

## Comparative aspects of the analysis of stationary time series, point processes and hybrids

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**Abstract:** This paper brings out comparative aspects of the analysis of time series, point processes and hybrids such as sampled time series and marked point processes. Second- and third-order moments and spectra prove useful tools for addressing certain scientific problems involving such processes. Illustrative analyses are presented for data on tides, neurons and earthquakes.

**Key words:** Bispectrum, earthquake, neuron, point process, spectrum, stationary increments, time series.

### 1. INTRODUCTION.

A time series,  $Y$ , is a wiggly line,  $Y(t)$ ,  $-\infty < t < \infty$ . A point process,  $N$ , is a collection of times,  $\{\tau_j, j = 0, \pm 1, \pm 2, \dots\}$ . (It will be assumed that the  $\tau_j$  are distinct.) A marked point process,  $J$ , is a collection of times and associated quantities, marks,  $\{(\tau_j, M_j), j = 0, \pm 1, \pm 2, \dots\}$ . There are also hybrids such as sampled time series,  $\{Y(\tau_j), j = 0, \pm 1, \pm 2, \dots\}$ . Time series techniques and time series data are common. Point process techniques appear less common, as do their analyses. Marked point process studies appear the rarest, but are under substantial current development, particularly for the spatial case. The paper seeks to bring out connections amongst these disparate processes. It will be seen that the second- and third-order moments can prove to be useful tools with which to grab onto scientific problems of interest. Estimates of such moments and corresponding spectra are provided for some particular time series, point process and marked point process data sets, specifically: ocean tides, nerve cell firings and earthquake occurrences. Section 4 lists some analytic methods useful for connecting the processes. The computational details are given in the Appendix.

### 2. STATIONARY INCREMENT PROCESSES.

In this section some specific processes are discussed. In the cases emphasized each is a process with stationary increments.

$X(\cdot)$  is called a process with stationary increments if the following holds:  $X(t)$ ,  $-\infty < t < \infty$ ,  $t$  is a random process such that the joint distribution of the increments  $X(t + b_1) - X(t + a_1), \dots, X(t + b_k) - X(t + a_k)$  does not depend on  $t$  for any  $a_1 < b_1, \dots, a_k < b_k$  and  $k = 1, 2, 3, \dots$ . The basic ideas are due to Kolmogorov and may be found pp. 551-559 in Doob (1953). There exists a

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statistical calculus for such processes, see Brillinger (1972).

A stationary time series  $Y$  corresponds to a stationary increment process,  $X$ , via

$$X(t) = \int_0^t Y(u) du \quad (2.1)$$

A point process,  $N$ , corresponds to a stationary increment process in which all the increments  $N(t+b) - N(t+a)$ ,  $a < b$ , are non-negative integers specifically,  $N(t+b) - N(t+a) = \#\{\tau_j \mid t+a < \tau_j \leq t+b\}$ . One can write

$$X(t) = \int_0^t dN(u)$$

A marked point process,  $J$ , with real-valued marks, may be represented via  $J(t) = \sum_{0 < \tau_j \leq t} M_j$  and there is the correspondence

$$X(t) = \int_0^t dJ(u)$$

The case of principal concern of the paper will be the stationary one. Then  $E\{dX(t)\} = c_X dt$  with  $c_X$  the mean. For simplicity suppose  $c_X$  to be 0. One then defines the autocovariance measure,  $C_{XX}$ , via

$$E\{dX(t+u)dX(t)\} = dC_{XX}(u)dt \quad (2.2)$$

and the third cumulant measure,  $C_{XXX}$ , via

$$E\{dX(t+u)dX(t+v)dX(t)\} = dC_{XXX}(u, v)dt \quad (2.3)$$

The process,  $X$ , has a spectral representation

$$X(t) = \int \frac{e^{it\lambda} - 1}{i\lambda} dZ(\lambda) \quad (2.4)$$

with  $Z$  a random function such that

$$E\{dZ(\lambda)dZ(\mu)\} = \delta(\lambda + \mu)f_{XX}(\lambda)d\lambda d\mu \quad (2.5)$$

and

$$E\{dZ(\lambda)dZ(\mu)dZ(\nu)\} = \delta(\lambda + \mu + \nu)f_{XXX}(\lambda, \mu)d\lambda d\mu d\nu \quad (2.6)$$

$\delta(\cdot)$  being the Dirac delta function and  $f_{XX}$ ,  $f_{XXX}$  the power spectrum and bispectrum respectively. The spectra themselves may be generalized functions containing Dirac deltas.

## 2.1 The Time Series Case

Consider a zero mean stationary time series  $Y(t)$ ,  $-\infty < t < \infty$ . Following (2.1-2.3) the autocovariance function is given by

$$E\{Y(t+u)Y(t)\} = \text{doverdu}C_{XX}(u) = c_{YY}(u) \quad (2.7)$$

and the third-order cumulant function by

$$E\{Y(t+u)Y(t+v)Y(t)\} = \frac{\partial}{\partial u} \frac{\partial}{\partial v} C_{XXX}(u, v) = c_{YYY}(u, v) \quad (2.8)$$

The spectral representation is

$$Y(t) = \int \epsilon^{it\lambda} dZ(\lambda)$$

with  $Z$  satisfying (2.5) and (2.6).

