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Quasi-Stationary Distributions: Continued Fraction and Chain Sequence Criteria for Recurrence

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Abstract: We study quasi-stationary distributions (q.s.d.) for Markov chains. We review some recent results relating q.s.d., Yaglom limit and limit distributions conditioned to non absorption. For birth-death chains we show new relations between the Markov chain associated to the minimal γ-invariant measure, continued fractions and uniqueness of parameters of chain sequences.

Key words: Quasi-stationary distribution, Yaglon limit, birth and death chain.

1 Introduction

The main result of this work is Theorem 4.1. There we show that the Markov chain associated to the minimal γ-invariant measure is recurrent if and only if some continued fraction is equal to 1 or equivalently if and only if the parameters of some chain sequence are uniquely determined. We use results on continued fractions obtained by (Wall 1948).

In section 2 we introduce γ-invariant measures and the stochastic matrices associated to them. In sections 3, 5 and 6 we review briefly some recent results on quasi-stationary distributions (q.s.d.), and their relations with Yaglom limit and the limit chain conditioned to non-absorption. Section 4 is devoted to the study of q.s.d. on birth-death chains.

2 Stochastic Matrices Related to γ-Invariant Measures

Let $(X_t : t \geq 0)$ be a Markov chain on $\mathbb{N} = \{0\} \cup \mathbb{N}^*$ defined by the transition matrix $P = (p_{xy} : x, y \in \mathbb{N})$. We assume that 0 is an absorbing state, $p_{00} = 1$, and that $P^* = (p_{xy} : x, y \in \mathbb{N}^*)$ is an irreducible matrix.

A non-trivial measure μ on $\mathbb{N}^*$ is γ-subinvariant if $\mu P^* \leq \gamma \mu$ and γ-invariant if $\mu P^* = \gamma\mu$. From irreducibility any γ-subinvariant measure μ is strictly positive.

Consider the series $P^*_{xy}(\gamma) = \sum_{n \geq 0} p^{(n)}_{xy} \gamma^{-n}$. The irreducibility of P^* implies that these series have a common convergence radii $\gamma_0^{-1} \in [1, \infty)$ wich does not depends on x, y. In (Vere-Jones 1962; Seneta 1973) it is shown that there exist γ-subinvariant measures if and only if $\gamma \geq \gamma_0$. The matrix P^* is called γ-transient if $P^*_{xy}(\gamma) < \infty$ and γ-recurrent otherwise. Notice that for any $\gamma > \gamma_0$ the matrix P^* is γ-transient.

Let us relate to a γ-subinvariant measure μ a matrix $M^{(\mu)} = (m_{xy} : x, y \in \mathbb{N}^*)$ defined by

$$m_{xy} = \gamma^{-1} \frac{\mu_y}{\mu_x} p_{yx} \text{ for } x, y \in \mathbb{N}^*. \tag{2.1}$$

$M^{(\mu)}$ is substochastic and it is stochastic if and only if μ is γ-invariant.

From the equality $m^{(n)}_{xy} = \gamma^{-n} \frac{\mu_y}{\mu_x} p^{(n)}_{yx}$, the following equivalence holds:

$$P^*_{xy}(\gamma) < \infty \text{ if and only if } M^{(\mu)}_{xy}(1) = \sum_{n \geq 0} m^{(n)}_{xy} < \infty.$$

Hence the condition at the right does not depends on μ and if

$\gamma > \gamma_0$ the substochastic matrix $M^{(\mu)}$ is transient. If P^* is γ_0-recurrent, there always exists a γ_0-invariant measure, let μ be one of them. P^* is γ_0-positive (resp. γ_0-null) if and only if the stochastic matrix $M^{(\mu)}$ is positive recurrent (resp. null recurrent). Also this last condition does not depend on the γ_0-invariant measure μ and can be characterized only in terms of coefficients p_{xy}, see (Seneta 1973).

If $h > 0$ is a right eigenvector, $P^* h = \gamma h$ for $\gamma > 0$, then we can associate to it the following stochastic matrix $N^{(h)} = (n_{xy})$,

$$n_{xy} = \gamma^{-1} \frac{h_y}{h_x} p_{xy} \text{ for } x, y \in \mathbb{N}^*. \tag{2.2}$$

Now assume that the matrix P^* is reversed by a strictly positive measure $\pi = (\pi_x : x \in \mathbb{N}^*)$ i.e. $\pi_x p_{xy} = \pi_y p_{yx}$ for any $x, y \in \mathbb{N}^*$. Let μ be a γ-invariant measure, then $\frac{\mu}{\pi} = (\frac{\mu_x}{\pi_x} : x \in \mathbb{N}^*)$ is a γ-invariant vector of P^*: $P^* \frac{\mu}{\pi} = \gamma \frac{\mu}{\pi}$ and it is easily shown that $M^{(\mu)} = N^{(\frac{\mu}{\pi})}$.

