb{R}^r, \tag{1.15}$$

$$u_k(0,x) = g_k(x); k = 1, \ldots, n.$$

If $G \neq \mathbb{R}^r$ some boundary conditions should be added to (1.15). For example, it can be the Neumann conditions

$$\left.\frac{\partial u_k(t,x)}{\partial n}\right|_{x \in \partial G} = 0, \tag{1.16}$$

or the Dirichlet conditions. Under mild assumptions on $f_k(x,u), g_k(x)$ there exists a unique solution of the problem (3.1)-(3.2) for all $t > 0$. This solution defines a semiflow $U_t g(.) = u(t,.), u(t,.) = (u_1(t,.), \ldots, u_n(t,.)),\ t \geq 0$ in the space of continuous functions. There are a number of interesting phenomena in the phase behavior of the semiflows corresponding to RDEs, for example, propagation of wave fronts and other structures, existence of stable equilibrium points with less symmetry than exists in the boundary problem.

An interesting problem - and one that is important for applications - is to study small random perturbations of these semiflows and semiflows defined by some other evolutionary partial differential equations.

One should say that in the case of PDEs there are, roughly speaking, more natural ways to introduce perturbations: one can consider perturbations of the equations, of the initial conditions, of boundary conditions, and perturbations of the domain where the initial-boundary problem is considered.

Consider, first, the white noise type perturbations of the equations:

$$\frac{\partial u_k^\epsilon(t,x)}{\partial t} = D_k \Delta u_k^\epsilon + f_k(x,u) + \sqrt{\epsilon}\,\frac{\partial^{r+1} W_k(t,x)}{\partial t \partial x^1 \dots \partial x^r}$$

$$t>0, x \in G, \left.\frac{\partial u_k^\epsilon}{\partial n}\right|_{\partial G} = 0,\ u_k^\epsilon(0,x) = g_k(x); k = 1,\dots,n.$$

Let $W_k(t,x), t > 0, x \in I\!R^r, k = 1,\dots,n$, be the independent Brownian sheets. This means that they are mean zero Gaussian random fields with correlation function $EW_k(s,x)W_k(t,y) = (s \wedge t)\prod_1^r (x^i \wedge y^i)$. The mixed derivative $\frac{\partial^{r+1} W_k(t,x)}{\partial t \partial x^1 \dots \partial x^r}$ is the natural counterpart of the classical white noise $\dot{W}_t$: it is the generalized mean zero Gaussian field with the correlation $\delta(t-s)\delta(x-y)$. The "statistical simplicity" of this field allows us to expect that one can have relatively explicit expressions for the characteristics of the perturbed semiflow. If $x \in G \subset I\!R^1$ it turns out this is the case: For example, if a function $F(x,u)$ exists such that $f_k(x,u) = -\frac{\partial F(x,u)}{\partial u_k,} k = 1,\dots,n$, one can introduce the potential

$$U[\varphi] = \frac{1}{2}\int_G \Big[\sum_{k=1}^n D_k \big\{\frac{d\varphi_k(x)}{dx}\big\}^2 + 2F(x,\varphi)\Big]dx. \tag{1.17}$$

It means that

$$-\frac{\delta U(\varphi)}{\delta \varphi_k} = D_k \Delta \varphi_k + f_k(x;\varphi_1,\dots,\varphi_n), k = 1,\dots,n.$$

One can write down a counterpart of formula (1.17) for the density of the invariant measure μ^ϵ of the perturbed semiflow with respect to an auxiliary Gaussian measure. The measure μ^ϵ concentrated near the points where the potential has its absolute minimum as $\epsilon \downarrow 0$. The hierarchy of the cycles, the logarithmic asymptotics of exit times and other characteristics of quasideterministic approximation also can be expressed through the potential $U[\varphi]$. The non-potential case for one dimensional space variable also is considered. Those results can be found in [FJ-L], [F4], [DPZ].

But if $G \subset I\!R^r, r > 1$, one cannot add the white noise $\sqrt{\epsilon}\ \frac{\partial^{r+1} W(t,x)}{\partial t \partial x^1 \dots \partial x^r}$ to equation (1.15): the perturbed equations, in general, have no solutions. The mean zero Gaussian field with correlation function $\epsilon\delta(t-s)B(x-y)$ can be considered as a perturbation. If $B(z)$ is smooth enough the perturbed equations have a solution and this solution defines a Markov process U_t^ϵ in the space of continuous functions. One can calculate the action functional for the family U_t^ϵ and to develop a theory similar to finite dimensional case. But, of course, we will not have such nice explicit expressions for the characteristics of the quasideterministic approximation as in the case of pure-white-noise perturbations. This makes quasideterministic approximation for the process U_t^ϵ less interesting. An interesting problem here, from my point fo view, is to consider perturbations of (1.15) by the mean zero Gaussian field with the correlation function $\epsilon_1\delta(t-s)B\{\frac{x-y}{\epsilon_2}\}, 0 < \epsilon_1, \epsilon_2 \ll 1$. The perturbed equations are solvable if $B(z)$ is smooth enough. And if $|B(z)| \to 0$ fast enough as $|z| \to \infty$, we can expected that the quasideterministic approximation for the solution of the perturbed equations as both parameters ϵ_1 and ϵ_2 tend to zero can be described in a more simple way. In particular, for the potential field $f(x,u) = -\nabla_u F(x,y)$ one can introduce the potential similar to (1.17)

$$U(\varphi) = A\int_G \Big[\sum_{k=1}^n D_k|\nabla\varphi_k(x)|^2 \ + 2F(x,\varphi(x))\Big]dx,$$

the constante A is defined by $B(z)$.

One can consider other types of perturbations: fast oscillating in time noise [Pa] impulse-like perturbations. Homogenization problem [Ko], [PV] for evolutionary equations also can be considered from point of view of perturbations of the limiting semiflows. One should mention also more specific problems connected with wave propagation in random media. Besides rigorous mathematical results there exists an extensive physical literature devoted to this topic.

Random perturbations of the boundary conditions considered in [FW2], [FS].