The autocovariance function (2.7) provides a measure of the dependence of values of the series lag  $u$  time units apart. An estimate is provided in Figure 1 for a tidal series from St. John, Canada. The data are for the time period 1 January to 31 March, 1991. There are  $T = 2160$  observations in all. The top left panel is an initial segment of the series. The autocovariance estimate here shows strong periodicity. The power spectrum of (2.5) is particularly useful in making inferences concerning periodicities and developing predictors. An estimate of a flattened version is given in the bottom display of Figure 1. Peaks are seen to stand out. The presence of periodic components in tidal series is basic and ascribed to the effects of the moon and the sun. A pertinent model is provided by

$$Y(t) = \mu + \sum_{k=1}^K \rho_k \cos(\omega_k t + \phi_k) + \epsilon(t) \quad (2.9)$$

with the  $\phi_k$  uniform,  $\phi_k, \phi_l, k \neq l$ , independent and with *epsilon* a stationary noise series with smooth spectrum  $f_{\epsilon\epsilon}$ . The power spectrum of  $Y$  is then

$$f_{YY}(\lambda) = \sum_k \frac{\rho_k^2}{4} [\delta(\lambda - \omega_k) + \delta(\lambda + \omega_k)] + f_{\epsilon\epsilon}(\lambda) \quad (2.10)$$

If for example  $\omega_3 = \omega_1 + \omega_2, \phi_3 = \phi_1 + \phi_2$ , then the bispectrum has a term

$$\frac{1}{8} \rho_1 \rho_2 \rho_3 \delta(\lambda - \omega_1) \delta(\mu - \omega_2) \quad (2.11)$$

Tidal analysis is discussed in Morettin and de Mesquita (1978), Wood (1978) and Forrester (1983). Wood (1978) lists various estimates of tidal frequencies. The model (2.9) was fit to the St. John data by least squares employing the  $K = 26$  frequencies of the final column of Figure 43 Wood (1978). The right

hand column of Figure 1 graphs the results. The residuals are much smaller. (Their standard error is .165 meters. The original standard error was 2.192 m.) The autocovariance estimate of the residuals and a flattened power spectrum are also given. Some things remain to be accounted for, there remain clear peaks with structure about them.

In nonGaussian circumstances and situations where a basic process has been transformed in a nonlinear fashion, the third order cumulant function,  $c_{YYY}(u, v)$ , and bispectrum,  $f_{YYY}(\lambda, \mu)$ , of (2.8) and (2.6) are of importance. For example squaring  $\rho_1 \cos(\omega_1 t + \phi_1) + \rho_2 \cos(\omega_2 t + \phi_2)$  of (2.9) leads to  $\rho_3 \cos(\omega_3 t + \phi_3)$  with  $\omega_3 = \omega_1 + \omega_2$  and  $\phi_3 = \phi_1 + \phi_2$  and the bispectrum term (2.11).

Figure 2 top presents estimates of the third moment function (2.8) for the original series and for the residuals from the least squares fit. Positive contours are graphed with a solid line, negative with a dashed line. The third-order cumulant estimate of the original data suggests periodicity and asymmetry. The structure in the case of the residuals is not apparent. The bottom displays of the figure provides estimates of

$$\min\{\rho_1^2/f_{\epsilon\epsilon}(\omega_1), \rho_2^2/f_{\epsilon\epsilon}(\omega_2), \rho_3^2/f_{\epsilon\epsilon}(\omega_3)\} \quad (2.12)$$

as a function of  $(\omega_1, \omega_2)$ , where  $\omega_3 = \omega_1 + \omega_2$ . This parameter is meant to examine the hypothesis that harmonic components at frequencies  $\omega_1, \omega_2, \omega_1 + \omega_2$  are all present, see Brillinger (1980). Further details are in the Appendix. Graphed are the values significant at the 1%. There is clear structure present in the original series and much of the structure remains in the residuals. There are strong suggestions of nonlinear interactions.

Cartwright (1969) discusses the generation of nonlinear interactions in tidal series. Marone and de Mesquita (1993) are concerned with estimating the bispectrum removing lower order information.

## 2.2 The Point Process Case

Suppose that the point process  $N$  is described via times  $\tau_j, j = 0, \pm 1, \pm 2, \dots$ . A step function description is provided by  $N(t) = \#\{\tau_j \mid 0 < \tau_j \leq t\}$ . There are other useful representations for a point process. A representation that suggests immediate extensions of corresponding time series procedures is

$$Y(t) = \frac{dN(t)}{dt} = \sum_j \delta(t - \tau_j) \quad (2.13)$$

with  $dN(t)/dt$  a symbolic derivative of the process. From the representation (2.13) one sees, for example, that a linear filtering is given by

$$\int a(t-u)Y(u)du = \sum_j a(t - \tau_j)$$

with  $a(\cdot)$  the impulse response of the filter. It can be convenient to consider a point process as a function of intervals, with  $N(I)$  counting the number of points in the interval  $I$ . Then one has

$$N(I) = \sum_{\tau_j \in I} 1 = \int_I dN(t)$$

and  $N$  is seen to be a counting measure on the line.

One basic parameter of a stationary point process is the rate,  $p_N$ , given by

$$\text{Prob}\{dN(t) = 1\} = p_N dt = E\{dN(t)\}$$

for small  $dt$ . A second is the autointensity function,  $h_{NN}(u)$  given by

$$\text{Prob}\{dN(t+u) = 1 \mid \text{point at } t\} = h_{NN}(u)du, \quad u \neq 0 \quad (2.14)$$

The autointensity is a more primitive concept than an autocovariance being based on a probability. It is a direct measure of the chance of a further point occurring  $u$  time units after an existing point.

Figure 3 presents an estimate  $h_{NN}(u)$  for each of two data sets. The top panels give illustrative segments of the data, whose collection is described in Bryant et al. (1973). The left hand column corresponds to a sea hare neuron firing as a pacemaker. The right column refers to a bursting neuron, also of the sea hare. The middle panels give the estimated autointensities. Complex periodic behavior is apparent in the pacemaker case. The autointensity estimate in the bursting case has a broad peak at a lag of about 25 seconds, presumably corresponding to the spacings of the bursts. The bottom panels provide estimates of the power spectra,  $f_{NN}(\lambda)$ , and bring out periodicities in an alternate fashion. The pacemaker firing is seen to have a complex structure not readily apparent in the basic data. The firing in the bursting case is seen to have a structure suggesting harmonics in the frequency domain.

