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# Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces[1]

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**Abstract:** We prove infinite-dimensional versions of the shadowing lemma and Smale's theorem ( for a transverse homoclinic orbit ) of a $C^1$ map, not a diffeomorphism, using the notion of an exponential dichotomy.

**Key words:** Exponential dichotomy, homoclinic orbit, shadowing lemma.

# Introduction

The title shows our indebtedness to Palmer's articles [12, 13]. Based on the notion of exponential dichotomies we prove infinite-dimensional versions of the shadowing lemma and Smale's theorem for a transverse homoclinic orbit of a $C^1$ map (not a diffeomorphism).

Blazquez [1] gives a shadowing lemma, Theorem 4.2, which would be interesting if it were proved. Chow, Lin and Palmer [3] prove an infinite dimensional shadowing lemma with a special notion of "hyperbolicity"; ours is a natural extension of that of Palmer [13].

We will need many results about exponential dichotomies, which are treated in Section 1. Some results are merely quoted from [7], but others – some new, some appearing only as exercises in [7], along with versions of results of Palmer [12] and Lin [10] – are completely proved. In fact, the treatment of dichotomies is more extensive than is strictly necessary here; I couldn't resist the temptation, and anyway I hope to extend also some results of Melnikov, Shilnikov and Deng in later publications.

# 1 Exponential dichotomies

Let $X$ be a Banach space, $J \subset \mathbb{R}$ an interval and $\{T(t,s); t \geq s \text{ in } J\} \subset \mathcal{L}(X)$ a family of **evolution operators**, i.e.,

$$T(s,s) = I \ , \quad T(t,s)T(s,r) = T(t,r) \ \text{ for } t \geq s \geq r \text{ in } J \ . \tag{1}$$

Sometimes we assume $\sup\{\|T(t,s)\| : 0 \leq t-s \leq 1\} < \infty$ and sometimes we assume $(t,s) \mapsto T(t,s)$ is strongly continuous; any such assumption is explicitly stated when needed.

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**Definition 1.1** *A family of evolution operators* $\{T(t,s); t \geq s \text{ in } J\}$ *has an* **exponential dichotomy** *(on $J$, with exponent $\beta$, bound $M$ and projections $P(t)$, $t \in J$) if there are constants $\beta > 0$, $M \geq 1$ and projections $P(t) = P(t)^2 \in \mathcal{L}(X)$ for $t \in J$ such that:*

*(i)* $T(t,s)P(s) = P(t)T(t,s)$ *for* $t \geq s$ *in* $J$;

*(ii) the restriction* $T(t,s)|\mathcal{R}(P(s)) \to \mathcal{R}(P(t))$ *is an isomorphism (bicontinuous bijection) for $t \geq s$ in $J$, and $T(s,t)$ is defined as the inverse from $\mathcal{R}(P(t))$ onto $\mathcal{R}(P(s))$;*

*(iii)* $\|T(t,s)(I - P(s))\| \leq Me^{-\beta(t-s)}$ *for* $t \geq s$ *in* $J$;

*(iv)* $\|T(t,s)P(s)\| \leq Me^{-\beta(s-t)}$ *for $t \leq s$ in $J$, where $T(t,s)P(s)$ is defined in (ii).*

*If* $\dim \mathcal{R}(P(t)) = m < \infty$ *for some $t \in J$, equality holds for all $t \in J$, by (ii), and we say the dichotomy has* **rank** *$m$. We sometimes call $\mathcal{R}(P(t)) = U(t)$ the* **unstable space** *and $\mathcal{N}(P(t)) = S(t)$ the* **stable space**.

**Remarks:** We only deal with *exponential* dichotomies and often say merely *dichotomy.* Lin [10], among others authors, requires $t \mapsto P(t)$ to be strongly continuous; this follows from strong continuity of the evolution operators, as we show in 1.12.

We have $\|P(t)\| \leq M$. Defining the **angle** $\langle\!\langle (E,F)$ between nonzero subspaces $E$, $F$, with $E \cap F = \{0\}$ by

$$\langle\!\langle (E,F) = \inf\{|e - f| : e \in E, f \in F, |e| = 1 = |f|\} ,$$

it is easy to see, for any non-trivial projection $P$, that $2/\|P\| \geq \langle\!\langle (\mathcal{R}(P), \mathcal{N}(P)) \geq 1/\|P\|$. Thus $\langle\!\langle (\mathcal{R}(P(t)), \mathcal{N}(P(t))) \geq 1/M$ for all $t \in J$, and the assumption to this effect in [8] is unnecessary. (In a Hilbert space, there is a geometrically natural angle $\theta(E,F)$, and $\langle\!\langle (E,F) = 2\sin\frac{1}{2}\theta(E,F)$.)

In general, the projection of a dichotomy is not unique. If $J \supset [\tau, \infty)$ for some $\tau$, the stable subspace $S(t) = \mathcal{N}(P(t))$, $t \geq \tau$, is unique: $S(t) = \{x | T(\theta,t)x \to 0$ [or, is bounded] as $\theta \to +\infty\}$. If $J \supset (-\infty, \tau]$ for some $\tau$, $U(t) = \mathcal{R}(P(t))$ is unique for $t \leq \tau$:

$$\begin{aligned} U(t) \quad = \quad & \{x | \text{ there is a bounded } \varphi : (-\infty, t] \to X \\ & \text{with } \varphi(t) = x \text{ and } \varphi(s) = T(s,r)\varphi(r) \text{ when } r \leq s \leq t\} . \end{aligned}$$

In this case, the "backward coninuation" $\varphi$ is unique, $\varphi(s) \in \mathcal{R}(P(s))$, and $\varphi(s) \to 0$ as $s \to -\infty$. If $J = \mathbb{R}$, the projection is uniquely determined.

Hale and Lin [6] define a **trichotomy** for $T$, which is equivalent to saying $\{e^{\lambda(t-s)}T(t,s) : t \geq s \text{ in } J\}$ has a dichotomy for both $\lambda = \pm\varepsilon$, some $\varepsilon > 0$, with different projections. (If the projections were equal, $T$ would have a dichotomy.)

**Examples**

(1) If $\{e^{At}, t \geq 0\} \subset \mathcal{L}(X)$ is a strongly continuous semigroup and we define $T(t,s) = e^{A(t-s)}$ for $t \geq s$, for any interval $J \subset \mathbb{R}$ $\{T(t,s), t \geq s \text{ in } J\}$ is a family of evolution operators. If also $\sigma(e^{At_0}) \cap S^1 = \emptyset$ for some (hence, every) $t_0 > 0$, define the projection $P$ by

$$I - P = \frac{1}{2\pi i} \int_{|\mu|=1} (\mu - e^{At_0})^{-1} \mathrm{d}\mu \ ;$$

Then $e^{At}P = Pe^{At}$ and we have an exponential dichotomy in $J$ with projection $P(t) = P$ constant. If $\beta > 0$ and $\sigma(e^{At_0}) \cap \{\mu : e^{-\beta t_0} \leq |\mu| \leq e^{\beta t_0}\} = \emptyset$, we may suppose the exponent is $\beta$. If the essential spectrum of $e^{At_0}$ is strictly inside the unit circle, $r_{ess}(e^{At_0}) < 1$, the dichotomy has finite rank.