In particular, the followning problem was studied in [FW]:

$$\frac{\partial u^\epsilon(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 u^\epsilon}{\partial x^2} + f(x,u^\epsilon), t > 0, |x| < 1,$$

$$u^\epsilon(0,x) = g(x), \frac{\partial u^\epsilon(t,x)}{\partial x}\Big|_{x=\pm 1} = \pm\xi(t/\epsilon).$$

Here $(\xi_+(t), \xi_-(t))$ is a stationary mean zero stochastic process satisfying some mixing proprerties. Then $u^\epsilon(t,x)$ defines a stochastic process in the functional space that converges as $\epsilon \downarrow 0$ to the semiflow corresponding to the problem

$$\frac{\partial u(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2} + f(x,u),\ t > 0, |x| < 1, u(0,x) = g(x), \frac{\partial u}{\partial x}(t,\pm 1) = 0.$$

Normal and large deviations of $u^\epsilon(t,x)$ from $u(t,x)$ are studies in [FW2]. The normalized difference $v^\epsilon(t,x) = \epsilon^{-1/2}(u^\epsilon(t,x) - u(t,x))$ under some assumptions concerning the mixing rate of the noise $(\xi_+(t), \xi_-(t))$ converges as $\epsilon \downarrow 0$ to the solution $v(t, x)$ of the problem

$$\frac{\partial v(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial f(x,u)}{\partial u}\Big|_{v,u=u(t,x)} \ t > 0, |x| < 1. \qquad (1.18)$$

$$v(0,x) = 0, \frac{\partial v}{\partial x}(t, \pm 1) = \dot{W}_\pm(t).$$

where $(W_+(t), W_-(t))$ is the Brownian motion with a covariance matrix expressed through the covariance matrix for the noise. Problem (1.18) is a partial stochastic differential equation with the noise in the boundary conditions. The general theory of such equations is not developed yet. The solution of problem (1.18) is a Gaussian random field, smooth inside the domain and equal to a generalized function on the boundary. Because of this one cannot consider weak convergence in the space of continuous functions. It turns out that "the right spaces" where v^ϵ converges weakly to v have a norm depending on the rate of mixing for the boundary noise. The same functional spaces should be considered when the moderate large deviations $u^\epsilon(t, x)$ from $u(t, x)$ are studied.

There are a number of interesting problems concerning perturbations of the domains. For example, the domain, where the boundary problem is considered, can have many randomly distributed small "holes" with some conditions on their boundaries. If the size of the holes tends to zero and the number of holes increases, the problem in the perforated domain can be replaced, under some conditions, by a problem in the domain without holes for non-random "effective" equation [PVZ], [Du]. These results can be looked on as a law-of-large-number-type results. The central-limit-theorem type results and large deviations theorems for this setting is also of interest.

Another class of problem related to the perturbations of the domains concerns perturbations of the boundaries.

§2 Dynamical systems with conservation laws

The large deviation theory gives satisfactory answers to many long time behavior problems. If, roughly speaking, the dynamical system has strong enough attractors, or, in other words, when the potential component of the vector field is strong enough. The rigorous sense of these assumptions was given in condition A: Not too many points of the phase space should be equivalent.

But there are important classes of dynamical systems where all points of the phase space are equivalent (say, with respect to white noise perturbations).

One says that the dynamical system

$$\dot{X}_t = b(X_t),\ X_0 = x \in \mathbb{R}^r, \tag{2.1}$$

has a first integral $H(x)$ if $H(X_t) = H(x) = const.$ for all t.

The first integral $H(x)$ is not necessarily a smooth function. If $H(x)$ is smooth then it is a first integral for system (2.1) if and only if

$$\nabla H(x) b(x) = 0, x \in \mathbb{R}^r.$$

Now, let us restrict ourselves to dynamical systems in the plane $\mathbb{R}^2$. We will make some remarks concerning the general case later.

Assume that system (2.1) on $\mathbb{R}^2$ has a smooth first integral $H(x)$ and let $\nabla H(x) = 0$ only if x is an equilibrium point of the field $b(x)$. Consider together with $\nabla H(x)$ the vector field

$$\overline{\nabla} H(x) = \left\{ \frac{\partial H(x)}{\partial x^2}, -\frac{\partial H(x)}{\partial x^1} \right\}, x = (x^1, x^2) \in \mathbb{R}^2.$$

The vectors $\nabla H(x)$ and $\overline{\nabla} H(x)$ form an orthogonal coordinate system at x if $b(x) \neq 0$, and

$$b(x) = \alpha(x) \nabla H(x) + \beta(x) \overline{\nabla} H(x),$$

where $\alpha(x), \beta(x)$ are some scalars. Since $H(x)$ is a smooth first integral for (2.1), $b(x) \nabla H(x) = 0$, and therefore

$$b(x) = \beta(x) \overline{\nabla} H(x). \tag{2.2}$$

Consider, first, the case of Hamiltonian systems $\beta(x) \equiv 1$;

$$\dot{X}_t = \overline{\nabla} H(X_t),\ X_0 = x \in \mathbb{R}^2. \tag{2.3}$$

We assume that $H(x) \to \infty$ as $|x| \to \infty$, that $H(x)$ has a finite number of non-degenerate critical points and $\min_{x \in \mathbb{R}^2} H(x) = 0$. then all level sets

$$C(y) = \{x \in \mathbb{R}^2 : H(x) = y\}, y \geq 0,$$

are compact. Each $C(y)$ consists of a finite number $n(y)$ of connected components $C_1(y) : C(y) = \cup_{i=1}^{n(y)} C_i(y)$. If y is not a critical value of $H(x)$ then each component $C_i(y)$ us a periodic trajectory of system (2.2).

For brevity let $H(x)$ be a generic function: all its critical points are non-degenerate and $C(y)$ contains at most 1 critical point. If $C(y)$ contains a critical point x_0, then $C(y)$ besides the trajectory $X_t \equiv x_0$ can contain two more trajectories having x_0 as their limit as $t \to \pm\infty$. It is easy to check that all points of the phase space for such a system are equivalent (with respect to the white noise perturbations).

The simplest example is given by the harmonic-oscillator-type Hamiltonian, when $H(x)$ has only one critical point: a minimum point, let us say, at the origin (Fig. 4).

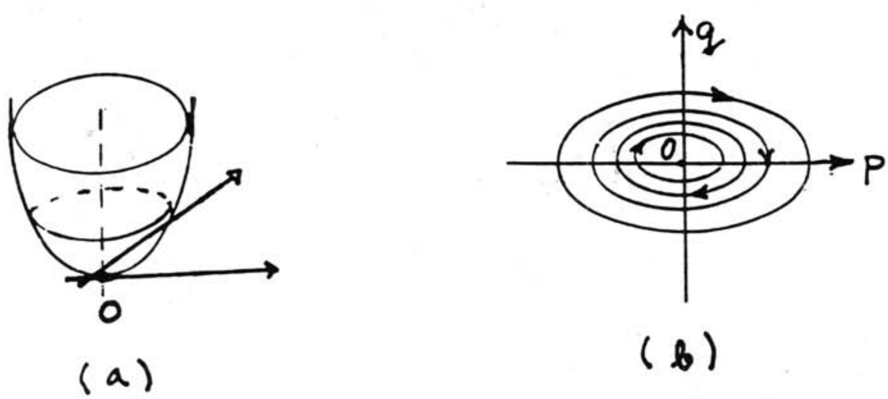

Fig. 4

The correspondent phase picture is given in Fig. 4(b): each level set consists of one periodic trajectory. Note that since $|\nabla H(x)| = |\overline{\nabla} H(x)|$, the normalized invariant density $M_y(x)$ on each periodic trajectory $C(y)$ has the form:

$$M_y(x) = \frac{1}{\int\limits_{C(y)} \frac{dl}{|\nabla H(x)|}} \frac{1}{|\nabla H(x)|}, \ x \in C(y),$$

dl is the length element on $C(y)$.