The autocovariance density of a stationary point process,  $N$ , at lag  $u$ ,  $q_{NN}(u)$ , is given by

$$\text{cov}\{dN(t+u), dN(t)\} = [\delta(u)p_N + q_{NN}(u)]dtdu$$

while the third-order cumulant density is given by

$$E\{[dN(t+u_1) - p_N dt] [dN(t+u_2) - p_N dt] [dN(t) - p_N dt]\} = q_{NNN}(u_1, u_2)dtdu_1du_2 \quad (2.15)$$

for  $u_1, u_2, 0$  distinct.

Estimates of (2.15) for the pacemaker and bursting cases are given in Figure 4, top row. Positive contours are graphed with a solid line, negative ones with a

dashed line. The periodic behaviors of Figure 3 show themselves in an alternate form. The bottom row of Figure 4 gives an estimate of the quantity (2.12), graphing points significant at the 1% level. There is a cluster at (1.40, .95) in the pacemaker case that might not have been suspected. The sum frequency, 2.35, is apparent in the periodogram. The burst statistic likewise shows some interesting structure.

### 2.3 Hybrid Cases

Consider a marked point process case with real-valued marks. Realizations of the process have the form  $\{(\tau_j, M_j), j = 0, \pm 1, \pm 2, \dots\}$ . A representation for the process, as a generalized ordinary time series, is provided by

$$Y(t) = \frac{dJ(t)}{dt} = \sum_j M_j \delta(t - \tau_j)$$

As a function of intervals  $J$  may be written

$$J(I) = \sum_{\tau_j \in I} M_j$$

and is seen to correspond to a discrete measure on the line. The autocovariance density at lag  $u$ ,  $q_{NN}(u)$ , of the process is given by

$$\text{cov}\{dJ(t+u), dJ(t)\} = [\delta(u)c_J + c_{JJ}(u)]dtdu$$

A hybrid process is provided by a sampled ordinary time series,  $\{Y(\tau_j)\}$ . This can be represented via  $dJ(t) = Y(t)dN(t)$ ,  $N$  being the process of sampling times. This  $J$  will have stationary increments when, for example, the processes  $Y$  and  $N$  are stationary and independent. A discrete time series corresponds to  $\tau_j = j$ . The spectral representation of  $J$  involves

$$dZ_J(\lambda) = \int dZ_Y(\lambda - \mu)dZ_N(\mu)$$

a relationship from which expressions for various spectra may be obtained.

Figure 5 presents the initial stretch of some California earthquake data and of the corresponding point process of times. The data set consists of the California earthquakes of magnitude 5 or greater occurring between 1931 and 1992. The second row left, presents an estimate of  $c_{JJ}(u)$ . Below is an estimate of the power spectrum,  $f_{JJ}$ . Approximate 95% confidence intervals are shown. No special structure is apparent.

An estimate of the third-order cumulant density is graphed in the top left of Figure 6. The bispectrum  $f_{JJJ}$  is given by

$$\text{cum}\{dZ_J(\lambda), dZ_J(\mu), dZ_J(\nu)\} = \delta(\lambda + \mu + \nu)f_{JJJ}(\lambda, \mu)d\lambda d\mu d\nu$$

and the bicoherence by

$$|f_{JJJ}(\lambda, \mu)|^2 / f_{JJ}(\lambda)f_{JJ}(\mu)f_{JJ}(\lambda + \mu)$$

Values significantly different from 0 at the 1

A question that arises when dealing with marked point processes is: are the series of marks,  $\{M_j\}$  and the inherent point process,  $N = \{\tau_j\}$ , independent of each other? This question may be addressed via a second-order moment analysis.

First some definitions pertinent to the bivariate case. The crosscovariance density at lag  $u$ ,  $c_{JN}(u)$ , between the jump process  $J$  and its inherent point process,  $N$ , is given by

$$cov\{dJ(t + u), dN(t)\} = c_{JN}(u)dtdu$$

for  $u \neq 0$ . Suppose that the marks  $M_j = Y(\tau_j)$  correspond to sampled values of a zero mean stationary series  $Y$ . In the case that  $Y$  and  $N$  are independent  $c_{JN}$  will be identically 0. So too will the cross-spectrum,  $f_{JN}$ , given by

$$E\{dZ_J(\lambda) dZ_N(\mu)\} = \delta(\lambda + \mu)f_{JN}(\lambda)d\lambda d\mu$$

Figure 5, middle right, graphs an estimate of  $c_{JN}(u)$  for the California earthquake data. The values fluctuate about 0. The sampling properties of an estimate of the coherence,  $|R_{JN}(\lambda)|^2_{sup} = |f_{JN}(\lambda)|^2_{sup} / f_{JJ}(\lambda)f_{NN}(\lambda)$  are simpler, hence this is the statistic employed to assess the independence. An estimate is graphed in Figure 5 bottom right. There is some evidence against independence, 21 points out of 128 exceed the 95

The third-order joint cumulant density may also be used to address the hypothesis of independence. It is given by

$$cum\{dJ(t + u), dN(t + v), dN(t)\} = c_{JNN}(u, v)dtdudv$$

for  $u, v, 0$  distinct. It will be 0 in the case of  $Y$  independent of  $N$ . An estimate is given in Figure 6 top right. The crossbispectrum  $f_{JNN}$  similarly will be 0 in the case of independence. An estimate based on the corresponding crossbicoherence is graphed in Figure 6. The points plotted are bifrequencies  $(\lambda, \mu)$  where the bicoherence estimate is significantly different from 0 at the 1%. There are many. A comparison of the two bicoherences of Figure 6 shows many more significant points in the  $JNN$  case. This goes along with the process  $(J, N, N)$  being more nonnormal.