Now consider the stochastic matrix $M^{(\mu)}$ associated to μ. From reversibility we get:

$$m_{xy} = \gamma^{-1} \frac{\mu_y}{\pi_y} \frac{\pi_x}{\mu_x} p_{xy}, \qquad \text{so } m^{(n)}_{xy} = \gamma^{-n} \frac{\mu_y}{\pi_y} \frac{\pi_x}{\mu_x} p^{(n)}_{xy}$$

Then the matrix $M^{(\mu)}$ is reversed by the measure $\frac{\mu^2}{\pi} = (\frac{\mu_x^2}{\pi_x} : x \in \mathbb{N}^*)$:

$$\frac{\mu_x^2}{\pi_x} m_{xy} = \frac{\mu_y^2}{\pi_y} m_{yx} \quad \text{for any } x, y \in \mathbb{N}^* \tag{2.3}$$

Hence $\frac{\mu^2}{\pi} = (\frac{\mu_x^2}{\pi_x} : x \in \mathbb{N}^*)$ is an invariant measure of $M^{(\mu)} : \frac{\mu^2}{\pi} M^{(\mu)} = \frac{\mu^2}{\pi}$.

Therefore, the matrix P^* is γ_0-positive if and only if $\sum_{x \in S^*} \frac{\mu_x^2}{\pi_x} < \infty$.

3 Quasi-Stationary Distributions

Probability measures which are γ-invariant are called quasi-stationary distributions (q.s.d). They verify:

$$\mu P^{*} = \gamma\mu, \ \mu > 0, \sum_{x \in S^{*}} \mu_x = 1,$$

and simply computations show that $\gamma = \gamma_\mu$ where $\gamma_\mu = 1 - \sum_{y \in \mathbb{N}^{*}} \mu_y p_{y0}$.

The q.s.d. are also characterized as the probability measures μ which satisfy:

$$\mathbb{P}_\mu\{X_t = x | \tau > t\} = \mu_x \quad \forall x \in \mathbb{N}^{*}, \tag{3.1}$$

where $\tau = \inf\{t : X_t = 0\}$ is the time of the first absorption of the chain.

From (3.1) it is easy to prove that if μ is a q.s.d. then the distribution $\mathbb{P}_\mu\{\tau \in \cdot\}$ is geometrically distributed. Observe that $\mathbb{E}_\mu(\gamma^{-\tau}) < \infty$ for any $\gamma > \gamma_\mu$ and it is infinite if $\gamma = \gamma_\mu$. A q.s.d. with $\gamma_\mu = \gamma_0$ is called a minimal q.s.d.

From the geometrical distribution of τ starting from μ, we get that a necessary condition for the existence of q.s.d. is that the chain is geometrically absorbed at 0:

$$\exists \gamma < 1 \text{ such that } \forall x \in \mathbb{N}^{*} \ : \ (1 - p_{x,0}^{(n)}) \le c_x \gamma^n \quad \forall n \ge 0 \text{ and some } c_x > 0.$$

The reciprocal has been recently shown for continuous Markov chains $(X_t : t \ge 0)$ which are honest process and the minimal ones associated to stable and conservative transition rates matrix which also verify
$\mathbb{P}_x\{\tau < t\} \xrightarrow[x \to \infty]{} 0$. More precisely for this kind of chains the condition of geometric absorption $(1 - p_{x,0}^{(t)}) \le c_x \gamma^t$ for any $t \ge 0$, is necessary and sufficient for the existence of q.s.d., see (Ferrari, Kesten, Martínez and Picco 1993).

For birth-death chains the set of γ-invariant measures is a continuous ordered by $\gamma \ge \gamma_0$, see (Cavender 1978) and (Ferrari, Martínez and Picco 1992). In this last work it is also shown, for discrete birth death chains, the equivalence between the existence of q.s.d. and geometrically absorption by using continued fraction techniques. In next section we use these tools to study the recurrence of $M^{(\tilde{\mu})}$ where $\tilde{\mu}$ is the minimal q.s.d. For continuous birth-death chains the equivalence mentioned above was shown in (van Doorn 1991; van Doorn and Schrijner 1992).