(2) Suppose $A$ is the generator of a strongly-continuous semigroup on $X$, $B : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous with $B(t+p) = B(t)$ for all $t$ and fixed $p > 0$. Let $\{T(t,s), t \geq s\} \subset \mathcal{L}(X)$ be the family of evolution operators such that $x(t) = T(t,s)x(s)$ when $t \geq s$ and $x(\cdot)$ is a mild solution of $\dot{x} = Ax + B(\cdot)x$ in $[s,t]$. Then for $t \geq s$ $T(t+p, s+p) = T(t,s)$ and $\sigma(T(s+p,s)) \backslash \{0\}$ is independent of $s$ (Lemma 7.2.2 of [7]).

Suppose $\sigma(T(s+p,s)) \cap S^1 = \emptyset$ for some (hence, every) $s \in \mathbb{R}$ and define

$$I - P(t) = \frac{1}{2\pi i} \int_{|\mu|=1} (\mu - T(t+p,t))^{-1} \mathrm{d}\mu \ ;$$

then $P(t)^2 = P(t) = P(t+p)$ for all $t$, $T(t,s)P(s) = P(t)T(t,s)$ for $t \geq s$. For any interval $J \subset \mathbb{R}$, $\{T(t,s), t \geq s \text{ in } J\}$ has an exponential dichotomy with projections $\{P(t)\}_{t \in J}$; in this case, $t \mapsto P(t)$ is strongly continuous. If $r_{ess}(T(t+p,t)) < 1$, the dichotomy has finite rank. (Most of the argument for this is in 7.2.3 of [7].)

(3) If $\|T(t,s)\| \leq Me^{-\beta(t-s)}$ for $t \geq s$ in $J$, and some $\beta > 0$, we have a trivial dichotomy with projection zero.

The theory is much simpler with discrete time and we see, in Theorem 1.3, there is little loss in restricting attention to this case.

If $\widehat{J}$ is an "interval" in $\mathbb{Z}$, $\{T_n\}_{n \in \widehat{J}} \in \mathcal{L}(X)$, define

$$T_{m,m} = I \ , \quad T_{n,m} = T_{n-1} \circ \cdots \circ T_{m+1} \circ T_m \quad \text{for } n > m \tag{2}$$

when $m$ and $n-1$ are in $\widehat{J}$; then $T_{n,m}T_{m,l} = T_{n,l}$ for $n \geq m \geq l$ with $l$ and $n-1$ in $\widehat{J}$. Let $\widehat{J}^+ = \widehat{J}$ if $\widehat{J}$ is not bounded above, $\widehat{J}^+ = \widehat{J} \cup \{1 + \max \widehat{J}\}$ otherwise, so $T_{n,m}$ is well defined for $n \geq m$ in $\widehat{J}^+$.

**Definition 1.2** *If $\widehat{J}$ is an interval in $\mathbb{Z}$, $\{T_n : n \in \widehat{J}\} \subset \mathcal{L}(X)$ and we define $\{T_{n,m} | n \geq m \text{ in } \widehat{J}^+\}$ as in (2) above, then $\{T_n\}_{n \in \widehat{J}}$ has a* **discrete dichotomy** *(with constants $M \geq 1$, $\theta \in (0,1)$ and projections $P_n$, $n \in \widehat{J}^+$) if:*

*(i)* $T_n P_n = P_{n+1} T_n$ *for* $n \in \widehat{J}$;

*(ii) the restriction* $T_n | \mathcal{R}(P_n) \to \mathcal{R}(P_{n+1})$ *is an isomorphism for* $n \in \widehat{J}$;

*(iii)* $\|T_{n,m}(I - P_n)\| \le M\theta^{n-m}$ *for* $n \ge m$ *in* $\widehat{J}^+$;

*(iv)* $\|T_{n,m} P_m\| \le M\theta^{m-n}$ *for* $n \le m$ *in* $\widehat{J}^+$, *where* $T_{n,m} P_m x = y \in \mathcal{R}(P_n)$ *is defined by* $P_m x = T_{m,n} y$ *(and well-defined, by (ii)).*

**Remarks:** We often say merely "dichotomy" when discreteness is evident. We have $T_{n,m} P_m = P_n T_{n,m}$ for all $n \ge m$ in $\widehat{J}^+$ and $T_{n,m} | \mathcal{R}(P_m) \to \mathcal{R}(P_n)$ is an isomorphism.

If we define $\widetilde{T}(t,s) = T_{n,m}$ when $t \in [t_n, t_{n+1})$, $s \in [t_m, t_{m+1})$, $t \ge s$ [$t_k = t_0 + kp$ for some fixed $p > 0$, $t_0 \in \mathbb{R}$] and if $\widetilde{J} = \bigcup_{k \in \widehat{J}} [t_k, t_{k+1})$, then $\{\widetilde{T}(t,s), t \ge s$ in $\widetilde{J}\}$ is a family of evolution operators and it has a dichotomy (with exponent $\beta$ and bound $M$) if and only if $\{T_n : n \in \widehat{J}\}$ has a dichotomy with constants $M$ and $\theta = e^{-\beta p}$.

It is clear that, for any family of evolution operators $\{T(t,s) | t \ge s \text{ in } J\}$, $t_0 \in J$ and $p > 0$, if we have an exponential dichotomy with exponent $\beta$, bound $M$ and projections $P(t)$, and if $T_n = T(t_{n+1}, t_n)$, $t_n = t_0 + np$, $\bigcup_{n \in \widehat{J}} [t_n, t_{n+1}) \subset J$, then $\{T_n : n \in \widehat{J}\}$ has a dichotomy with constants $M$, $\theta = e^{-\beta p}$ and projections $P_n = P(t_n)$.

The converse also holds provided $\sup\{\|T(t,s)\| : 0 \le t - s \le 1\} < \infty$. The following is a stronger version of Exercise 10, Section 7.6 of [7].

**Theorem 1.3** *Let* $\{T(t,s) | t \ge s \text{ in } J\} \subset \mathcal{L}(X)$ *be a family of evolution operators with* $\sup\{\|T(t,s)\| : 0 \le t - s \le 1\} < \infty$, $J$ *a closed interval,* $p > 0$, $t_n = t_0 + np$ *and* $\widehat{J} \subset \mathbb{Z}$ *an interval such that* $\bigcup_{n \in \widehat{J}} [t_n, t_{n+1}) = J$ *(or* $J \backslash \{\max J\}$, *if* $J$ *is bounded above). Let* $T_n = T(t_{n+1}, t_n)$ *for* $n \in \widehat{J}$.

*If* $\{T_n : n \in \widehat{J}\}$ *has a discrete dichotomy with constants* $M \ge 1$, $\theta = e^{-\beta p} \in (0,1)$ *and projections* $\{P_n, n \in \widehat{J}^+\}$, *then* $\{T(t,s), t \ge s \text{ in } J\}$ *has an exponential dichotomy with exponent* $\beta$, *bound* $M'$, *and projections* $\{P(t), t \in J\}$ *such that* $P(t_n) = P_n$ *for* $n \in \widehat{J}^+$. *Writing* $K_p = \sup\{\|T(t,s)\| : 0 \le t - s \le p\}$, *we have* $K_p \le K_1^{p+1}$ *and may use*

$$M' = \max(K_p^2 M \theta^{-2}, K_p^2 M^2 + K_p \theta^{-1}) \ .$$

*The projections for the "interpolated" dichotomy are uniquely determined by the* $\{P_n, n \in \widehat{J}^+\}$ *and* $\{T(t,s), t \ge s \text{ in } J\}$, *when we require* $P(t_n) = P_n$.