Vere-Jones (1970) discussed point and marked point processes associated with earthquakes. The theory of point processes and marked point processes is presented in Daley and Vere-Jones (1988).

A question related to the present context is: assuming  $Y$  and  $N$  independent, how does one estimate  $c_{YY}$  and  $f_{YY}$ ? One answer is given in Brillinger (1972). Some results of applying the technique are given in Moore et al. (1987).

### 3. CONNECTIONS.

There are several methods for relating time series, point and marked point processes and techniques. Advantages of employing these include: computing programs available for one type may be used with the others, models and theoretical results may be transferred, and generally further insight and understanding may be obtained.

A point process on the line may be studied via ordinary time series methods through picking a small cell width  $\delta$  and setting up the discrete time series

$$Y(t) = N(t, t + \delta) \quad (3.1)$$

for  $t = 0, \pm\delta, \pm2\delta, \dots$ . This 0-1 series may be fed to either moment or likelihood based techniques. For example the second-order moments are connected via

$$c_{YY}(u) = p_N (\delta - |u|)_+ + q_{NN}(u)\delta^2$$

for small  $\delta$ . The power spectrum of the discrete series (3.1) is given by

$$f_{YY}(\lambda) = 4(\sin \delta \frac{\lambda}{2})^2 \sum_{j=-\infty}^{\infty} (\lambda + \frac{2\pi j}{\delta})^{-2} f_{NN}(\lambda + \frac{2j\pi}{\delta})$$

References include Vere-Jones and Davies (1966), Lewis (1970), Guttorp (1986).

The use of 0-1 series for point process likelihoods occurs in Brillinger and Segundo (1979) and Berman and Turner (1992). When the model is correct, the likelihood approach may be anticipated to be the more efficient. However the moment approach has the advantage of being broadly applicable and of having the same form for distinct types of processes. Indeed if one moves to the frequency domain, the moment procedures are essentially the same for time series, point processes and marked point processes.

A discrete time series,  $Y(t)$ ,  $t = 0, \pm 1, \pm 2, \dots$ , may be set up as a planar point process via the correspondence  $Y(t) \rightarrow (t, Y(t))$ . A marked point process, with marks in  $R_{supp}$ , may similarly be considered a point process lying in  $R_{supp} + 1$  through the expedient of simply viewing  $(\tau_j, M_j)$  as a point in  $R_{supp} + 1$ . One reference is Karr (1976).

A jump process,  $J$ , may be associated with a time series in continuous time through the correspondence

$$Y(t) = \int a(t-u)dJ(u)$$

see Priestley (1963), Jowett and Vere-Jones (1972). The spectra are related by

$$f_{X\dots X}(\lambda_1, \dots, \lambda_{k-1}) =$$

$$A(\lambda_1)\dots A(\lambda_{k-1})\overline{A(\lambda_1 + \dots + \lambda_{k-1})}f_{J\dots J}(\lambda_1, \dots, \lambda_{k-1})$$

on which estimates may be based. The 0-1 time series above corresponds to  $a(\cdot)$  a boxcar function of width  $\delta$ . Hence  $A(\lambda) = 2(\sin \lambda\delta/2)/\lambda$ , which is approximately  $\delta$  for small  $\delta$ .

Parallel development of the time series and point process cases is provided in Brillinger (1978).

#### 4. DISCUSSION AND SUMMARY

In her functioning, Nature appears to make use of each of time series, point processes and marked point processes. This work has sought to bring out some parallel definitions and methods for these concepts. The models and techniques employed are mainly nonparametric and moment based. Another aspect has been the illustration of both time-side and frequency-side analyses. Generally speaking the (approximate) sampling properties are simpler in the frequency domain.

Various displays were presented for each data type. In particular the tool of stacking has been highlighted as being of use in some particular circumstances.

A new statistic (A.6) has been employed in the study of discrete components in a bispectrum. The statistic has advantages over the biperiodogram for the biperiodogram will be large in amplitude when any of the frequency components involved is large. The statistic (A.6) standardizes for this.

Analyses were provided of data taken from three fields: oceanography, neurophysiology and seismology. In all studies it is good practice to ask: "What is the question?" Questions going along with the examples of this paper include: .LP 1. Tides. How to predict? The analyses presented were in part directed at understanding if an existing model was satisfactory. .LP 2. Nerve Firings. How to describe? Description is needed because there are so many types of behavior. .LP 3. Earthquake Times and Sizes. How to predict? One focus was on whether magnitudes were related to occurrence times, a second was on the presence of periodicities.

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### Figure legends

Figure 1. St. John tides and residuals. The first column presents statistics for the tidal series, the second statistics for the residuals of a least squares fit to the series. The top displays are initial sections of the series themselves. The middle row provides estimated autocovariance functions. The final row provides an estimate of the spectrum with the continuous component flattened. The dashed line is an approximate upper 95

Figure 2. Top left is an estimate of the third-order cumulant function for the St. John tidal data. Negative values are plotted as dashed lines. Top right provides the same for the residual series. The points significant at the 1

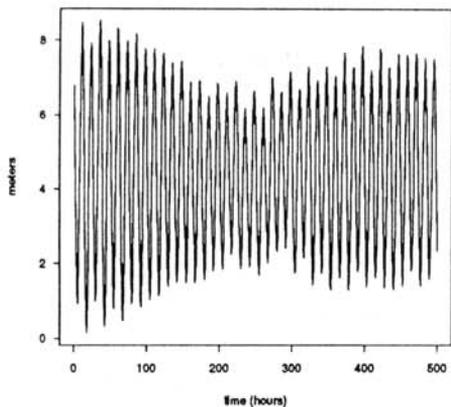
Figure 3. The left hand panels give statistics for a neuron firing regularly, the right hand panels for a second neuron firing in bursts. The middle displays are estimates of the autointensity (of (2.14)). The bottom row provides the periodograms (A.5).