4 Birth-Death Chains

Now let us consider the birth-death chain on $\mathbb{N}$ with 0 an absorbing state. Denote $q_x = p_{x,x-1}, \ p_x = p_{x,x+1}$ for $x \in \mathbb{N}^{*}$, we assume that they are strictly positive.

This irreducible chain is reversed by the measure:

$$\pi_1 = 1, \quad \pi_x = \prod_{y=1}^{x-1} \frac{p_y}{q_{y+1}} \text{ for } x \geq 2 \tag{4.1}$$

A measure μ is γ-invariant if

$$\forall x \in \mathbb{N}^* : \quad q_{x+1}\mu_{x+1} + (1 - p_x - q_x)\mu_x + p_{x-1}\mu_{x-1} = \gamma\mu_x.$$

For any γ for which there exists a solution μ it is unique up to the multiplicative constant $\mu_1 > 0$. When the γ-invariant measure μ is finite, it is a probability measure when we fix $\mu_1 = \frac{1}{q_1}(1 - \gamma)$.

In (Ferrari, Martínez and Picco 1992) it was shown that if there exist γ-invariant measure for $\gamma < 1$, then the property of being finites or infinites does not depend on $\gamma < 1$. In fact, it was proven that if $\mathbb{E}_x(\tau) < \infty$ (that is if $\sum_{x \geq 1} (\prod_{y=1}^{x} \frac{p_y}{q_{y+1}}) = \infty$) and if there exists a γ-invariant measure μ with $\gamma_\mu < 1$, then μ is a q.s.d. Reciprocally, if μ is a q.s.d. the chain is geometrically absorbed, so $\mathbb{E}_x(\tau) < \infty$.

Let μ be a γ-invariant measure. Consider the stochastic matrix $M^{(\mu)}$ as in (2.1), so with coefficients $m_{xy} = \gamma^{-1} \frac{\mu_y}{\mu_x} p_{yx}$. It is also a birth-death chain and for the description of its coefficients it is useful to introduce the variables:

$$W_x^{(\gamma)} = \gamma^{-1} \frac{\mu_{x+1}}{\mu_x} q_{x+1}.$$

Then:

$$\begin{aligned} m_{x,x+1} &= W_x^{(\gamma)} \text{ for } x \in \mathbb{N}^*, \\ m_{x,x} &= \gamma^{-1}(1 - p_x - q_x), \\ m_{x,x-1} &= \gamma^{-2} \frac{p_{x-1} q_x}{W_{x-1}^{(\gamma)}} \text{ for } x \geq 2. \end{aligned} \tag{4.2}$$

From the equality $m_{x,x-1} + m_{x,x} + m_{x,x+1} = 1$ it follows:

$$\begin{aligned} W_x^{(\gamma)} &= g_{\gamma,x}(W_{x-1}^{(\gamma)}) \text{ for } x \geq 2, \\ \text{where} \quad g_{\gamma,x}(w) &= 1 - \gamma^{-1}(1 - p_x - q_x) - \gamma^{-2} \frac{p_{x-1} q_x}{w} \text{ for } x \geq 2 \\ \text{and} \quad W_1^{(\gamma)} &= 1 - \gamma^{-1}(1 - p_1 - q_1). \end{aligned} \tag{4.3}$$

A necessary and sufficient conditions for the existence of a γ-invariant measure is that the sequence $(W_x^{(\gamma)} : x \in \mathbb{N}^*)$ is strictly positive. In this case the associated γ-invariant measure is defined by

$$\mu_x^{(\gamma)} = \mu_1^{(\gamma)} \gamma^{x-1} \prod_{y=1}^{x-1} \frac{W_y^{(\gamma)}}{q_{y+1}} \text{ for } x \geq 2 \text{ with } \mu_1^{(\gamma)} > 0.$$

Now, consider

$$\gamma^* = \inf \Gamma, \text{ where } \Gamma = \{\gamma : \text{ there exists } \mu^{(\gamma)} \text{ a } \gamma\text{-invariant measure}\}.$$

Then $\gamma^* > 0$ and the set Γ is either empty or the interval $[\gamma^*, \infty)$.