Let us consider now the case when $H(x)$ has more than one critical point (Fig. 5)

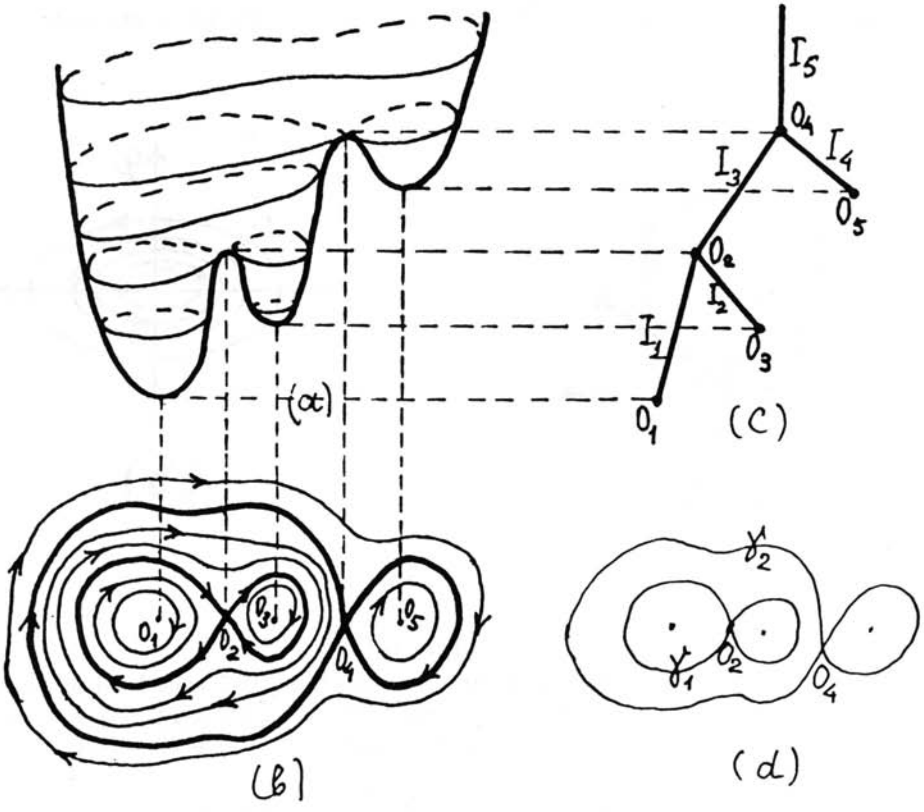

Fig. 5

Then the set of trajectories consists of several families of periodic orbits divided by the separatrises. For example in Fig. 5(b) there are five families: rotations around O_1, around O_3 and around O_5, rotations around O_1, O_2, O_3, and periodic orbits around all five critical points. These families are separeted by two ∞−shaped curves: γ_1 with the crossing point at O_2 and γ_2 with the crossing point at O_4.

An important feature of the system with Hamiltonian having many critical points is the appearance of a new first integral independent of $H(x)$. This integral $\hat{H}(x)$ is the number of the family of the periodic orbits: if X^x. denotes the orbit containing the point $x \in I\!R^2$, then

$$\hat{H}(x) = \begin{cases} 1, & \text{if } X^x. \text{ belongs to the left loop of } \gamma_1; \\ 2, & \text{if } X^x. \text{ belongs to the right loop of } \gamma_1; \\ 3, & \text{if } X^x. \text{ belongs to the right loop of } \gamma_2; \\ 4, & \text{if } X^x. \text{ belongs to the left loop of } \gamma_2, \text{and } \gamma_1 \text{ is inside } X^x.; \\ 5, & \text{if } \gamma_2 \text{ is located inside } X^x.. \end{cases}$$

Consider now the white noise perturbations of the system (2.3):

$$\dot{\tilde{X}}_t^\epsilon = \overline{\nabla} H(\tilde{X}_t^\epsilon) + \sqrt{\epsilon}\, \dot{W}_t.$$

It is more convenient to rescale time: Let $X_t^\epsilon = \tilde{X}_{t/\epsilon}^\epsilon$. Then we have for X_t^ϵ the following equation

$$\dot{X}_t^\epsilon = \frac{1}{\epsilon} \overline{\nabla} H(X_t^\epsilon) + \dot{W}_t. \tag{2.4}$$

Let us consider, first, the case of $H(x)$ with one critical point (Fig. 4). The motion X_t^ϵ consits of two components: the fast rotation according to the non-perturbed dynamics and the motion with a speed of order 1 (as $\epsilon \downarrow 0$) in the tranversal direction. The fast rotation for $\epsilon \ll 1$ can be characterized by the invariant density $M_y(x)$ on the orbit $C(y)$. To describe the slow component for $\epsilon \ll 1$ one can use the averaging principle:

Applying the Ito formula, we have

$$\begin{aligned} H(X_t^\epsilon) - H(x) = & \frac{1}{\epsilon} \int_0^t \nabla H(X_s^\epsilon)\, \overline{\nabla} H(X_s^\epsilon) ds + \\ & + \int_0^t \nabla H(X_s^\epsilon) dW_s + \frac{1}{2} \int_0^t \Delta H(X_s^\epsilon) ds \end{aligned} \tag{2.5}$$

The first term in the eight hand side of (2.4), actually, is equal to zero since $\nabla H(x)\, \overline{\nabla} H(x) \equiv 0$. Now, before the slow component moves on a small but fixed distance δ the fast component makes as $0 < \epsilon \ll 1$ many (of order ϵ^{-1}) rotations along the deterministic orbit. Because of this the "diffusion" and "drift" coefficients in (2.5) should be averaged with respect to the density $M_y(x)$. Therefore one can prove that the process $H(X_t^\epsilon) = Y_t^\epsilon$, $0 \le t \le T$, converges weakly in C_{0t} as $\epsilon \downarrow 0$ to a one-dimensional diffusion process Y_y. The diffusion coefficient $\sigma^2(y)$ and the drift $B(y)$ of the limiting process are given by averaging:

$$\sigma^2(y) = \frac{1}{\int\limits_{C(y)} \frac{dl}{|\nabla H(x)|}} \int\limits_{C(y)} |\nabla H(x)| dl,$$

$$B(y) = \frac{1}{2 \int\limits_{C(y)} \frac{dl}{|\nabla H(x)|}} \int\limits_{C(y)} \frac{\Delta H(x)}{|\nabla H(x)|} dl. \tag{2.6}$$

The limiting process Y_t is defined by the equation

$$dY_t = \sigma(Y_t) d\tilde{W}_t + B(Y_t)dt, \; Y_0 = H(x),$$

on set $\{y \geq 0\}$. The point $y = 0$ is inaccessible for the process Y_t.