**Proof:** Let $K = \sup\{\|T(t,s)\| : 0 \le t - s \le p \text{ in } J\}$. If $t \in [t_n, t_{n+1}] \subset J$, define $X(t) = T(t, t_n)\mathcal{R}(P_n)$, so $X(t_n) = \mathcal{R}(P_n)$, $X(t_{n+1}) = \mathcal{R}(P_{n+1})$. Then

$$T_n \Big|_{\mathcal{R}(P_n) \to \mathcal{R}(P_{n+1})} = T(t_{n+1}, t) \Big|_{X(t) \to \mathcal{R}(P_{n+1})} \circ T(t, t_n) \Big|_{\mathcal{R}(P_n) \to X(t)} . \qquad (3)$$

By definition, $T(t, t_n)|\mathcal{R}(P_n) \to X(t)$ is surjective, and it is injective by (3) and condition (ii) for a discrete dichotomy, so both factors on the right side of (3) are continuous bijections. Furthere, if $y \in \mathcal{R}(P_n)$, $y = T_{n,n+1}P_{n+1}x$ for some $x$ and $|y| \leq M\theta|P_{n+1}x|$ by condition (iv), $P_{n+1}x = T_n y$ and $|y| \leq M\theta|T_n y| \leq KM\theta|T(t, t_n)y|$ for all $y \in \mathcal{R}(P_n)$. Thus both factors on the right side of (3) are isomorphisms, $X(t)$ is a closed space and

$$\|(T(t_{n+1}, t)|X(t) \to \mathcal{R}(P_{n+1}))^{-1}\| \leq KM\theta ,$$

$$\|(T(t, t_n)|\mathcal{R}(P_n) \to X(t))^{-1}\| \leq KM\theta .$$

For $t \in [t_n, t_{n+1}]$, define

$$P(t) = (\text{inclusion } X(t) \subset X) \circ (T(t_{n+1}, t)|X(t) \to \mathcal{R}(P_{n+1}))^{-1} \circ P_{n+1} \circ T(t_{n+1}, t) . \tag{4}$$

It is then easy to show that $\mathcal{R}(P(t)) = X(t)$, $P(t)^2 = P(t) \in \mathcal{L}(X)$, $P(t_{n+1}) = P_{n+1}$, $P(t_n) = P_n$ and $\|P(t)\| \leq K^2M^2\theta$. If $t \geq s$ are in $[t_n, t_{n+1}]$, $T(t, s)X(s) = X(t)$ by definition and

$$T(t_{n+1}, s)\Big|_{X(x) \to \mathcal{R}(P_{n+1})} = T(t_{n+1}, t)\Big|_{X(t) \to \mathcal{R}(P_{n+1})} \circ T(t, s)\Big|_{X(s) \to X(t)}$$

so $T(t, s)|X(s) \to X(t)$ is also an isomorphism. Further, by (4),

$$\begin{aligned} T(t_{n+1}, t)\Big|_{X(t)} \circ T(t, s)P(s) &= T(t_{n+1}, s)P(s) = P_{n+1}T(t_{n+1}, t) \circ T(t, s) \\ &= T(t_{n+1}, t)\Big|_{X(t)} \circ P(t)T(t, s) \end{aligned}$$

so $T(t, s)P(s) = P(t)T(t, s)$. The equality holds for any $t \geq s$ in $J$, by an easy calculation, so (i) and (ii) of Definition 1.1 hold.

Verification of (iii) and (iv) is now straight-forward. For example, if $t \geq s$, $t \in [t_{n+1}, t_n]$, $s \in [t_{m+1}, t_m]$ with $n \geq m$, $y = T(s, t)P(t)x$, we have

$$y = (T(t_{n+1}, s)\Big|_{X(s) \to \mathcal{R}(P_{n+1})})^{-1} T_{m+1, n+1}P_{n+1}T(t_{n+1}, t)x$$

so $|y| \leq M^2K^2\theta^{n-m+1}|x| \leq M^2K^2e^{-\beta(t-s)}|x|$, proving (iv).

Most of the following results treat only discrete dichotomies, and often with $\hat{J} = \mathbb{Z}$ so the projections are uniquely determined.

**Theorem 1.4** *Let $\{T_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$; then the following are equivalent.*

*(i) $\{T_n\}_{-\infty}^{\infty}$ has discrete dichotomy.*

*(ii) For every bounded sequence $\{f_n\}_{-\infty}^{\infty} \subset X$, there is a unique bounded sequence $\{x\}_{-\infty}^{\infty} \subset X$ with $x_{n+1} = T_n x_n + f_n$ for all $n$.*

**Proof:** See [7], Theorem 7.6.5.

**Remarks:** The unique bounded solution is $x_n = \sum_{-\infty}^{\infty} G_{n,k+1} f_k$ where $G_{n,m} = T_{n,m}(I - P_m)$ for $n \geq m$, $G_{n,m} = -T_{n,m} P_m$ for $n < m$ so $\|G_{n,m}\| \leq M\theta^{|n-m|}$; the double sequence $\{G_{n,m}\}$ is the Green function.

O. Perron [14], for ordinary differential equations on $\mathbb{R}_+$, and T. Li [9], for difference equations on $\mathbb{Z}_+$, obtained analogous conditions for finite dimensions, though the exponential bounds were not recognized until 1954 (Maizel). These results were greatly generalized by Massera and Schäffer [11]. Coffman and Schäffer [4] treated infinite-dimensional difference equations on $\mathbb{Z}_+$, with a more general notion of dichotomy. Slyusharchuk [16] gives a result like 1.4 (with partial proof) when $T_n \in \mathcal{L}(X_n, X_{n+1})$, the spaces depending on $n$.

**Simple examples** (with $X = \mathbb{C}$).

(1) $T_n = a$, $|a| \neq 1$; the only bounded solution of $x_{n+1} = ax_n + f_n$ [$f$ bounded] is $x_n = \sum_{n}^{\infty} a^{n-k-1} f_k$, if $|a| > 1$, or $x_n = \sum_{-\infty}^{n-1} a^{n-k-1} f_k$, if $|a| < 1$.

(2) $T_n = a$, $|a| = 1$: $x_n = a^n$ is a bounded non-trivial solution of $x_{n+1} = ax_n$, so there is no dichotomy. If $x_{n+1} - T_n x_n = a^n$ $(\forall n)$ then $x_n = x_0 a^n + na^{n-1}$ is unbounded for any $x_0$.

(3) $T_n = 2$ $(n \geq 0)$, $T_n = \frac{1}{2}$ $(n < 0)$: $\{T_n\}_{n \geq 0}$ and $\{T_n\}_{n \leq 0}$ both have dichotomies, but there is no dichotomy on all $\mathbb{Z}$ since $x_{n+1} - T_n x_n = \delta_{n,0}$ has no bounded solution. In this example, $x_{n+1} = T_n x_n$ $(\forall n)$ with $x_n$ bounded only when all $x_n = 0$. (Example of Slyusharchuk.)

**Theorem 1.5** *Suppose $\{T_n\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants $M \geq 1$, $\theta \in (0,1)$ and suppose $M_1 > M$, $\theta < \theta_1 < 1$ and*

$$0 < \varepsilon \leq \frac{\theta_1 - \theta}{1 + \theta\theta_1} \left( \frac{1}{M} - \frac{1}{M_1} \right) .$$

*Then any sequence $\{S_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$ with $\sup_n \|S_n - T_n\| \leq \varepsilon$ has a discrete dichotomy with constants $M_1$, $\theta_1$. If $\{P_n^S\}$, $\{P_n^T\}$ are the corresponding projections, as $\sup_n \|S_n - T_n\| \to 0$,*

$$\sup_n \|P_n^S - P_n^T\| = O\left(\sup_n \|S_n - T_n\|\right) .$$

**Proof:** See [7], Theorem 7.6.7.

The argument for Theorem 1.5 uses the following lemma, stated as exercise 11 in Section 7.6 of [7].