Let μ be a γ-invariant measure. From (2.3) and (4.1) the birth - death chain $M^{(\mu)}$ is reversed by the measure $\frac{\mu^2}{\pi}$. We have:

$$\frac{\mu_x^2}{\pi_x} = \mu_1^2 \gamma^{2(x-1)} \left(\prod_{y=1}^{x-1} W_y^{(\gamma)} \right)^2 \left(\prod_{y=1}^{x-1} p_y q_{y+1} \right)^{-1},$$

and from classical results on birth - death chains, for instance see (Karlin and Taylor 1975),

$$M^{(\mu)} \text{ is transient if and only if } \sum_{x \in \mathbb{N}^*} \left(\frac{\mu_x^2}{\pi_x} W_x^{(\gamma)} \right)^{-1} < \infty \text{ and as it occurs}$$

with reversed chains, it is positive recurrent if and only if $\sum_{x \in \mathbb{N}^*} \frac{\mu_x^2}{\pi_x} < \infty$.

To analyse the existence of q.s.d. we introduce the sequence $W^{(\gamma)} = (W_x^{(\gamma)} : x \in \mathbb{N}^*)$ defined by $W_x^{(\gamma)} = g_{\gamma,x}(W_{x-1}^{(\gamma)})$ for $x \geq 2$ and $W_1^{(\gamma)} = 1$ and we consider the quantities:

$$h_\gamma(x, y) = g_{\gamma,x}^{-1} \circ \cdots \circ g_{\gamma,y}^{-1}(0), \quad \text{for} \quad y \geq x \geq 2.$$

It can be proven, by using monotonicity arguments, that $W^{(\gamma)}$ is strictly positive if and only if:

$$\begin{aligned} \forall y \geq x \geq 3 \quad &: \quad 0 < h_\gamma(x, y) < 1 - \gamma^{-1}(1 - p_{x-1} - q_{x-1}) \\ \forall y \geq 2 \quad &: \quad 0 < h_\gamma(2, y) < 1. \end{aligned}$$

From this last relation it was shown in (Ferrari, Martínez, Picco 1992) that the equality $\gamma_0 = \gamma^* \in (0, 1]$ holds. Moreover $\Gamma = [\gamma^*, \infty)$ is also verified, in particular Γ is non empty.

In the case $p_x + q_x = 1$ for any $x \in \mathbb{N}^*$ the equations (4.3) take the form:

$$\begin{aligned} W_1^{(\gamma)} &= 1 \text{ and} \\ W_x^{(\gamma)} &= g_{\gamma,x}(W_{x-1}^{(\gamma)}) \text{ for any } x \geq 2 \\ &\quad \text{where } g_{\gamma,x}(\omega) = 1 - \gamma^{-2} \frac{p_{x-1} q_x}{\ } \end{aligned} \tag{4.4}$$

It turns that this form is well-adapted to be studied with techniques on continued functions. In this case $h_\gamma(x,y)$ is the following approximant of a continued fraction:

$$h_\gamma(x,y) = \cfrac{p_{x-1}q_x/\gamma^2}{1 - \cfrac{p_x q_{x+1}/\gamma^2}{1 - \cfrac{p_{x+1}q_{x+2}/\gamma^2}{\ddots \over 1 - p_{y-1}q_y/\gamma^2}}}$$

From previous discussion we get that there exist γ-invariant measures if and only if there exists $\gamma < 1$ such that $h_\gamma(x,y) < 1$ for any $y \geq x \geq 2$. In particular this implies that the continued fraction $h_\gamma(2,\infty) = \lim_{y\to\infty} h_\gamma(2,y)$ is ≤ 1. We shall prove below that the continued fraction $h_{\gamma_0}(2,\infty)$ gives us all the information to analyse the recurrence of the matrix $M^{(\mu)}$ associated to the minimal γ_0-invariant measure.

Before to supply the result we shall introduce another concept that we will relate to γ-invariant measures.

A sequence $(a_x)_{x\geq 2}$ is called a chain sequence if it is of the form $a_x = (1 - \tilde{q}_{x-1})\tilde{q}_x$ for some sequence $(\tilde{q}_x \in [0,1])_{x\in\mathbf{N}^*}$. This last sequence is called the parameters of the chain sequence. The chain sequences were introduced in (Wall 1948) and in the next paragraph we summarize some of the main results that were obtained on them.