Let us now turn to the case when the Hamiltonian $H(x)$ has more than one critical point. In this case $C(y)$ consist, at least for some y, of several components $C_1(y), C_2(y), \ldots, C_{n(y)}(y)$. One should average the coefficients of (2.5) not over the whole level set but only over the connected component containing the initial point. The behavior of the process $H(X_t^\epsilon)$ before a time t_0 can help us to identify the connected component at time t_0. Therefore one cannot expect the $H(X_t^\epsilon)$ converges in this case to a Markov process. To have in the limit a Markov process, we have to extend the phase space: to remember not only the level set (or $H(X_t^\epsilon)$) but the connected component of this level set $(H(X_t^\epsilon))$ where the non-perturbed system has mixing.

The set of connected components of the Hamiltonian $H(x)$ provided with the natural topology is homeomorphic to a graph Γ. For example, in Fig. 5 each minimum point of $H(x)$ corresponds to an exterior vertex O_1, O_3 or O_5 of the graph. The saddle points together with the ∞−shaped curve correspond to the interior vertices O_2, O_4. The points of open edges $I_1, \ldots, I_5$ correspond to the periodic orbits. Say I_1 counts all orbits around O_1 up to the energy level $H(O_2)$, and I_2 corresponds to the rotations around O_3 up to the energy $H(O_2)$. The points of I_3 correspond to the orbits in the region where $H(x) = 4$.

If some critical point of $H(x)$ degenerates, more than 3 edges can meet at the vertex of the graph corresponding to that point. For systems in $\mathbb{R}^2$ the corresponding graph always has the structure for a tree. For Hamiltonians on other manifolds, say on a torus, the graph can have loops.

Denote by Y the mapping of the set of connected components of the level sets of $H(x)$ to the graph $\Gamma : Y(C_i(y))$ is the point of I_i corresponding to $C_i(y)$. One can consider the value of $H(x)$ as a coordinate in I_i, so that $Y(C_i(y))$ is the pair (y, i), which characterizes a point of the graph.

One can consider the mapping Y from $\mathbb{R}^2$ to $\Gamma : Y(x), x \in \mathbb{R}^2$, is defined as the point of Γ corresponding to the connected component of the level set $C(H(x))$ containing the point x.

Consider the family of stochastic process on the graph Γ

$$Y_t^\epsilon = Y(X_t^\epsilon),\ t \geq 0.$$

It turns out the process Y_t^ϵ, $0 \leq t \leq T$, converge weakly in the space of continuous functions with the value in the graph Γ as $\epsilon \downarrow 0$ to a continuous Markov process on Γ.

It is worth mentioning that there exists a number of "classical" asymptotic problems where the limiting process has values in a graph. One can find some of such problems in [FW]. In the end of this section we will mention some other problems.

It is important for all these problems to have a description of continuous Markov process on graphs.

Consider a graph $\Gamma = \{I_1, \ldots, I_m; O_1, \ldots, O_l\}$. Let $L_1, \ldots, L_m$ be elliptic second order differential operators.

$$L_i = \frac{\sigma_i^2(x)}{2}\ \frac{d^2}{dy^2} + B_i(y)\frac{d}{dy}, y \in I_i.$$

We assume that the coefficients are, say, Lipschitz continuous and bounded, $\sigma_i(y) \geq \sigma > 0$. Then a diffusion process $X_t^{(i)}$ in I_i corresponds to $L_i, i = 1, \ldots, m$. The process $X_t^{(i)}$ is defined up to the first exit from interval I_i. How can one describe continuous Markov processes on Γ coinciding with the processes $X_t^{(i)}$ inside the edges? We should define behavior of the process after reaching the vertices.

Here the situation is similar to the well know problem considered by Feller, [Fe1], [Fe2]: Describe all possible continuations of a continuous Markov process on an open interval to a process on the closed interval preserving the continuity and the Markov property. The most convenient way to describe all such continuations is to describe the domain of definition of the infinitesimal operator of the extended Markov process. If the process inside the interval I was governed by operator $L = \frac{1}{2}a(x)\frac{d2}{dx^2} + b(x)\frac{d}{dx}, a(x) > 0$, then each possible continuation is defined by boundary conditions in the ends of the interval. For example, if the process in the closed interval has instantaneous reflection in boundary then the corresponding boundary conditions are $\left.\frac{df(x)}{dx}\right|_{x\in\partial I} = 0$. This means that the infinitesimal operator A of the extended process is defined for smooth $f(x), x \in I$, such that $\left.\frac{df(x)}{dx}\right|_{x\in\partial I} = 0$, and inside the interval $Af(x) = Lf(x)$. Feller described all boundary conditions corresponding to Markovian continuations of the process inside the interval I.

In our case the boundary conditions should be replaced by some gluing conditions at the vertices. Any smooth in $\Gamma\backslash\{O_1, \ldots, O_l\}$ function satisfying these conditions should belong to the domain of the generator of the extended process.

We will write $I_k \sim O_i$ to say that the edge I_k has O_i as one its ends. For any set of constants $\alpha_i, \beta_{ij} \geq 0, j \in \{k : I_k \sim O_i\}, i = 1, \dots, l,\ \alpha_i + \sum_{j:I_j \sim O_i} \beta_{ij} \neq 0$ there exists a unique continuous Markov process Y_t on Γ such that its generator A defined for continuous functions $f(y), y \in \Gamma$, satisfying the conditions:

1. $f(y)$ is twice continuously differentiable inside the edges $I_1, \dots, I_m$;
2. If $I_i \sim O_k$ then $\lim_{y \to O_k} L_i f(y)$ exists and is independent of i; we denote that limit $Lf(O_k)$;
3. $\alpha_k Lf(O_k) + \sum_{j:I_j \sim O_k} \beta_{kj} \frac{df}{dy_j}(O_k) = 0, i = 1, \dots, l$;

   Here y_j is the coordinate on I_j such that $y_j = 0$ for the point O_k and $y_j > 0$ inside I_j.
   If $f(y)$ satisfies these conditions, then $Af(y) = L_i f(y)$ for $y \in I_i$.

Moreover, for any conditions Markov process on the graph coinciding with process $X_t^{(i)}, i = l, \dots, m$, inside the edges one can find constants $\alpha_i, \beta_{ij} \geq 0,\ \sum_{j:I_j \sim O_i} \beta_{ij} + \alpha_i \neq 0$, such that its generator A is defined for functions $f(y), y \in \Gamma$, satisfying conditions 1-3 and $Af(y) = L_i f(y)$ for $y \in I_i$.

If our graph consists of one segment, this statement coincides with Feller's result. This was proved for processes on graphs in [FW3],[FW4].