**Lemma 1.6** *If $a \geq 0$, $b \geq 0$, $0 < r < r' \leq r_1, r_2 \leq 1$ and*

$$b < (r' - r)/(1 + rr')$$

*and if $\{g_n\}_{-\infty}^{\infty} \subset \mathbb{R}$ satisfies*

$$0 \leq g_n \leq ar_1^{|n|} + b\sum_{-\infty}^{\infty} r^{|n-k-1|} g_k \quad \textit{for all } n \in \mathbb{Z}\ ,$$

*and $g_n = 0(r_2^{-|n|})$ as $n \to \pm\infty$, then*

$$g_n \leq ar_1^{|n|}/(1 - b(1 + rr_1)/(r_1 - r)) \quad \textit{for all } n \in \mathbb{Z}\ .$$

**Proof:** As suggested in [7], we consider the map $\Phi$ of real sequences

$$\{f_n\} \xrightarrow{\Phi} \left\{ b\sum_{-\infty}^{\infty} r^{|n-k-1|} f_k \right\}$$

and show it is a contraction in the norm $\|\cdot\|_q$, $\|f\|_q = \sup_n |f_n| q^{|n|}$, when $r' \leq q \leq 1/r'$. If $S_n = \sum_{-\infty}^{\infty} r^{|n-k-1|} q^{|n|-|k|}$, then $\|\Phi f\|_q / \|f\|_q \leq \sup_n S_n$. We have $S_n = q^{-2} S_{2-n}$ for $n \leq 0$, $\sup_n S_n = S_0 = (1 + qr)/(q - r)$ if $r < q \leq 1$, $\sup_n S_n = S_{+\infty} = q^2(1-r^2)/[(q-r)(1-qr)] \leq (q+r)/(1-rq)$ if $1 \leq q < r^{-1}$. Thus if $\theta = b(1+rr')/(r'-r)$, $\theta < 1$ and $\|\Phi f\|_q \leq \theta \|f\|_q$ for $r' \leq q \leq 1/r'$. If $f_n = ar_1^{|n|}$, $0 \leq g \leq f + \Phi g \leq f + \Phi f + \cdots + \Phi^k f + \Phi^{k+1} g$, and $\|\Phi^{k+1} g\|_{r_2} \to 0$ as $k \to \infty$. For each $n$, $g_n \leq \sum_{k=0}^{\infty} (\Phi^k f)_n$ and $\|f\|_{1/r_1} = a$, $\|\Phi^k f\|_{1/r_1} \leq \frac{b(1+rr_1)}{r_1 - r} \|\Phi^{k-1} f\|_{1/r_1}$, which gives the result.

**Remark:** Theorem 1.5, on the "roughness" of exponential dichotomies, may also be proved by continuity using Theorem 1.4 (as in [16]) or (at least for finite dimensions and invertible operators) by direct calculation as in Palmer [13] (or Coppel [5] for ODEs), where it is the beginning of the theory. "Roughness" theorems seem to start with Massera and Schäffer [11].

Sakamoto [15] gives a non-symmetric version of the lemma, for sequences $O(\theta_+^n)$ in $n \geq 0$, $O(\theta_-^{|n|})$ in $n \leq 0$.

**Theorem 1.7** *Suppose $\{T_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$ is a bounded sequence and $\{P_n\}$, $\{\widetilde{P}_n\}$ are bounded sequences of projections in $\mathcal{L}(X)$, and $M \geq 1$, $\theta \in (0,1)$ are constants such that, for all $n$,*

$$\|P_n\| \leq M\ ,\quad \|\widetilde{P}_n\| \leq M\ ,\quad \|I - \widetilde{P}_n\| \leq M\ ;$$
$$T_n P_n = \widetilde{P}_{n+1} T_n\ ,\quad \mathcal{R}(T_n P_n) = \mathcal{R}(\widetilde{P}_{n+1})\ ;$$
$$\|T_n x\| \leq \theta \|x\| \textit{ if } P_n x = 0\ ,\quad \|T_n x\| \geq \theta^{-1} \|x\| \textit{ if } P_n x = x\ .$$

*If $\theta < \theta_1 < 1$ and $M_1 > M$, there exists $\varepsilon > 0$ depending on $\theta, \theta_1, M, M_1$ and $\sup_k \|T_k\|$ such that: for any $\{S_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$, if $\|S_n - T_n\| \leq \varepsilon$ and $\|\widetilde{P}_n - P_n\| \leq \varepsilon$ for all $n$, $\{S_n\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants $M_1$, $\theta_1$.*

**Proof:** See [7], Theorem 7.6.8. It suffices that $4\varepsilon \leq \frac{\theta_1-\theta}{1+\theta_1\theta}(\frac{1}{M} - \frac{1}{M_1})/(1 + M\sup_k \|T_k\|)$.

**Remark:** Palmer proves a similar result for ODEs in [12] and for the case of finite-dimensional invertible operators $T_n$ in [13].

The following simple result allows us to apply Theorems 1.4, 1.5 to dichotomies defined only on $\mathbb{Z}_+$ or $\mathbb{Z}_-$. It is a simpler version of ex. 15, sec. 7.6 of [7].

**Theorem 1.8** *If $\{T_n\}_{n\geq 0}$ has a discrete dichotomy with projections $\{P_n\}_{n\geq 0}$ and constants $M, \theta$, define $\widetilde{T}_n = T_n$ for $n \geq 0$, $\widetilde{T}_n = \theta^{-1}P_0 + \theta(I - P_0)$ for $n < 0$, $\widetilde{P}_n = P_n$ for $n \geq 0$, $\widetilde{P}_n = P_0$ for $n \leq 0$. Then $\{\widetilde{T}_n\}_{-\infty}^{\infty}$ has a dichotomy with projections $\{\widetilde{P}_n\}$ and constants $M, \theta$.*

*If $\{T_n\}_{n<0}$ has a discrete dichotomy with projections $\{P_n\}_{n\leq 0}$ and constants $M, \theta$, define $\widetilde{T}_n = T_n$ for $n < 0$, $\widetilde{T}_n = \theta^{-1}P_0 + \theta(I - P_0)$ for $n \geq 0$, $\widetilde{P}_n = P_n$ for $n \leq 0$, $\widetilde{P}_n = P_0$ for $n \geq 0$. Then $\{\widetilde{T}_n\}_{-\infty}^{\infty}$ has a dichotomy with projections $\{\widetilde{P}_n\}$ and constants $M, \theta$.*

**Proof:** A straight-forward calculation. We only note that the condition for a dichotomy in $\mathbb{Z}_-$ uses $P_n$ for $n \leq 0$ but only $T_n$ for $n \leq -1$, so we may define $\widetilde{T}_0$ conveniently in the second part. (This was overtooked in [7] 7.6, ex. 15, so our result is simpler.)