Figure 4. The lefthand panels are the estimated cumulant densities, as estimated from (A.2). Negative contours are plotted as dashed. The bottom panels are the statistics (A.6) significant at the 1

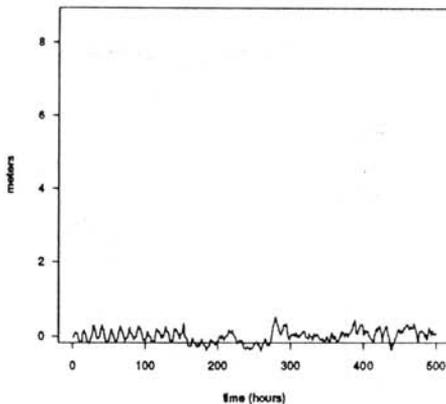
Figure 5. Analyses of the California earthquake data. The top displays are the magnitudes and times and just the times respectively. The bottom left is the power spectrum estimate with an approximate 95The bottom right is the estimated coherence with an upper 95

Figure 6. Estimates of third-order cumulant densities for the marked point process and point process. The bottom two are points of the estimated bicoherence significant at the 1

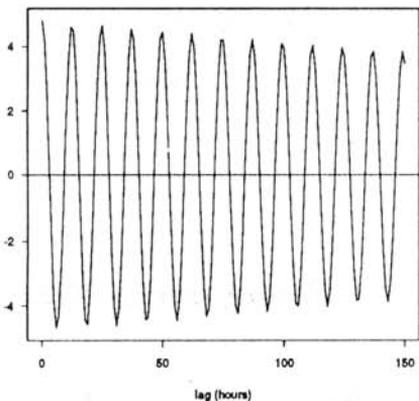
Saint John tides



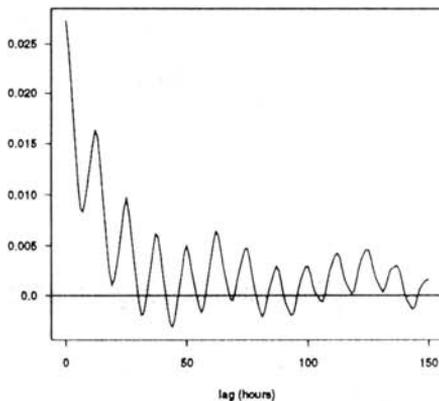
Residuals (26 components)