Any chain sequence $(a_x)_{x\geq 2}$ has minimal parameters $(\underline{q}_x)_{x\in\mathbf{N}^*}$ and maximal parameters $(\bar{q}_x)_{x\in\mathbf{N}^*}$ such that for any other parameters $(\tilde{q}_x)_{x\in\mathbf{N}^*}$ of the chain sequence: $\underline{q}_x \leq \tilde{q}_x \leq \bar{q}_x$, $x \in \mathbf{N}^*$. The minimal parameters verify

$$\underline{q}_1 = 0,\ \underline{q}_{x+1} = \begin{cases} 0 & \text{if } \underline{q}_x = 1 \\ \frac{a_{x+1}}{1-\underline{q}_x} & \text{if } \underline{q}_x < 1 \end{cases} \text{ for } x \in \mathbf{N}^*;$$

and the maximal parameters are given by the continued fractions

$$\bar{q}_x = 1 - \cfrac{a_{x+1}}{1 - \cfrac{a_{x+2}}{1 - \cfrac{a_{x+3}}{\ddots}}} \text{ for } x \in \mathbf{N}^*.$$

The maximal parameters are equal to the maximal ones if and only if $\sum_{y=1}^{\infty} \prod_{x=2}^{y} \frac{\underline{q}_x}{(1-\underline{q}_x)} = \infty$, or equivalently if and only if $\bar{q}_1 = 0$ (Wall, 1948).

The following relations hold between γ-invariant measures μ and chain sequences. Furthermore the next result also gives a criterion for recurrence of the matrix $M^{(\mu)}$ associated to μ by (2.1).

THEOREM 4.1. *An absorbed birth-death chain has a γ-invariant measure if and only if the sequence $(a_x^{(\gamma)} = \gamma^{-2}(1-q_{x-1})q_x)_{x\geq 2}$, is a chain sequence. Moreover if this condition holds and μ is the γ-invariant measure then the matrix $M^{(\mu)} = (m_{xy})$ associated to μ verifies $m_{x,x-1} = \underline{q}_x$ for $x \in \mathbb{N}^*$, the minimal parameters of the chain sequence $(a_x^{(\gamma)})_{x\geq 2}$.*

For $\gamma > \gamma_0$ the minimal and maximal parameters of the chain sequence are different. For $\gamma = \gamma_0$ the chain induced by $M^{(\tilde{\mu})}$, with $\tilde{\mu} = \mu^{(\gamma_0)}$, is recurrent if and only if the minimal and the maximal parameters of $(a_x^{(\gamma_0)})_{x\geq 2}$ are equal, and this holds if and only if $h_{\gamma_0}(2,\infty) = 1$. When $M^{(\tilde{\mu})}$ is recurrent then $W_x^{(\gamma_0)} = h_{\gamma_0}(x+1,\infty)$ for $x \in \mathbb{N}^*$, so $\tilde{\mu} = \mu^{(\gamma_0)}$ can be computed as simple continued fractions.

Proof. From (4.2), (4.4) it follows that if μ is a γ-invariant measure and $M^{(\mu)} = (m_{xy})$ is its induced transition matrix, then $a_x^{(\gamma)} = m_{x,x-1}m_{x-1,x}$. Set $\tilde{q}_x = m_{x,x-1}$ for $x \in \mathbb{N}^*$, so $m_{x-1,x} = (1-\tilde{q}_{x-1})$ and $a_x^{(\gamma)} = (1-\tilde{q}_{x-1})\tilde{q}_x$, then $(a_x^{(\gamma)})_{x\geq 2}$ is a chain sequence. Since $\tilde{q}_1 = m_{10} = 1 - m_{12} = 0$ we get that $\tilde{q}_x = m_{x,x-1}$ with $x \in \mathbb{N}^*$, are the minimal parameters of the chain sequence.

Reciprocally if $(a_x^{(\gamma)})_{x\geq 2}$ is a chain sequence we can write $a_x^{(\gamma)} = (1-\tilde{q}_{x-1})\tilde{q}_x$ for some parameters $(\tilde{q}_x)_{x\in\mathbb{N}^*}$. Hence:

$$(1-\tilde{q}_x) = 1 - \frac{a_x^{(\gamma)}}{1-\tilde{q}_{x-1}} = 1 - \frac{\gamma^{-2}(1-q_{x-1})q_x}{(1-\tilde{q}_{x-1})}.$$

Consider $(\underline{q}_x)_{x\in\mathbb{N}^*}$ the minimal parameters of the chain sequence and call $W_x^{(\gamma)} = 1 - \underline{q}_x$. Then it verifies the dynamical equation $W_x^{(\gamma)} = 1 - \gamma^{-2}\frac{(1-q_{x-1})q_x}{W_{x-1}^{(\gamma)}}$, with initial condition $W_1^{(\gamma)} = 1 - \underline{q}_1 = 1$. From unicity of solution of (4.2) we deduce $m_{x,x+1} = 1 - \underline{q}_x$, $m_{x,x-1} = \underline{q}_x$ for $x \in \mathbb{N}^*$. Then the first part of the theorem holds.