**Remarks:** A similar extension is possible for $\{T_n\}$ defined only in a finite interval, $a \leq n \leq b$. We may also treat continuous time. For example, suppose $\{T(t,s), t \geq s \geq 0\}$ has a dichotomy with exponent $\beta$, bound $M$ and projections $\{P(t), t \geq 0\}$. Define $\widetilde{T}(t,s) = T(t,s)$ for $t \geq s \geq 0$, $\widetilde{T}(t,s) = e^{\beta(t-s)}P(0) + e^{\beta(s-t)}(I - P(0))$ for $0 \geq t \geq s$, $\widetilde{T}(t,s) = T(t,0)\widetilde{T}(0,s)$ for $t \geq 0 \geq s$. Then $\widetilde{T}$ is a family of evolution operators which has a dichotomy with exponent $\beta$, bound $M$ and projections $\{\widetilde{P}(t), t \in \mathbb{R}\}$, $\widetilde{P}(t) = P\ (\max\{t, 0\})$.

In each of the following corollaries, we extend the sequence to $\{T_n\}_{-\infty}^{\infty}$ as in Theorem 1.8 (for appropriate $P_0$), prove the extended sequence has a dichotomy (by Theorem 1.5, in the first case, or by Theorem 1.4), and then restrict to $\mathbb{Z}_\pm$.

**Corollary 1.9** *Assume $\{T_n\}_{n\geq 0} \subset \mathcal{L}(X)$ [or $\{T_n\}_{n<0}$] has a dichotomy with constants $M, \theta$, and $M_1 > M$, $\theta < \theta_1 < 1$, and $0 < \varepsilon \leq (1/M - 1/M_1)(\theta_1 - \theta)/(1 + \theta\theta_1)$. If $S_n \in \mathcal{L}(X)$ with $\|S_n - T_n\| \leq \varepsilon$ for all $n \geq 0$ [or $n < 0$], then $\{S_n\}_{n\geq 0}$ [or $\{S_n\}_{n<0}$] has a dichotomy with constants $M_1, \theta_1$ and the corresponding projections satisfy $\sup_n \|P_n^S - P_n^T\| = O\ (\sup_n \|S_n - T_n\|)$ as $\sup_n \|S_n - T_n\| \to 0$.*

**Corollary 1.10** *Given $\{T_n\}_{n\geq 0} \subset \mathcal{L}(X)$, the following are equivalent:*

*(i) $\{T_n\}_{n\geq 0}$ has a dichotomy.*

(ii) $S_0 = \{x_0 | \exists$ *bounded* $(x_n)_{n\geq 0} \subset X$ *with* $x_{n+1} = T_n x_n,\ \forall n \geq 0\}$ *splits in* $X$ *(i.e., there is a closed subspace* $U_0$ *so that* $S_0 \oplus U_0 = X$*) and, for every bounded* $\{f_n\}_{n\geq 0} \subset X$*, there is a bounded* $\{X_n\}_{n\geq 0} \subset X$ *with* $x_{n+1} = T_n x_n + f_n$, $n \geq 0$.

**Corollary 1.11** *Given* $\{T_n\}_{n<0} \subset \mathcal{L}(X)$*, the following are equivalent:*

(i) $\{T_n\}_{n<0}$ *has a dichotomy.*

(ii) $U_0 = \{x_0 | \exists$ *bounded* $\{x_n\}_{n\leq 0} \subset X$ *with* $x_{n+1} = T_n x_n,\ \forall n < 0\}$ *splits in* $X$ *and, for every bounded* $\{f_n\}_{n<0} \subset X$*, there exists a bounded* $\{x_n\}_{n\leq 0} \subset X$ *with* $x_{n+1} = T_n x_n + f_n$ *for all* $n < 0$*, and*

$$\left\{(x_n)_{-\infty}^{\infty} \in \ell_\infty(\mathbb{Z}, X) \mid x_n = 0 \text{ for } n \geq 0,\ x_{n+1} = T_n x_n \text{ for all } n < 0\right\}$$

*consists only of the zero sequence.*

**Remark:** The final hypothesis of (ii) in Corollary 1.11 is ugly but inevitable unless we change the definition of a dichotomy. With only the first two hypotheses, the map $(x_n)_{-\infty}^{\infty} \mapsto (x_{n+1} - \widetilde{T}_n x_n)_{-\infty}^{\infty}$ in $\ell_\infty(\mathbb{Z}, X)$ (for the extended sequence) is surjective; its kernel is the set required to be zero by the final hypothesis. I can't find an example with the first two hypotheses true and the last false, nor prove the last unnecessary.

The more general notion of a dichotomy in Coffman and Schäffer [4] gives a result like 1.10 without assuming $S_0$ splits.

It is sometimes useful to know that the projections of a (continuous time) dichotomy are strongly continuous; this certainly holds if the evolution operators are strongly continuous.

**Theorem 1.12** *Suppose* $\{T(t,s), t \geq s \text{ in } J\} \subset \mathcal{L}(X)$ *is a family of evolution operators which has an exponential dichotomy with projections* $\{P(t), t \in J\}$ *and assume, for some interval* $[a,b] \subset J$ *and* $p$ *with* $0 < p < b - a$*, that* $s \mapsto T(s+p, a)$ $(a - p \leq s \leq a)$*,* $s \mapsto T(s+p, s)$*,* $(a \leq s \leq b - p)$*, and* $s \mapsto T(b, s)$ $(b - p \leq s \leq b)$ *are strongly continuous. Then* $t \mapsto P(t) : [a,b] \to \mathcal{L}(X)$ *is strongly continuous.*

**Proof:** Suppose the dichotomy has exponent $\beta$ and bound $M$. We may extend $T, P$ from $[a,b]$ to all $\mathbb{R}$, as in the remark following Theorem 1.8, so $T$ has a dichotomy on all $\mathbb{R}$ with exponent $\beta$ bound $M$ and projections $\{P(t), t \in \mathbb{R}\}$, and for the extension, $s \mapsto T(s+p, s)$ is strongly continuous and $\sup\{\|T(s+p,s)\| : s \in \mathbb{R}\} = K < \infty$. The new $T, P$ agree with the original $T, P$ in $[a,b]$.

For each $t$ and $n \in \mathbb{Z}$, define $T_n(t) = T(t + np + p, t + np)$, $P_n(t) = P(t + np)$; each $t \mapsto T_n(t)$ is strongly continuous, $\|T_n(t)\| \leq K$, and $\{T_n(t)\}_{-\infty}^{\infty}$ has a discrete dichotomy with constants $M, \theta = e^{-\beta p}$ and projections $\{P_n(t)\}_{-\infty}^{\infty}$. The Green functions satisfy

$$G_{n,m}(t) - G_{n,m}(s) = \sum_{-\infty}^{\infty} G_{n,k+1}(t)\,(T_k(t) - T_k(s))\,G_{k,m}(s)\ .$$

With $m = n = 0$ and fixed $x \in X$, $s \in \mathbb{R}$, and any $N$

$$|P(t)x - P(s)x| \leq$$

$$\sum_{|k|\leq N} M\theta^{|k+1|}|(T_k(t) - T_k(s))G_{k;0}(s)x| + 2KM^2(1+\theta^2)(1-\theta^2)^{-1}\theta^{2N+1}|x| .$$

Given $\varepsilon > 0$, choose $N$ large so the second term is $< \varepsilon/2$; for $t$ near $s$, the first is also $< \varepsilon/2$. Thus $P(t)x \to P(s)x$ as $t \to s$.

We return to discrete time.

More-or-less the following result has appeared in various places – we only mention Coppel [5] and ex. 22, sec. 7.6 of [7].