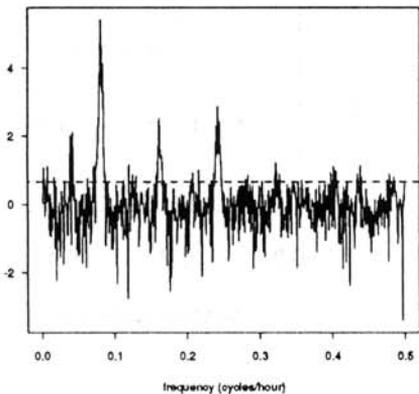
Autocovariance estimate



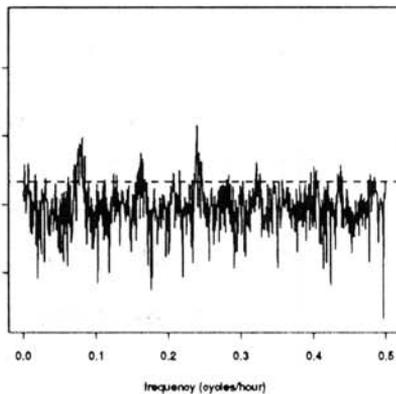
Autocovariance estimate



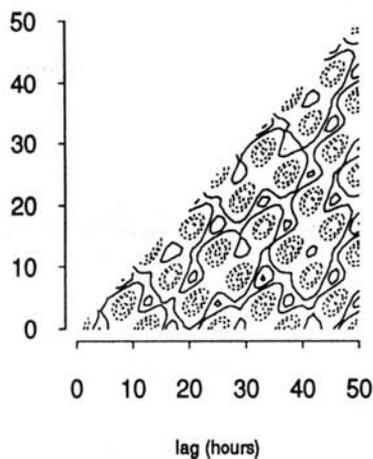
Periodogram



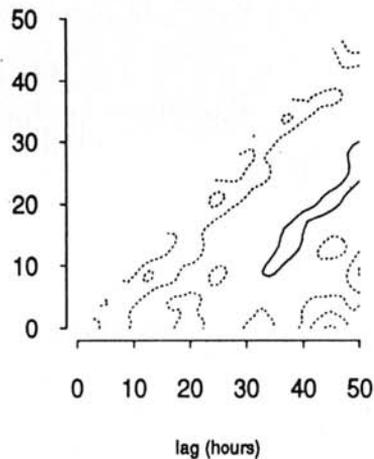
Periodogram



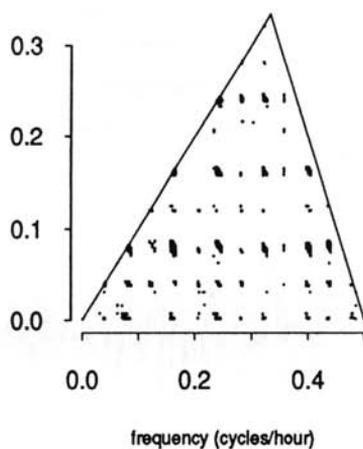
Third cumulant tides



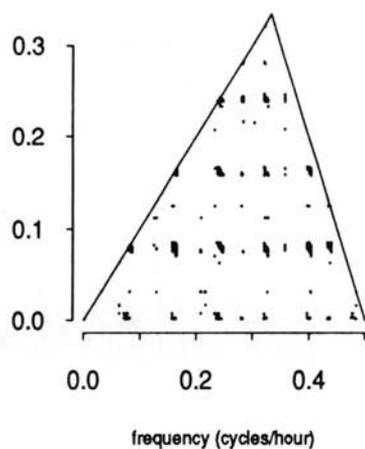
Third cumulant residuals



Minimum periodogram tides



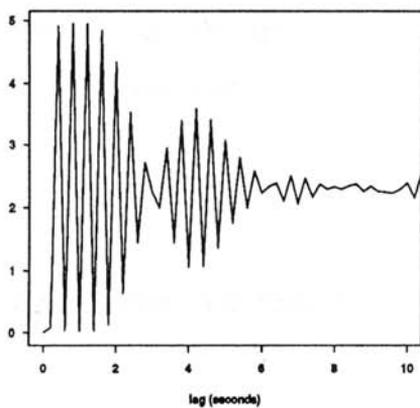
Minimum periodogram residuals



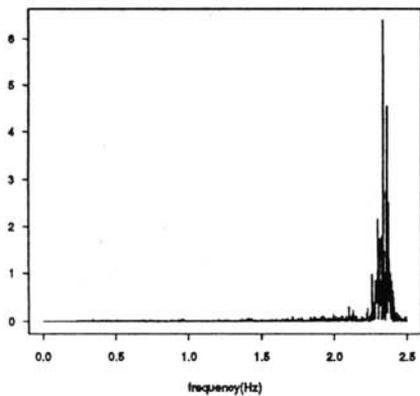
L10 pacemaker



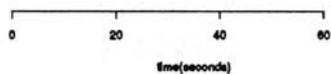
Autointensity estimate



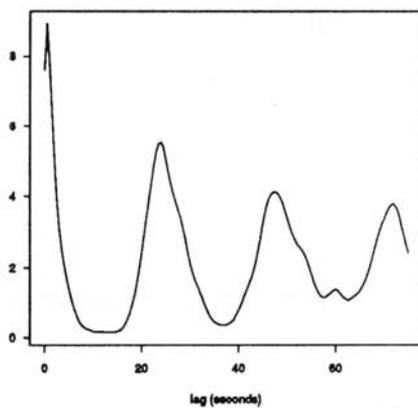
Periodogram



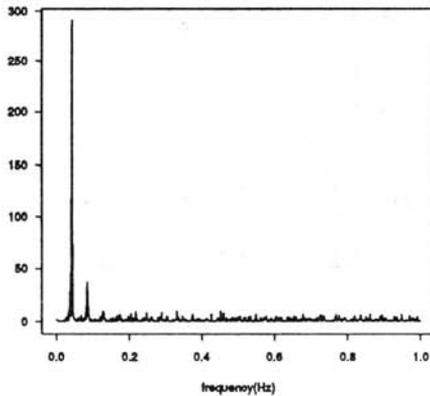
L10 bursting & accelerando



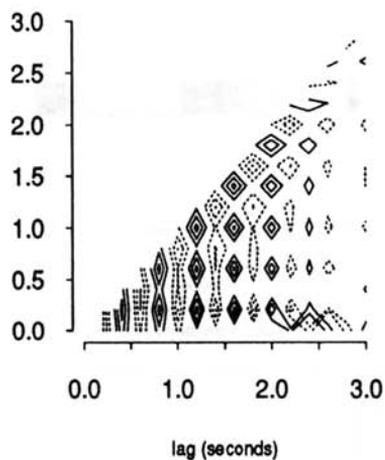
Autointensity estimate



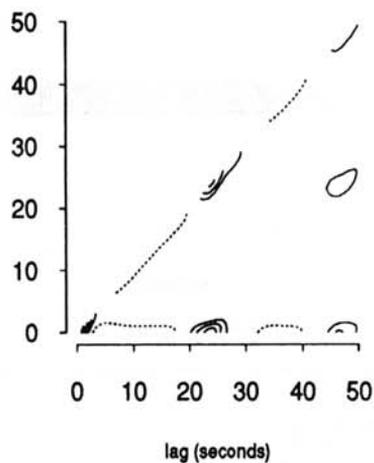
Periodogram



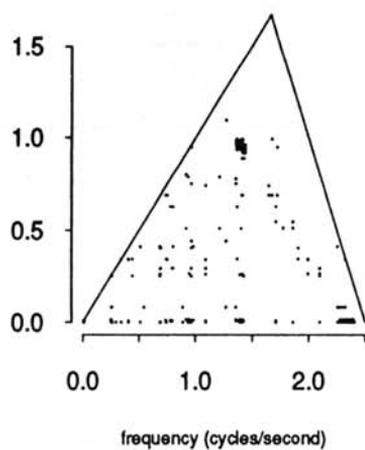
Third cumulant pacemaker



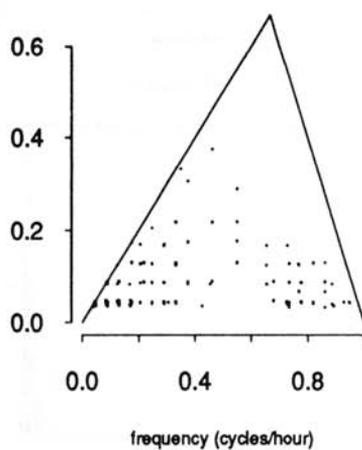
Third cumulant bursts