For $\gamma > \gamma_0$ the chain induced by the matrix $M^{(\mu)}$, with $\mu = \mu^{(\gamma)}$, is transient. From the usual criterion on birth-death chains (see Karlin and Taylor 1975) we get that the transient property is equivalent to $\sum_{y=1}^{\infty}\prod_{x=2}^{y}\frac{m_{x,x-1}}{m_{x,x+1}} = \infty$. Since $m_{x,x-1} = \underline{q}_x$, $m_{x,x+1} = 1 - \underline{q}_x$ we deduce, from the referred result of (Wall 1948), that the minimal parameters of $(a_x^{(\gamma)})_{x\geq 2}$ are different from the maximal ones.

For $\gamma = \gamma_0$ the last discussion gives that the chain induced by $M^{(\tilde{\mu})}$ is recurrent if and only if the maximal and the minimal parameters of $(a_x^{(\gamma_0)})_{x\geq 2}$ are equal. Consider $M^{(\tilde{\mu})} = (m_{xy})$ and denote $\tilde{q}_x = m_{x,x-1}, \tilde{p}_x = 1-\tilde{q}_x$. Since $a_x^{(\gamma_0)} = \tilde{p}_{x-1}\tilde{q}_x$ we get:

$$h_{\gamma_0}(2,\infty)=\cfrac{p_1q_2\gamma_0^{-2}}{1-\cfrac{p_2q_3\gamma_0^{-2}}{\ddots\cfrac{}{1-\cfrac{p_xq_{x+1}\gamma_0^{-2}}{\ddots}}}}=\cfrac{\tilde p_1\tilde q_2}{1-\cfrac{\tilde p_2\tilde q_3}{\ddots\cfrac{}{1-\cfrac{\tilde p_x\tilde q_{x+1}}{\ddots}}}}$$

From results of (Scott and Wall 1940) and more precisely Theorem 11.2 of (Wall 1948), we get:

$$h_{\gamma_0}(2,\infty)=1-\frac{\tilde q_1}{1-\tilde S^{-1}}\ \text{where}\ \tilde S=1+\sum_{\ell=0}^{\infty}\prod_{r=0}^{\ell}\frac{\tilde q_{1+r}}{\tilde p_{1+r}}$$

Now, the chain given by matrix M is recurrent if and only if $\tilde S=\infty$, which is equivalent to $h_{\gamma_0}(2,\infty)=1-\tilde q_1$. Since $\tilde p_1=1$, we get the result.

Finally, for any $\gamma\geq\gamma_0$ the following equations are verified by the continued fractions

$$h_\gamma(x,\infty)=\frac{\tilde p_{x-1}\tilde q_x}{1-h_\gamma(x+1,\infty)},\ \text{so}\ h_\gamma(x+1,\infty)=1-\frac{p_{x-1}q_x}{\gamma^2h_\gamma(x,\infty)}\ \text{for}\ x\geq 2.$$

Let $\tilde\mu=\mu^{(\gamma_0)}$. If the chain associated to $M^{(\tilde\mu)}$ is recurrent, then $h_{\gamma_0}(2,\infty)=1$ and the result holds because $W_x^{(\gamma_0)}$ and $V_x=h_{\gamma_0}(x+1,\infty)$ obey the same evolution equations with the same initial condition $W_1^{(\gamma_0)}=V_1=1$. ■

Example. Random Walks. In this case $q_x=q, p_x=p=1-q\quad\forall x\in\mathbb{N}^*$. Consider the non symmetric case, $q\neq\frac{1}{2}$. We have $\gamma_0=2\sqrt{q(1-q)}$ and $g_{\gamma_0,x}(w)=1-\frac{1}{4w}$. Then $h_{\gamma_0}(2,\infty)=\frac{1}{2}$ and $\mu^{(\gamma_0)}$ is transient.

5 Relation with Yaglom Limits

Intimately related to q.s.d. is the concept of Yaglom limit. Let us define it. Denote by d the period of the Markov chain (X_t) and consider $C(x,r)=\{y:p_{xy}^{(nd+r)}>0\ \text{for some}\ n\}$. The measure μ^Y is the Yaglom limit of (X_t) if the following limit exists,

$$\mu_y^Y=\lim_{n\to\infty}\frac{p_{xy}^{(nd+r)}}{1-p_{x0}^{(nd+r)}}\ \text{for}\ y\in C(x,r),$$

if it does not depends on $x \in \mathbf{N}^*$ and $\sum_{y \in C(x,r)} \mu_y^Y = 1$ for any $0 \leq r < d$.