**Theorem 1.13** *Suppose $\{T_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$. We have a discrete dichotomy on $\mathbb{Z}$ if and only if the restrictions in both $\mathbb{Z}_+$ and $\mathbb{Z}_-$ have dichotomies and also $X = S_0 \oplus U_0$ where*

$$\begin{aligned} U_0 &= \{x_0 \mid \exists \text{ bounded } \{x_n\}_{n\leq 0} \subset X \text{ with } x_{n+1} = T_n x_n \text{ for } n < 0\} \\ S_0 &= \{x_0 \mid \exists \text{ bounded } \{x_n\}_{n\geq 0} \subset X \text{ with } x_{n+1} = T_n x_n \text{ for } n \geq 0\} . \end{aligned}$$

*In case the dichotomies in $\mathbb{Z}_+$, $\mathbb{Z}_-$ have finite rank, $X = S_0 \oplus U_0$ means they have the same rank and also the only bounded solution of $x_{n+1} = T_n x_n$ (all $n$) is the zero sequence.*

**Proof:** If we have a dichotomy on $\mathbb{Z}$ with projections $\{P_n\}_{-\infty}^{\infty}$, it is clear the restrictions in $\mathbb{Z}_+$ and $\mathbb{Z}_-$ have dichotomies. If $P_0 x_0 = 0$, $x_n = T_{n,0}(I - P_0)x_0$ is bounded as $n \to +\infty$ so $x_0 \in S_0$. If $x_0 \in S_0$, $x_n = T_{n,0}x_0$ is bounded and $P_0 x_0 = T_{0,n}P_n x_n \to 0$ as $n \to \infty$ so $x_0 \in \mathcal{N}(P_0) : S_0 = \mathcal{N}(P_0)$. Similarly $U_0 = \mathcal{R}(P_0)$.

Now assume we have dichotomies in $\mathbb{Z}_+$, $\mathbb{Z}_-$ with projections $\{P_n^+\}_{n\geq 0}$, $\{P_n^-\}_{n\leq 0}$. As above, $\mathcal{N}(P_0^+) = S_0$ and $\mathcal{R}(P_0^-) = U_0$, and we assume $U_0 \oplus S_0 = X$. Given bounded $\{f_n\}_{-\infty}^{\infty} \subset X$, we show there is a unique bounded solution of $x_{n+1} = T_n x_n + f_n$ $(\forall n)$. In fact

$$\begin{aligned} x_n &= T_{n,0}(1 - P_0^+)x_0 + \sum_0^{\infty} G_{n,k+1}^+ f_k \quad \text{for } n \geq 0 \\ x_n &= T_{n,0}P_0^- x_0 + \sum_{-\infty}^{-1} G_{n,k+1}^- f_k \quad \text{for } n \leq 0 \end{aligned}$$

is the only candidate, and we only need to show these equations are consistent for a (unique) choice $x_0$, i.e.,

$$(P_0^+ x_0, (1 - P_0^-)x_0) = \left( -\sum_0^{\infty} T_{0,k+1}P_k^+ f_k,\ \sum_{-\infty}^{-1} T_{0,k+1}(1 - P_{k+1}^-) f_k \right)$$

has a unique solution $x_0$. It suffices to show

$$x_0 \mapsto (P_0^+ x_0, (I - P_0^-)x_0) : X \to \mathcal{R}(P_0^+) \times \mathcal{N}(P_0^-)$$

is a bijection – which is equivalent to $S_0 \oplus U_0 = X$.

If $P_0^+ x_0 = 0$, $(I - P_0^-)x_0 = 0$ then $x_0 \in \mathcal{N}(P_0^+) \cap \mathcal{R}(P_0^-) = S_0 \cap U_0 = \{0\}$.

If $a = P_0^+ a$, $P_0^- b = 0$, $a - b \in X = S_0 + U_0$ so $a - b = s + u$ for some $s \in S_0$, $u \in U_0$, and then $x_0 = a - s = b + u$ satisfies $P_0^+ x_0 = P_0^+(a - s) = a$, $(I - P_0^-)x_0 = (I - P_0^-)(b + u) = b$.

The next result is due to X.-B. Lin [10] for continuous time.

**Theorem 1.14** *Given $\{T_n\}_{n<n_1} \subset \mathcal{L}(X)$ and $n_0 < n_1$, suppose $\{T_n\}_{n<n_0}$ has a dichotomy with finite rank and projections $\{P_n\}_{n\le n_0}$ and assume $T_{n_1,n_0}|\mathcal{R}(P_{n_0})$ is injective. Then $\{T_n\}_{n<n_1}$ has a dichotomy with the same rank and projections $\{\widetilde{P}_n\}_{n\le n_1}$ such that $\|P_n - \widetilde{P}_n\| \to 0$ exponentially when $n \to -\infty$.*

*Given $\{T_n\}_{n\ge n_0} \subset \mathcal{L}(X)$ and $n_0 < n_1$, suppose $\{T_n\}_{n\ge n_1}$ has a dichotomy with finite rank and projections $\{P_n\}_{n\ge n_1}$ and assume the adjoint $T^*_{n_1,n_0}|\mathcal{R}(P^*_{n_1})$ is injective. Then $\{T_n\}_{n\ge n_0}$ has a dichotomy with the same rank and with projections $\{\widetilde{P}_n\}_{n\ge n_0}$ such that $\|P_n - \widetilde{P}_n\| \to 0$ exponentially as $n \to +\infty$.*

*If the constants of the original dichotomy are $M, \theta$, we may use the same "$\theta$" for the extended dichotomy but a larger "$M$"; the exponential convergence is $O(\theta^{2|n|})$.*

**Proof:** For the first case, define $U_n = \mathcal{R}(P_n)$ for $n \le n_0$, $U_n = T_{n,n_0}\mathcal{R}(P_{n_0})$ for $n_0 \le n \le n_1$. By hypothesis, $\dim U_{n_1} = \dim U_{n_0} < \infty$ so $\dim U_n$ is independent of $n$ and each $T_n|U_n \to U_{n+1}$ is an isomophism. Choose a closed space $S_{n_1}$ so that $S_{n_1} \oplus U_{n_1} = X$ and define $S_n = T^{-1}_{n_1,n}S_n$ for $n \le n_1$, a closed subspace of $X$ with $T_n S_n = S_{n+1} \cap \mathcal{R}(T_n) \subset S_{n+1}$ for $n < n_1$. If $x \in S_n \cap U_n$, $T_{n_1,n}x \in S_{n_1} \cap U_{n_1} = \{0\}$ and $T_{n_1,n}|U_n$ is injective so $x = 0$. If $x \in X$, $T_{n_1,n}x = u + s$ for some $u \in U_{n_1}$, $s \in S_{n_1}$ and $u = T_{n_1,n}u_n$ for some $u_n \in U_n$ so $T_{n_1,n}(x - u_n) \in S_{n_1}$ or $x \in S_n + U_n$. Thus $X = U_n \oplus S_n$ and there is a projection $\widetilde{P}_n$ with $\mathcal{R}(\widetilde{P}_n) = U_n$, $\mathcal{N}(P_n) = S_n$. We have

$$\widetilde{P}_{n+1}T_n = \widetilde{P}_{n+1}T_n\widetilde{P}_n + \widetilde{P}_{n+1}T_n(1 - \widetilde{P}_n) = \widetilde{P}_{n+1}T_n\widetilde{P}_n = T_n\widetilde{P}_n$$

and for $n \le n_0$, $\mathcal{R}(\widetilde{P}_n) = \mathcal{R}(P_n)$ so $\widetilde{P}_n P_n = P_n$, $P_n\widetilde{P}_n = \widetilde{P}_n$.