The coefficients α_k, β_{ki} characterize the behavior of the process at O_k. For example if $\beta_{ki} = 0, i \in \{j : I_j \sim O_k\}$, then the process stops at O_k.. The coefficients β_{ki}, roughly speaking, characterize the probabilities of going to I_k from O_k. If $\alpha_k \neq 0$ then the trajectory spends a positive time at the point O_k.

We assumed that the operators L_i that govern the process inside the edges are non-degenerate. This condition is fulfilled in a number of asymptotic problems where the limiting process is a Markov process on a graph [FW]. However, we have to consider degenerate processes if we study the white noise perturbations of Hamiltonian system in $\mathbb{R}^2$.

To describe the process Y_t on the graph Γ limiting for the family $Y_t^\epsilon = Y(X_t^\epsilon)$ as $\epsilon \downarrow 0$, we should calculate the operators L_i for each edge $I_i \subset \Gamma$ and the gluing conditions at the vertices. The calculations of the operators L_i, actually, are similar to the case of Hamiltonians with one critical point (see formulas (2.6)). The only difference is that in the general case the level set $C(y)$ is a sum of several connected components: $C(y) = \cup_{i=1}^{n(y)} C_i(y)$. Now the averaging should be carried out only over corresponding component $C_i(y)$. Therefore the operator $L_i = \frac{\sigma_i^2(y)}{2} \frac{d^2}{2} + B_i(y)\frac{d}{dy}$ governing the limiting process inside I_i has coefficients

$$\sigma_i^2(y) = \frac{1}{\int\limits_{C_i(y)} \frac{dl}{|\nabla H(x)|}} \int\limits_{C_i(y)} |\nabla H(y)| dl,$$

$$B_i(y) = \frac{1}{2 \int\limits_{C_i(y)} \frac{dl}{|\nabla H(x)|}} \cdot \int\limits_{C_i(y)} \frac{\Delta H(y)}{|\nabla H(y)|} dl. \tag{2.7}$$

One can see from formulas (2.7) that the diffusion coefficients $\sigma_i^2(y)$ degenerate at the vertices $O_k, k = 1, \ldots, l$. Simple calculations show that the order of degeneration of the diffusion coefficients at the vertices corresponding to the extremums of the Hamiltonian exterior-vertices and the signs of the drift coefficients at these points are such that the exterior vertices are inaccessible for the limiting process on graph. It means that no additional conditions should be imposed at the points.

The situation is different at the interior vertices corresponding to the saddle points of the Hamiltonian. Although the diffusion is degenerate at these points, the degeneration is slow enough. All such points are accessible and gluing conditions should be imposed at these vertices.

To formulate the gluing conditions at an interior vertex O_k, consider the ∞–shaped curve γ_k corresponding to the saddle point O_k. This curve consists of two loops γ_k^1 and γ_k^2. Let the edge I_k^1 correspond to the orbits located inside γ_k^1, I_k^2 correspond to the orbits located inside γ_k^2 and I_k^3 correspond to the orbits containing γ_k inside themselves (Fig. 6).

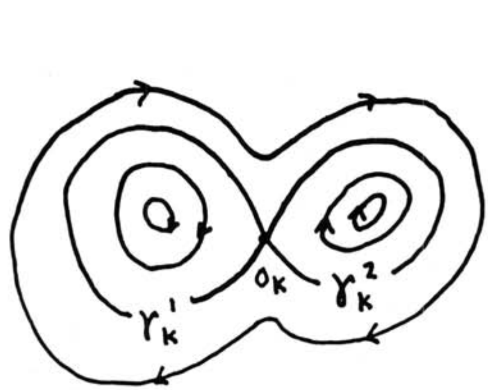

Fig. 6

Denote

$$\beta_{ki} = \int_{\gamma_k^i} |\nabla H(x)|\, dl, i = 1, 2, \ \beta_{k3} = \beta_{k1} + \beta_{k2}.$$

Then the gluing conditions at the point O_k have the form: $\alpha_k \equiv 0$ (it means that the trajectory has no delay at the vertices), and

$$\beta_{k1} \frac{df}{dy_1}(O_k) + \beta_{k2} \frac{df_2}{dy_2}(O_k) + \beta_{k3} \frac{df}{dy_3}(O_k) = 0. \tag{2.8}$$

The operators $L_i, i = 1, \dots, m$ defined by (2.7) and gluing conditions (2.8) at the interior vertices define the limiting process in a unique way. These results were proved in [FW4].

How can one prove the convergence of the process Y_t^ϵ and calculate the gluing conditions (2.8) for the limiting process?

First, one should check the tightness of the process $Y_t^\epsilon, 0 \le t \le T$, in the weak topology, then calculate the operators $L_i, i = 1, \dots, m$, using the mentioned above averaging procedure. The next step is to prove that the limiting process is Markovian. Now, since we have a description of all continuous Markov process on the graph Γ, we should find the gluing conditions. To do this one can use the fact that uniform distribution is the invariant measure for X_t^ϵ in $\mathbb{R}^2$ for any $\epsilon > 0$. Using this fact it is simple to calculate the invariant measure for the process $Y_t^\epsilon = Y(X_t^\epsilon)$ on Γ. This measure is "the projection" of the uniform distribution in $\mathbb{R}^2$ on the graph Γ. It is independent of ϵ. Now one should choose the constants β_{ij} at any vertex O_i so that the process on Γ with given operators L_i and given gluing conditions has the prescribed invariant density. A plan close to this one (but slightly different) was realized in [FW4].

The result is similar if we consider a more general class of perturbations:

$$\dot{\tilde{X}}_t^\epsilon = \overline{\nabla} H(\tilde{X}_t^\epsilon) + \sqrt{\epsilon}\sigma(\tilde{X}_t^\epsilon)\dot{W}_t + \epsilon b(\tilde{X}_t^\epsilon), X_0 = x \in \mathbb{R}^2,$$

where $\sigma(x)\sigma^*(x)$ is a nondegenerate matrix.

So, the evolution of the energy under white-noise-type perturbations can be described in a proper time scale as a diffusion process on the graph corresponding to the Hamiltonian H. The limiting process has no delay at the vertices. In general, it is not necessary that the evolution of the first integrals has no delay at the vertices of the graph; some vertices can correspond to a set where the non-perturbed trajectory spends a positive time.