The existence of this limit was established for branching process by (Yaglom 1947), for finite chains by (Darroch and Seneta 1965) and for birth-death chains by (Good 1968; Kijima and Seneta 1991; van Doorn 1991).

If P^* is also aperiodic and such that for any $y \in \mathbf{N}$ the set $\{x \in \mathbf{N}^* : p_{xy} > 0\}$ is finite, then if the Yaglom limit μ^Y exists, it is a q.s.d. In Proposition 6.1 of (Ferrari, Kesten, Martínez and Picco 1993) it was shown that μ^Y is in this case, the minimal q.s.d.

Let X_t be a birth-death chain with 0 an absorbing state and such that $p_x + q_x = 1$, $\forall x \in \mathbf{N}^*$. From solidarity property arguments, it is easy to show that if the following quantities exist

$$\mu_y^Y = \lim_{\substack{n\to\infty \\ n+y-1 \text{ even}}} \frac{p_{1y}^{(n)}}{1 - p_{10}^{(n)}} \text{ for any } y \in \mathbf{N}^*$$

and $\sum_{y \text{ odd}} \mu_y^Y = \sum_{y \text{ even}} \mu_y^Y = 1$, then the measure μ^Y is the Yaglom limit of the birth-death chain. For the absorbed random walk $q_x = q > \frac{1}{2}$ constant for $x \in \mathbf{N}^*$, it was shown in (Seneta and Vere-Jones 1966) that the Yaglom limit exists, being

$$\mu_y^Y = y\left(\sqrt{\frac{p}{q}}\right)^{y-1}\left(\frac{1-4pq}{q}\right) \text{ if } y \text{ is odd,}$$
$$\mu_y^Y = y\left(\sqrt{\frac{p}{q}}\right)^{y-1}\left(\frac{1-4pq}{q}\right)\frac{1}{2\sqrt{pq}} \text{ if } y \text{ is even.}$$

Now, for the random walk the minimal q.s.d. is:

$$\tilde{\mu}_y = y\left(\sqrt{\frac{p}{q}}\right)^{y-1}\frac{1-2\sqrt{pq}}{q}$$

Due to the fact that the period of the chain is $d = 2$ the Yaglom limit μ^Y is not a q.s.d. In order to analyse, for birth-death chains the relation between μ^Y and the minimal q.s.d. $\tilde{\mu}$ it is useful the following observations.

Denote τ^x first absorption time when the chain starts at x. $\{\tau^x = n\}$ is non-empty if and only if $n + x$ is even. Then:

$$\text{if } n + x \text{ is odd, } \frac{\mathbf{P}_x\{\tau > n-1\}}{\mathbf{P}_x\{\tau > n\}} = 1$$

When $n + x$ is even and the Yaglom limit μ^Y exists the following relation hold:

$$\lim_{\substack{n\to\infty \\ n+x \text{ even}}} \frac{\mathbf{P}_x\{\tau > n-1\}}{\mathbf{P}_x\{\tau > n\}} = \frac{1}{1 - q_1\mu_1^Y}$$

By using these relations in (Martínez and Vares 1992) it was shown that if the Yaglom limit μ^Y exists and if we define $\gamma_Y = \sqrt{1 - q_1\mu_1^Y}$ and

$$\tilde{\mu}_x = \frac{1}{1+\gamma_Y}\mu_x^Y \text{ if } x \text{ is odd and } \tilde{\mu}_x = \frac{\gamma_Y}{1+\gamma_Y}\mu_x^Y \text{ if } x \text{ is even,}$$

then $\tilde{\mu}$ is a q.s.d. and $\gamma_\mu = 1 - q_1\mu_1 = \gamma_Y$. Moreover $\tilde{\mu}$ is the minimal q.s.d. so $\gamma_{\tilde{\mu}} = \gamma_0$.