If $n \le n_0$

$$\widetilde{P}_n = \widetilde{P}_n P_n + \widetilde{P}_n(1 - P_n) = P_n + T_{n,n_0}P_{n_0}\widetilde{P}_{n_0}T_{n_0,n}(1 - P_n)$$

so $\|\widetilde{P}_n - P_n\| \le \|\widetilde{P}_{n_0}\| M^2\theta^{2(n_0-n)} \to 0$ as $n \to -\infty$.

In particular $K = \sup_{n\le n_0} \|\widetilde{P}_n\| < \infty$.

If $n \le m \le n_0$

$$\|T_{n,m}\widetilde{P}_m\| = \|T_{n,m}P_m\widetilde{P}_m\| \le KM\theta^{m-n}$$

and if $m \le n \le n_0$, similarly

$$\|T_{n,m}(I - \widetilde{P}_m)\| \le (K+1)M\theta^{n-m}$$

There are finitely many other indices in $(n_0, n_1]$ and each $T_n|\mathcal{R}(\widetilde{P}_n) \to \mathcal{R}(\widetilde{P}_{n+1})$ is an isomorphism, so we get a dichotomy for $\{T_n\}_{n<n_1}$.

For the second case, let $S_n = \mathcal{N}(P_n)$ for $n \ge n_1$, $S_n = T_{n_1,n}^{-1} S_{n_1}$ for $n_0 \le n < n_1$; we show $\operatorname{codim} S_n = \operatorname{codim} S_{n_1} < \infty$ for all $n \ge n_0$, initially for $n = n_0$. Suppose $u_1, \ldots, u_m$ are independent relative to $S_{n_0}$ : $\sum_1^m c_k u_k \in S_{n_0}$ $(c_k \in \mathbb{R})$ implies all $c_k = 0$. Then $\sum_1^m c_k T_{n_1,n_0} u_k \in S_{n_1}$ implies $\sum_1^m c_k n_k \in S_{n_0}$ so the $T_{n_1,n_0} u_k$ are independent relative to $S_{n_1}$, $\operatorname{codim} S_{n_1} \ge \operatorname{codim} S_{n_0}$ (and similarly $\operatorname{codim} S_{n_1} \ge \operatorname{codim} S_n$ for $n_0 \le n \le n_1$). Let $\xi_1, \ldots, \xi_m \in X^*$ be a basis for $S_{n_1}^{\perp} = \mathcal{R}(P_{n_1}^*)$. If $x \in S_{n_0}$, $T_{n_1,n_0} x \in S_{n_1} \perp \xi_k$ so $T_{n_1,n_0}^* \xi_k \in S_{n_0}^{\perp}$. By hypothesis, the $T_{n_1,n_0}^* \xi_k$ are independent so $\operatorname{codim} S_{n_0} \ge \operatorname{codim} S_{n_1}$ and we have equality. If $n_0 < n \le n_1$, $T_{n_1,n}^* | \mathcal{R}(P_{n_1}^*)$ is also injective so $\operatorname{codim} S_n = \operatorname{codim} S_{n_1}$ for $n_0 \le n < n_1$, and equality is obvious for $n \ge n_1$.

Choose $U_{n_0}$ with $U_{n_0} \oplus S_{n_0} = X$ and define $U_n = T_{n,n_0} U_{n_0}$ for $n > n_0$. If $x \in S_n \cap U_n$, $x = T_{n,n_0} x_0$ for some $x_0 \in U_{n_0}$ and also $x_0 \in S_{n_0}$ so $x_0 = 0$, $x = 0$. Since $\mathcal{N}(T_{n,n_0}) \subset S_{n_0}$, $T_{n,n_0}|U_{n_0} \to U_n$ is a bijection and $\dim U_n = \dim U_{n_0} = \operatorname{codim} S_{n_0} = \operatorname{codim} S_n$ for all $n \ge n_0$, $T_n|U_n \to U_{n+1}$ is an isomorphism and $U_n \oplus S_n = X$. If $\widetilde{P}_n$ is the projection with $\mathcal{R}(\widetilde{P}_n) = U_n$, $\mathcal{N}(\widetilde{P}_n) = S_n$, we have $\widetilde{P}_{n+1} T_n = T_n \widetilde{P}_n$ for $n \ge n_0$ and $\mathcal{N}(\widetilde{P}_n) = S_n = \mathcal{N}(P_n)$ for $n \ge n_1$ so $\widetilde{P}_n P_n = \widetilde{P}_n$, $P_n \widetilde{P}_n = P_n$ for $n \ge n_1$. Then for $n \ge n_1$,

$$\widetilde{P}_n = P_n \widetilde{P}_n + (I - P_n)\widetilde{P}_n = P_n + T_{n,n_1}(I - P_{n_1})\widetilde{P}_{n_1} T_{n_1,n} P_n = P_n + O(\theta^{2(n-n_1)})$$

and the proof is completed as in the first cases.

The last result of this section is due to Palmer [12] for ODEs. A continuous-time version for retarded FDEs is given by Lin [10]. I am unable to find a continuous-time version for PDEs which is not a disguised version of discrete time; Lemma 3.2 of [1] and Theorem 2.2 of [2] are certainly false as stated.

**Theorem 1.15** *Let $\{T_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$ and assume the restrictions in $\mathbb{Z}_+$ and $\mathbb{Z}_-$ both have dichotomies of finite rank. Define $S_0, U_0$ as in Theorem 1.13 and define $L : \mathcal{D}(L) \subset \ell_\infty(X) \to \ell_\infty(X)$ by*

$$x = (x_n)_{-\infty}^{\infty} \in \ell_\infty(\mathbb{Z}, X) \text{ is in } \mathcal{D}(L) \text{ if } \sup_n |x_{n+1} - T_n x_n| < \infty \ ,$$

*and then $Lx = (x_{n+1} - T_n x_n)_{-\infty}^{\infty}$.*

*Then $L$ is a closed operator,* $\dim \mathcal{N}(L) = \dim(S_0 \cap U_0) < \infty$, *$\mathcal{R}(L)$ is closed with* $\operatorname{codim} \mathcal{R}(L) = \operatorname{codim}(S_0 + U_0) < \infty$, *and $L$ is Fredholm with index*

$$\operatorname{ind} L = \dim \mathcal{N}(L) - \operatorname{codim} \mathcal{R}(L) = \dim U_0 - \operatorname{codim} S_0 = (\operatorname{rank} \mathbb{Z}_-) - (\operatorname{rank} \mathbb{Z}_+) \ .$$

*Finally, $f \in \mathcal{R}(L)$ if and only if $0 = \sum_{-\infty}^{\infty}\langle \xi_{k+1}, f_k\rangle$ for all $\xi \in \ell_\infty(\mathbb{Z}, X^*)$ with $\xi_k = T_k^*\xi_{k+1}$ ($\forall k$). We remark that any such $\xi$ has $|\xi_k| \to 0$ exponentially as $k \to \pm\infty$, and there are only finitely many linearly independent $\xi$.*

**Proof:** Let $(x_n)_{-\infty}^{\infty} \in \mathcal{N}(L)$; $(x_n)$ is a bounded solution of $x_{n+1} = T_n x_n$ ($\forall n$) or $x_n = T_{n,0}(1 - P_0^+)x_0$ for $n \geq 0$, $x_n = T_{n,0}P_0^- x_0$ for $n \leq 0$ with $x_0 \in \mathcal{R}(P_0^-) \cap \mathcal{N}(P_0^+) = U_0 \cap S_0$. The sequence is determined by $x_0$ so $\dim \mathcal{N}(L) = \dim(U_0 \cap S_0)$.