Consider, for example, a Hamiltonian system on the two-dimensional torus. It has the following behavior in the case of general position ([A],[KhS]): There exists a finite number of loops like those shown in Fig. 7a; inside such loops the system may have other equilibrium points and behave like a system in a region of $\mathbb{R}^2$. The trajectories outside the loops have an ergodic behavior (Fig. 7b): each of them is dense outside the loops. Therefore, if we

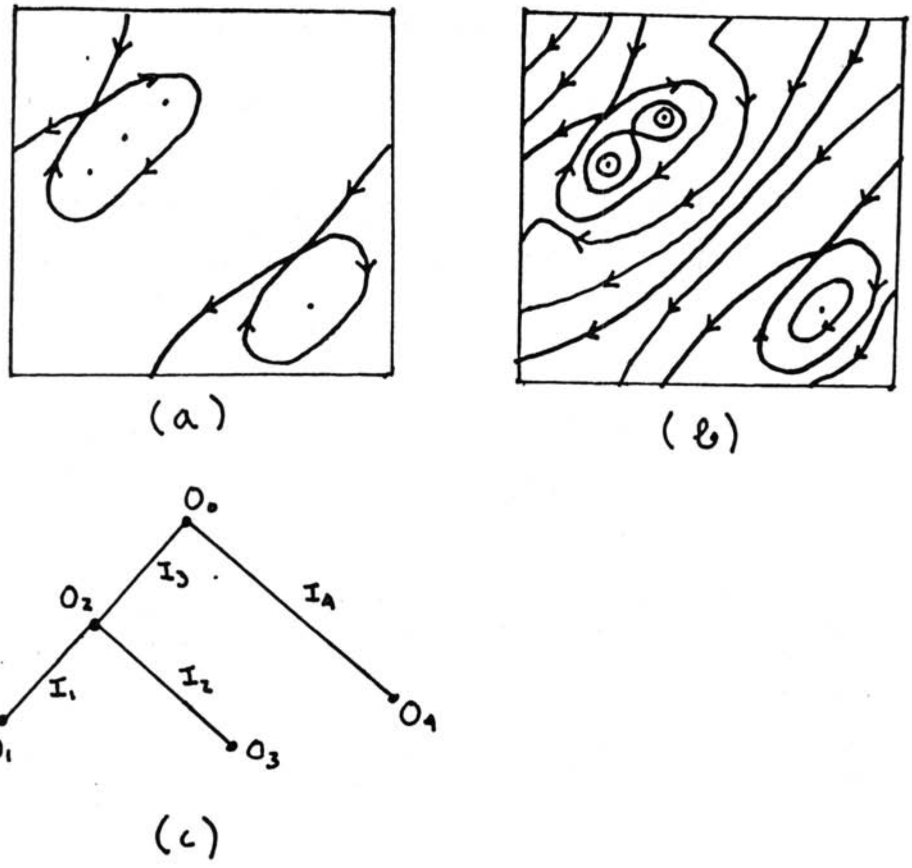

Fig. 7

Consider white noise perturbations of such a system, all trajectories outside the loops should be glued in one point (point O_0 in Fig. 7c). The segments I_1, I_2, I_3, I_4 correspond to different families of periodic orbits I_4 corresponds to the orbits inside γ_2; I_3 counts the orbits inside but outside the $\infty-$shaped curve inside γ_1; I_1 and I_2 correspond to the periodic trajectories inside the two parts of the $\infty-$ shaped curve.

The vertices O_1, O_3, O_4 correspond to the stable equilibrium points inside the loops γ_1 and γ_2. The vertex O_2 corresponds to the $\infty-$shaped curve.

Using the averaging principle one can calculate the operators $L_i, i = 1, \ldots, 4$, governing the limiting process inside I_i.

The gluing conditions at the vertex O_2 are calculated in the same way as above; the exterior vertices O_1, O_3, O_4 are inaccessible and no condition should be imposed there. But at the vertex O_0 the situation is different from what we had before: this point corresponds to the shadowed area in Fig. 7(a). This area has a positive measure and therefore the limiting process will spend at O_0 a positive time.

Thus the gluing conditions at O_0 will have a positive coefficient α_0. The coefficients α_0, β_{0i} of the gluing conditions at O_0 can be again calculated using the fact that the Euclidian area is the invariant measure for the process X_t^ϵ for all ϵ.

Up to now we considered system (2.2) with $\beta(x) \equiv 1$. If $\beta(x) \not\equiv 1$ but preserves the sign the results will be more or less similar. But if $\beta(x)$ changes the sign we will have a number of new effects.

Consider the case when the first integral $H(x)$ has only one minimum (Fig. 4). Supose that $\beta(x)$ is negative inside the loop γ =ABCDEFA in

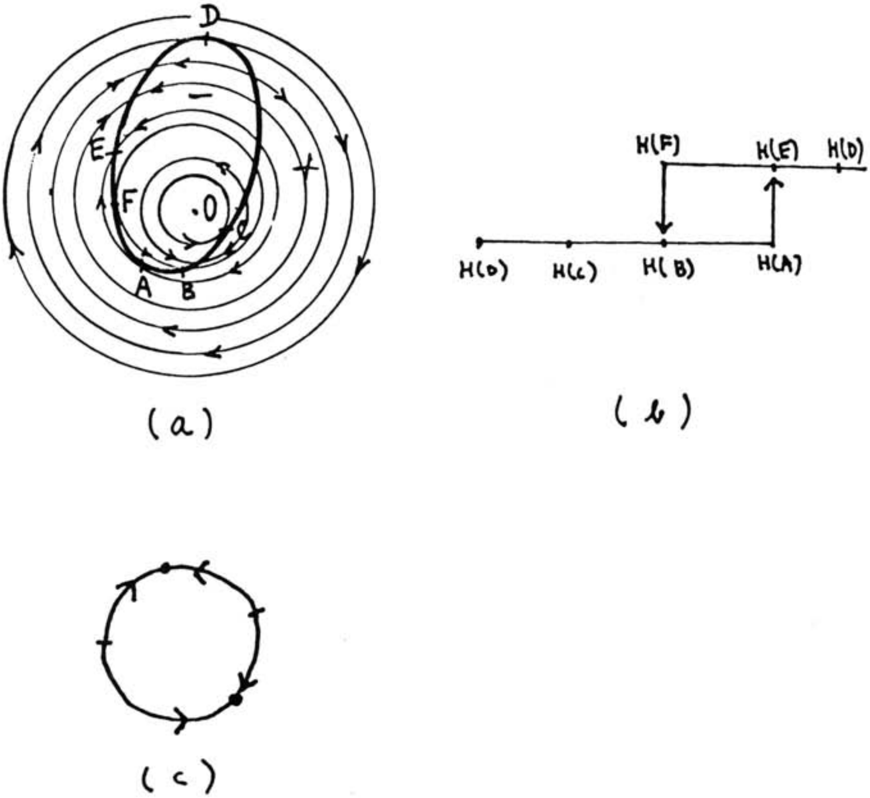

Fig. 8

Fig. 8a and positive outside this loop. Then, at least on some of the level sets of $H(x)$, the dynamical system has four equilibrium points: two stable and two unstable, if considered on the level set (Fig. 8c). This results in appearance of a new independent of $H(x)$ first integral, and it is necessary to consider the limiting process on a graph if we want to preserve the Markov property. Namely, consider the perturbed process

$$\dot{\tilde{X}}_t^\epsilon = \beta(\tilde{X}_t^\epsilon)\overline{\nabla}H(\tilde{X}_t^\epsilon) + \sqrt{\epsilon}\dot{W}_t, \ \tilde{X}_0^\epsilon = x, \tag{2.8}$$

and the rescaled process $X_t^\epsilon = \tilde{X}_{t\backslash\epsilon}^\epsilon$. Denote by $x_0^{(1)}(y)$ the point of intersection of $C(y) = \{x : H(x) = y\}$ and of the arc FED of the curve $\gamma = \{x : \beta(x) = 0\}$. The point of intersection of $C(y)$ and of the arc ABC denote $x_0^{(2)}(y)$. Define $\nu(x), x \in I\!R^2$, as follows:

$\nu(x) = i$, if x belongs to the domain of attraction of $x_0^i(H(x))$;
$\nu(x) = 1$, if $H(x) \geq H(D)$ or if x belongs to the arc CPD of γ;
$\nu(x) = 2$, if $H(x) \leq H(C)$ or if x belongs to the arc AGF of γ.

The function $\nu(x)$ is the first integral for system (2.2) independent of $H(x)$.

Denote

$$a_1(y) = \begin{cases} \frac{1}{\int_{C(y)} |\beta(X)\nabla H(X)|^{-1}dl} \int_{C(y)} \frac{|\nabla H(x)|dl}{|\beta(x)|}, \ y > H(D), \\ |\nabla H(x_0^{(1)}(y))|^2, \ H(F) \leq y \leq H(D); \end{cases}$$

$$B_1(y) = \begin{cases} \frac{1}{2\int_{C(y)} |\beta(X)\nabla H(X)|^{-1}dl} \int_{C(y)} \frac{\Delta H(x)dl}{|\beta(X)\nabla H(X)|}, y > H(D), \\ \frac{1}{2}|\Delta H(X_0^{(1)}(y))|, \ H(F) \leq y \leq H(D); \end{cases}$$

$$a_2(y) = \begin{cases} \frac{1}{\int_{C(y)} |\beta(X)\nabla H(X)|^{-1}dl} \int_{C(y)} \frac{|\nabla H(x)|dl}{|\beta(x)|}, \ y < H(C), \\ |\nabla H(X^{(2)}(y))|^2, \ H(C) \leq y \leq H(A); \end{cases}$$

$$B_2(y) = \begin{cases} \frac{1}{2\int_{C(y)} |\beta(X)\nabla H(X)|^{-1}dl} \int_{C(y)} \frac{\Delta H(x)dl}{|\beta(X)\nabla H(X)|}, y < H(C), \\ \frac{1}{2}|\Delta H(X^{(2)}(y))|, \ H(C) \leq y \leq H(A); \end{cases}$$

$$L_1 = \frac{A_1(y)}{2}\ \frac{d^2}{dy^2} + B_1(y)\frac{d}{dy}, \ L_2 = \frac{a_2(y)}{2}\ \frac{d^2}{dy^2} + B_2(y)\frac{d}{dy}.$$

Consider the mapping $Y(x) = \{H(x), \nu(x)\}$ of $I\!R^2$ to the graph drawn in Fig. 8c. Note that $x_0^{(1)}(y)$ is a stable equilibrium point for system (2.2) when $H(F) < y < H(D)$, and $x_0^{(2)}(y)$ is a stable equilibrium point for $H(C) < y < H(A)$. At point F the branch FED loses stability and X_t^ϵ "jumps" from the point F to B along the deterministic trajectory as $\epsilon \ll 1$. Similarly, X_t^ϵ "jumps" from the point A to E. Therefore "from the point of view of the process X_t^ϵ for $\epsilon \ll 1$" the points $(H(A), 2)$ and $(H(E), 1)$ as well as $(H(F), 1)$ and $(H(B), 2)$ should be identified. Denote by Γ the graph in Fig. 8b with identified points $(H(A), 2) \sim (H(E), 1)$ and $(H(F), 1)$ as well as $(H(F), 1)$ and $(H(F), 1) \sim (H(B), 2)$.

Consider the diffusion process Y_t on Γ by L_1 on the edge $I_1 = \{y \in \gamma, \nu = 1\}$ and by L_2 on $I_2 = \{y \in \Gamma, \nu = 2\}$ with the gluing conditions: $f(y), y \in \Gamma$, is continuous, $f(y)$ continuously differentiable at $(H(B), 2)$ alon I_2 and at $(H(E), 1)$ along I_1. Obviously, these gluing conditions have the form described above.

One can prove that the process $(H(X_t^\epsilon, \nu(X_t^\epsilon)))$ converge weakly as $\epsilon \downarrow 0$ to the process Y_t on the graph Γ. As it is in the case of Hamiltonian systems, such an approximation allows one to calculate main forms as $\epsilon \downarrow 0$ of a number of interesting characteristics of the process X_t^ϵ explicity.

Consider now fast oscillating perturbations (1.5). Assume that the process ξ_t has good enough mixing properties (see [BF]). Let $Eb(x, \xi_t) = b(x)$ and assume that the system

$$\dot{X}_t = b(X_t), \ X_0 = x$$

has a smooth first integral $H(X) : \nabla H(X) b(X) \equiv 0$. Then $H(X_t^\epsilon)$ tends to $H(x)$ as $\epsilon \downarrow 0$ for any finite t. But after rescaling of time $t \to t\backslash\epsilon$ we can expect that $H(X_{t\backslash\epsilon}^\epsilon)$ converges weakly as $\epsilon \downarrow 0$ to a diffusion process, if $H(X)$ has only one critical point as in Fig. 4. The convergence is the result of double averaging on the fast oscillating noise and fast rotation along the deterministic trajectories. This is a generalization of the central-limit-theorem-type result mentioned above when condition (1.7) is fulfilled: If $Eb(x, \xi_t) \equiv 0$ then the function $H_1(x) = x^1, \ldots, H_r(x) = x^r$ are first integrals. The problem of convergence of $H(X_{t\backslash\epsilon}^\epsilon)$ to a diffusion process and its generalizations were studied in [BF]. Note, that if $H(x)$ has many critical points then it is necessary to consider the limiting process on the graph corresponding to $H(x)$ if we want to have the Markov property. This problem is still open.

The limit theorems described in this section can be used for studying the asymptotic behavior of boundary problems for PDEs connected with diffusion processes ([FW3], [FW4]) for problems of optimal stabilization of dynamical systems perturbed by a noise [DF]. Since the limiting process is one-dimensional one can expect explicit expressions for many interesting characteristics of the perturbed process.

If we consider perturbations of s Hamiltonian system with more than one degree of freedom, but the system has good enough ergodic properties on non-critical level sets, one can expect a result very close to the case of one degree of freedom.