Minimum periodogram pacemaker



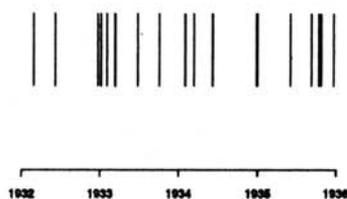
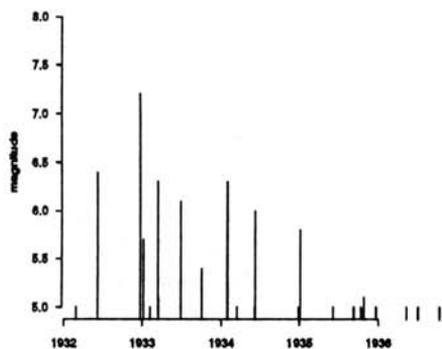
Minimum periodogram bursts



## California earthquakes 1932-1992

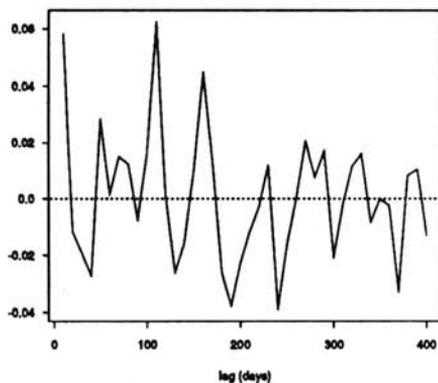
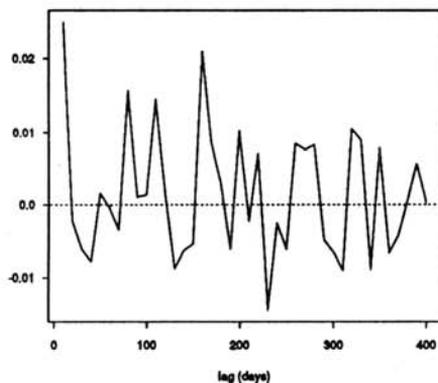
Marked point process

Point process



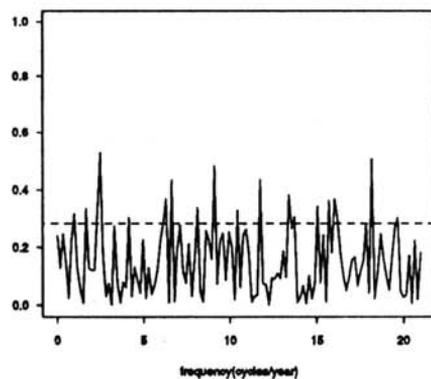
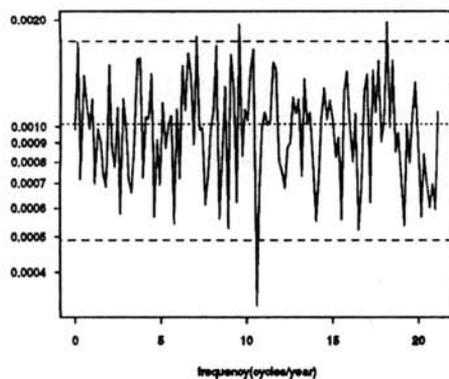
Autocovariance density

Crosscovariance density

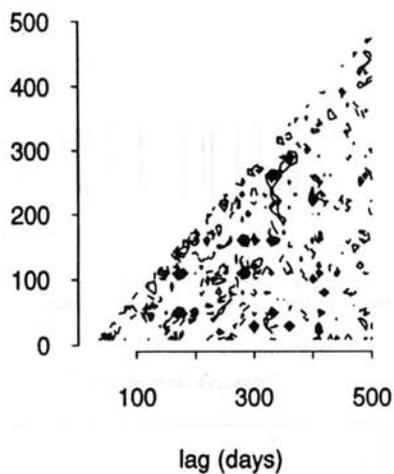


Power spectrum

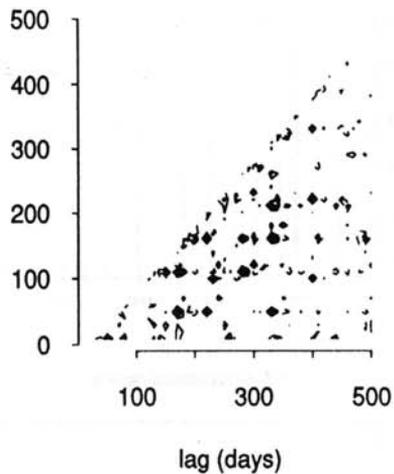
Coherence



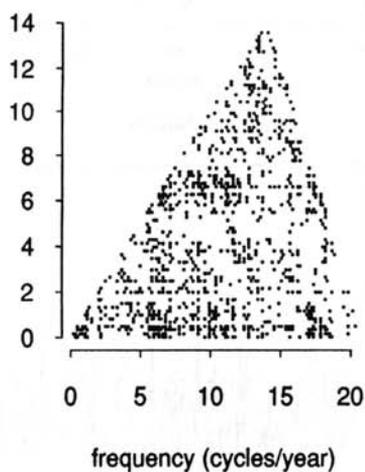
Third cumulant



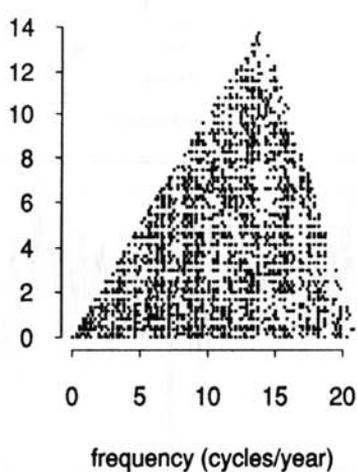
Third joint cumulant



Bicoherence



Crossbicoherence



## APPENDIX

This section provides some details of the estimates and computations. Given data,  $X(t)$ ,  $0 \leq t < T$ , general estimates of the  $f_{X \dots X}$  and  $M_{X \dots X}$  are indicated in Brillinger (1972) for processes with stationary increments.