Kesten has recently made a big progress in the study of Yaglom limit. He has proved that for a wide class of substochastic matrices P^* there exists the Yaglom limit. The matrices P^* studied in (Kesten 1993) verify the following hypotheses:

(K1) $\exists L < \infty$ such that $p_{xy} = 0$ if $|x-y| > L$, with $x, y \in \mathbb{N}$;

(K2) $\exists \delta_0 > 0$, $M < \infty$ such that for any $x \in \mathbb{N}^*$ there exist $r \geq 1$ and $1 \leq k_1, ..., k_r \leq M$ depending on x, verifying $p_{xx}^{(k_s)} \geq \delta_0$ for $s = 1, ..., r_x$ and g.c.d. $(k_1, ..., k_r) = 1$;

(K3) $\exists \delta_1 > 0, N < \infty$ such that for any $x \in \mathbb{N}^*$ there exist $1 \leq n, m \leq N$ depending on x, verifying $p_{x,x+1}^{(n)} \geq \delta_1$ and $p_{x,x-1}^{(m)} \geq \delta_1$;

(K4) $\exists \gamma < 1$ such that $\mathbb{E}_x(\gamma^{-\tau}) < \infty$ for some $x \in \mathbb{N}^*$ (then for all $x \in \mathbb{N}^*$).

Kesten proved that conditions (K1-K3) imply the existence of a measure μ and a function h, both strictly positive, such that

$$\mu P^* = \gamma_0 \mu, \quad P^* h = \gamma_0 h.$$

When (K4) is also satisfied then μ can be taken as a probability measure, so $\gamma = 1 - \sum_{z \in \mathbb{N}^*} \mu_y p_{z0}$ and μ is the Yaglom limit,

$$\lim_{n\to\infty} \frac{p_{xy}^{(n)}}{1 - p_{x0}^{(n)}} = \mu_y \text{ for any } y \in \mathbb{N}^*$$

6 Non Absorbed Conditioned Process

Let us introduce the general framework. Consider a Markov chain (X_t) taking values on $\mathbb{N}$ with 0 an absorbing state.

For $n > k \geq 1$ we have:

$$\mathbb{P}\{X_k = x_k, X_{k-1} = x_{k-1}, \ldots, X_1 = x_1 | \tau > n, X_0 = x_0\} =$$
$$\prod_{\ell=1}^{k} \frac{\mathbb{P}_{x_\ell}\{\tau > n - \ell\}}{\mathbb{P}_{x_{\ell-1}}\{\tau > n - \ell + 1\}} p_{x_{\ell-1}, x_\ell}$$

From this relation it was shown in (Martínez and Vares 1992) that if there exists $\lim_{n\to\infty} \frac{\mathbf{P}_y\{\tau>n-1\}}{\mathbf{P}_x\{\tau>n\}}$ for any couple x, y such that $p_{xy} > 0$ then $\lim_{n\to\infty} \mathbf{P}\{X_0 = x_0, ..., X_k = x_k | \tau > n\}$ exists.

Set

$$\hat{m}_{xy} = \left(\lim_{n\to\infty} \frac{\mathbf{P}_y(\tau > n-1)}{\mathbf{P}_x(\tau > n)} \right) p_{xy}.$$

Then if the matrix $\hat{M} = (\hat{m}_{xy})_{x,y\in\mathbf{N}^*}$ is stochastic the limit distribution $\lim_{n\to\infty} \mathbf{P}_{x_0}(X_1 = x_1, ..., X_k = x_k | \tau > n)$ defines a Markov chain on the $\mathbf{N}^*$, with transition matrix $\hat{M}$ and initial state x_0. A particular situation where $\tilde{M}$ is stochastic occurs when for each $x \in \mathbf{N}^*$ the set $\{y \in \mathbf{N}^* : p_{xy} > 0\}$ is finite.

If P^* is aperiodic, reversed by some measure $\pi > 0$, the Yaglom limit μ^Y exists and $\{x \in \mathbf{N}^* : p_{xy} > 0\}$ is finite for any $y \in \mathbf{N}^*$, then $\hat{M} = M^{(\mu^Y)}$. For birth-death chains it can be also proven that if μ^Y exists then $\hat{M} = M^{(\tilde{\mu})}$ where $\tilde{\mu}$ is the minimal q.s.d. All these results were shown in (Martínez and Vares 1992).

In all these results the reversibility plays an important role. To clarify this fact recall that under π-reversibility, if μ a γ-invariant measure the function $h = \frac{\mu}{\pi}$ is a γ-right eigenvector and $N^{(h)} = M^{(\mu)}$. Hence, if P^* can be reversed the matrices associated to right or left eigenvectors are the same.

When there is no reversibility the matrix $N^{(h)}$ is the relevant one. In fact it follows from the computations made in (Kesten 1993) that for the matrices verifying conditions (K1) - (K4) the transition matrix $\hat{M}$ defining the limit distribution conditioned to non absorption is $\hat{M} = N^{(h)}$, where h is the γ_0-right eigenvector.

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