If the system has several first integrals $H_1(x), \ldots, H_l(x)$ and the dynamical system is ergodic on the non-singular level sets $\{x \in \mathbb{R}^r : H_1(x) = y_i, \ldots, H_l(x) = y_l\} = C(y) = C(y^1, \ldots, y^l)$ ($C(y)$ is non-singular if the Jacobian $\left\{\frac{\partial H_k(x)}{\partial x^i}\right\}$ has maximal rank for all $x \in C(y)$ one can expect that $(H_1(X_t^\epsilon), \ldots, H_l(X_t^\epsilon))$ converges weakly to a process Y_t on a set consisting of glued l-dimensional pieces.

Inside these pieces Y_t is a l-dimensional diffusion process. The coefficients of this process can be calculated using the averaging. Some gluing conditions similar to those considered above should be imposed at the places where several l-dimensional pieces are glued. This problem in the case $l > 1$ is still open.

Now I will mention one more problem that leads to a jump in Markov process on the graph. Let ν_t be the Poisson process with the parameter 1 and $\xi(x)$ be a random vector in $I\!R^2$; $I\!P(x_1,\gamma) = I\!P\{\xi(x) \in \gamma\}, \Gamma \subset I\!R^2$. Consider the following perturbations of system (2.3):

$$\dot{\tilde{X}}^\epsilon_t = \overline{\nabla} H(\tilde{X}^\epsilon_t) + \dot{\nu}_{\epsilon t}\xi(\tilde{X}^\epsilon_t),\ X^\epsilon_0 = x \in I\!R^2. \tag{2.9}$$

It is clear that $\tilde{X}^\epsilon_t = \tilde{X}_t$ as $\epsilon \downarrow 0$, but after the time rescaling the process $X^\epsilon_t = \tilde{X}^\epsilon_{t\backslash\epsilon}$ has a non-trivial limit of the slow component. To describe this limit consider the mapping $Y : I\!R^2 \to \Gamma$, which was introduced when we studied equation (2.4); Γ is the graph corresponding to $H(x), C_i(y)$ are the same as before. Let $M_{y,i}(dz), (y,i) \in \Gamma$, be the normalized invariant measure for system (2.4) on $C_i(y)$; $G = Y(\gamma), \gamma \subset I\!R^2, G \subset \Gamma$. Denote

$$p\big[(i,y),G\big] = \int\limits_{C_i(y)} I\!P(x,\gamma) M_{y,i}(dx).$$

One can prove that $Y(X^\epsilon_t)$ converges as $\epsilon \downarrow 0$ to the continuous time Markov chain on Γ such that

$$I\!P_y\{\tau > t\} = e^{-t},\ I\!P_y\{Y_\tau \subset G\} = p(y,G),$$

where τ is the time of the first jump; $\tau = \inf\{s : Y_s \neq Y_0\}$.

One can introduce the notion of the first integral for a Markov process X_t in $I\!R^r : H(x), x \in I\!R^r$, is a first integral for X_t if $I\!P_x\{H(X_t) = H(x)\} = 1, x \in I\!R^r$.

If X_t is the diffusion process corresponding to an operator

$$L = \frac{1}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x)\frac{\partial}{\partial x^i},$$

then a smooth function $H(x), x \in I\!R^r$, is a first integral for the process X_t if and only if

$$Lf(x) = 0,\ \sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial H(x)}{\partial x^i}\ \frac{\partial H(x)}{\partial x^j} = 0, x \in I\!R^r.$$

Of course, a non-trivial first integral can exist only if this diffusion process degenerates.

Assume that the process X_t corresponding to L has a smooth first integral $H(x)$, $H(x) \geq 0$ and the level sets $C(y) = \{x : H(x) = y\}$ are compact for any $y \geq 0$. Furthermore, let $H(X)$ be generic and process X_t non-degenerate on any non-singular component $C_i(y)$ of level set $C(y)(C_i(y)$ is singular if $\nabla H(x) = 0$ for some $x \in C_i(y))$. Then the process X_t has on $C_i(y)$ a normalized invariant measure $\nu_{y,i}(dx)$. Such a measure is unique. If y is a critical value and x is the corresponding critical point, then $\nu_{y,i}$ is concentrated at $x \in C_i(y)$. Let Γ be the graph corresponding to $H(x)$ and Y be the corresponding mapping: $Y : I\!R^r \to \Gamma$.

Consider perturbations of the process X_t. Let, for example, the perturbed process $\tilde{X}_t^\epsilon$ be the Markov process X_t. Let, for example, the perturbed process $\tilde{X}_t^\epsilon$ be the Markov process in $I\!R^r$ governed by the operator L^ϵ :

$$L^\epsilon f(x) = Lf(x) + \epsilon \int\limits_{I\!R^r \setminus \{0\}} [f(x+\beta) - f(x) - \sum \beta_i \frac{\partial f(x)}{\partial x^i}] \mu_x(d\beta).$$

Here $\mu_x(.)$ is a measure in $I\!R^r$, $\epsilon > 0$ is a small parameter. Assume, for brevity, that μ_x is finite for all $x \in I\!R^r$, and $\sup\limits_x \mu_x(I\!R^r) = \overline{\mu} < \infty$. Denote motion in the level sets of the function $H(x)$, and the slow component $H(X_t^\epsilon)$.

One can prove that the slow component $X_t^\epsilon = H(X_t^\epsilon), 0 \leq t \leq T$, converges weakly as $\epsilon \downarrow 0$ to a continuous time jumping Markov process Y_t on the graph Γ. The density of the jumps for the limiting process Y_t at a point $(y, i) \in \Gamma$ is equal to

$$\int\limits_{C_i(y)} \mu_x(I\!R^r \setminus \{0\}) \; \nu_{y,i}(dx) = \overline{\mu}(y, i).$$

The probability of the jump from $(y, i) \in \Gamma$ to set $\gamma \subset \Gamma$ is equal to

$$I\!P((y, i), \gamma) = \int\limits_{C_i(y)} \frac{\mu_x(\gamma)}{\mu_x(I\!R^r \setminus \{0\})} \nu_{y,i}(dx).$$

One can consider white-noise perturbations of the process X_t corresponding to operator L. The perturbed process $\tilde{X}_t^\epsilon$ in this case is governed by the operator $\tilde{L}^\epsilon = L + \frac{\epsilon}{2}\Delta$. Then the slow component of $X_t^\epsilon = \tilde{X}_{t \setminus \epsilon}^\epsilon$ under some additional assumptions also converges to a diffusion process on the graph corresponding to $H(x)$. Proper gluing conditions at the vertices should be imposed. Note that if the operator L is non-degenerated in some domain $D \subset I\!R^r$, the function $H(x)$ should be equal to a constant for $x \in D$. It can result in a delay of the limiting process at the point of our graph corresponding to D.

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