If  $\bar{Y}$  denotes the mean of the data  $Y(t)$ ,  $t = 0, \dots, T-1$  of a discrete time series, then an estimate of the autocovariance function is

$$c_{YY}^T(u) = \frac{1}{T} \sum_{t=0}^{T-|u|} [Y(t+u) - \bar{Y}][Y(t) - \bar{Y}]$$

and of the third cumulant function is

$$c_{YYY}^T(u, v) = \frac{1}{T} \sum_{0 \leq t, t+u, t+v \leq T-1} [Y(t+u) - \bar{Y}][Y(t+v) - \bar{Y}][Y(t) - \bar{Y}]$$

These appear in Figures 1 and 2.

An estimate of the rate of a point process,  $N$ , is  $p_N \text{sup} T = N(T)/T$ , while an estimate of the autointensity is

$$h_{NN}^T = \# \{ |\tau_j - \tau_k - u| < b \} / 2bN(T) \quad (\text{A.1})$$

The estimate (A.1) was introduced in Griffith and Horn (1963) and considered in Cox (1965). It appears in Figure 3.

Following the discussion of Section 4 an estimate of the third-order cumulant density at  $u, v, 0$  distinct is given by

$$q_{NNN}^T(u, v) = c_{YYY}^T(u, v) / \delta^3 \quad (\text{A.2})$$

where  $Y$  is the corresponding 0-1 time series based on cells of small width  $\delta$ . This appears in Figure 4.

In the marked point process case one can consider the statistic

$$\sum_{|\tau_j - \tau_k - u| < b} M_j M_k = \iint_{|t-s-u| < b} dJ(t) dJ(s) \quad (\text{A.3})$$

(with  $j \neq k$  and  $t \neq s$ ) in analogy with (A.2). One bases an estimate of  $\text{cov}\{dJ(t+u), dN(t)\}$  on

$$\sum_{|\tau_j - \tau_k - u| < b} M_j = \iint_{|t-s-u| < b} dJ(t) dN(s) \quad (\text{A.4})$$

These appear in Figure 5, having adjusted the marks to mean 0.

In estimating frequency domain parameters it can be convenient to work with the empirical Fourier transform

$$d_X^T(\lambda) = \int_0^T e^{-i\lambda t} dX(t)$$

In the cases of a discrete time series, a point process, a marked point process this becomes

$$\sum_t Y(t)e^{-it\lambda}, \quad \sum_j e^{-i\tau_j\lambda}, \quad \sum_j M_j e^{-i\tau_j\lambda}$$

respectively. These satisfy central limit theorems in various circumstances allowing approximate distributions of derived statistics to be set down.

A crude estimate of the power spectrum is provided by the periodogram

$$I^T(\lambda) = \frac{1}{2\pi T} |d^T(\lambda)|^2 \quad (\text{A.5})$$

This appears in Figure 3.

The spectrum (2.10) shows lines superposed on a (smooth) curve. To make the lines stand out more: the data is tapered prior to Fourier transforming and the curve is flattened. The flattening was done by applying a resistant heavy smoother to the log periodogram values to obtain an estimate of the spectrum, which is then divided out. In a related context Tukey (1963) suggests dividing the periodogram by the result of a repeated running median and in a testing situation Chiu (1989) suggests dividing by trimmed means of periodograms.

A crossspectral estimate  $f_{JN}^T$  may be computed by breaking a data set of length  $T$  into  $L$  segments of length  $V$ , computing the crossperiodogram,  $(2\pi V)^{-1} d_J^V \overline{d_N^V}$ , for each and averaging. The coherence may then be estimated by  $|f_{JN}^T|^2 / f_{JJ}^T f_{NN}^T$ . Likewise a bispectrum estimate may be obtained by averaging the biperiodograms

$$\text{loever}(2\pi)^2 V d^V(\lambda) d^V(\nu) \overline{d^V(\lambda + \mu)}$$

The bicoherence may be estimated via

$$|f_{JJJ}^T|^2 / f_{JJ}^T f_{JJ}^T f_{JJ}^T$$

Its distribution, in the case that the population value  $f_{JJJ}$ , is 0 is exponential with mean  $V/2\pi L$ . See Huber et al. (1971).

In the case of a line in the bispectrum it can be more useful to consider the statistic

$$\min\{|I^T(\lambda)|^2 / f^T(\lambda), |I^T(\mu)|^2 / f^T(\mu), |I^T(\lambda + \mu)|^2 / f^T(\lambda + \mu)\} \quad (\text{A.6})$$

with  $I^T$  the periodogram and  $f_{sup}T$  a heavily smoothed resistant estimate of the power spectrum. The large sample distribution of (A.6) under the null hypothesis,  $\rho_1, \rho_2, \rho_3 = 0$ , is that of

$$\min\{e_1, e_2, e_3\}, \min\{e_1, e_2\}, e_1$$

where the  $e$ 's are independent exponentials depending on whether all  $\rho$ 's are 0, two are 0 or just one. The critical value employed is based on the last.

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