Let $f \in \mathcal{R}(L) : f_n = x_{n+1} - T_n x_n$ ($\forall n$) for some bounded $(x_n)_{-\infty}^{\infty}$, so

$$x_n = \begin{cases} T_{n,0}(1 - P_0^+)x_0 + \sum_0^\infty G_{n,k+1}^+ f_k & \text{for } n \geq 0 \\ T_{n,0}P_0^- x_0 + \sum_{-\infty}^{-1} G_{n,k+1}^- f_k & \text{for } n \leq 0 \end{cases}$$

and $x_0$ satisfies

$$(P_0^+ x_0, (I - P_0^-)x_0) = \left( -\sum_0^\infty T_{0,k+1}P_{k+1}^+ f_k \ , \ \sum_{-\infty}^{-1} T_{0,k+1}(I - P_{k+1}^-)f_k \right) .$$

Conversely, any solution $x_0$ of the last equation determines a bounded $(x_n)_{-\infty}^{\infty} = x$ with $Lx = f$.

An argument similar to that in Theorem 1.13 shows $(a, b) \in \mathcal{R}(P_0^+) \times \mathcal{N}(P_0^-)$ is in the range of $x_0 \mapsto (P_0^+ x_0, (I - P_0^-)x_0)$ if and only if $a - b \in \mathcal{N}(P_0^+) + \mathcal{R}(P_0^-) = S_0 + U_0$.

Thus $f \in \mathcal{R}(L)$ if and only if, for all $\xi \perp (S_0 + U_0)$,

$$0 = \sum_{-\infty}^{-1}\langle \xi, T_{0,k+1}(I - P_{k+1}^-)f_k\rangle + \sum_0^\infty \langle \xi, T_{0,k+1}P_{k+1}^+ f_k\rangle$$

or $0 = \sum_{-\infty}^{\infty}\langle \xi_{k+1}, f_k\rangle$ where $\xi_k = T_{0,k}^*(1 - P_0^{-*})\xi$ for $k \leq 0$, $\xi_k = (T_{0,k}P_k^+)^*\xi$ for $k > 0$. Now $\xi \perp (S_0 + U_0)$ means $\xi \in \mathcal{N}(P_0^+)^\perp \cap \mathcal{R}(P_0^-)^\perp = \mathcal{R}(P_0^{+*}) \cap \mathcal{N}(P_0^{-*})$ or $\xi = P_0^{+*}\xi = (I - P_0^{-*})\xi$, which is $\xi_0$, so $\xi_{k-1} = T_{k-1}^*\xi_k$ for all $k$. Conversely, a bounded solution of this equation has $\xi_0 \in (S_0 + U_0)^\perp$. It only remains to note that

$$\text{codim}\mathcal{R}(L) = \dim(S_0 + U_0)^\perp = \text{codim}(S_0 + U_0) = \text{codim} S_0 - \dim U_0 + \dim(U_0 \cap S_0) .$$

## 2 The shadowing lemma

Let $X$ be a Banach space, $V$ open in $X$ and $f : V \to X$ of class $C^1$. A set $C \subset V$ is **invariant** if $f(C) = C$.

**Definition 2.1** *A compact invariant $C \subset V$ is* **hyperbolic** *if, for every orbit $(y_n)_{-\infty}^{\infty} \subset C$ [$y_{n+1} = f(y_n)$ for all $n$], $\{Df(y_n)\}_{-\infty}^{\infty}$ has a discrete dichotomy with finite rank $m$ and constants $M, \theta$, where $m, M, \theta$ are independent of the orbit considered.*

This is not the usual definition of "hyperbolic structure" but it is equivalent.

**Lemma 2.2** *Assume $f$ is $C^1$ on a neighborhood of the compact invariant set $C$ and $f|C$ is injective. Then $C$ is hyperbolic if and only if there exists continuous $P : C \to \mathcal{L}(X)$ with $P(y)^2 = P(y)$, rank$P(y)$ = constant, $Df(y)P(y) = P(f(y))Df(y)$ and $Df(y)|\mathcal{R}(P(y)) \to \mathcal{R}(P(f(y)))$ an isomorphism for all $y \in C$, and for some constants $M \geq 1$, $0 < \theta < 1$, and any integer $n \geq 0$, $y \in C$,*

$$|Df^n(y)(I - P(y))| \leq M\theta^n ,$$
$$|Df^{-n}(y)P(y)| \leq M\theta^n$$

*where, by definition,*

$$Df^{-n}(y)P(y)z = w \in \mathcal{R}(P(f^{-n}y)) \quad when \quad P(y)z = Df^n(f^{-n}y)w.$$

**Remark:** $\mathcal{N}(P) = \{(y, z) | y \in C, P(y)z = 0\}$ is the stable vector bundle, and $\mathcal{R}(P)$ the unstable bundle, of the usual definition.

**Proof:** It is clear that any such $P(\cdot)$ gives us a dichotomy on any orbit $\{f^n(y)\}_{-\infty}^{\infty} \subset C$. Suppose $C$ is hyperbolic and $y \in C$: there is a dichotomy for $\{Df(f^n(y))\}_{-\infty}^{\infty}$ with projections $\{P_n(y)\}_{-\infty}^{\infty}$.
If $z = f(y)$, there is also a dichotomy for $\{Df(f^n(z)\}_{-\infty}^{\infty}$ with projections $\{P_n(z)\}_{-\infty}^{\infty}$, and $f^n(z) = f^{n+1}(y)$, so $P_n(z) = P_{n+1}(y)$ or $P_n(f(y)) = P_{n+1}(y)$, $P_n(y) = P_0(f^n(y))$ for all $n \in \mathbb{Z}$. Define $P(y) = P_0(y)$; then $P(f(y))Df(y) = Df(y)P(y)$ and $Df(y)|\mathcal{R}(P(y)) \to \mathcal{R}(P(f(y)))$ is an isomorphism for each $y \in C$. If $n \geq 1$,

$$Df^n(y) = Df(f^{n-1}(y)) \cdots Df(f(y))Df(y) = T_{n,0} \text{ when } T_k = Df(f^k(y)) ,$$

which gives the estimates claimed.

Proof of continuity of $P(\cdot)$ is more interesting. Since $C$ is compact and $f|C \to C$ is a continuous bijection, the inverse is also continuous. Suppose $\varepsilon > 0$, $N$ is a positive integer and $y \in C$; there is a neighborhood $V_y$ of $y$ so that, if $y' \in V_y \cap C$, $|f^n(y) - f^n(y')| \leq \varepsilon$ when $|n| \leq N$. Also there is a neighborhood $U$ of $C$, $K > 0$ and an increasing function $\omega_0(\cdot)$ with $\omega_0(t) \to 0$ as $t \to 0^+$ such that

$$|Df(x)| \leq K \text{ for } x \in U \ , \quad |Df(x) - Df(y)| \leq \omega_0(|x - y|) \text{ for } x \in U, y \in C \ .$$

We also assume $K$ exceeds the Lipschitz constant of $f|C$. Then $|Df(f^n(y)) - Df(f^n(y'))| \leq \omega_0(\varepsilon)$ for $|n| \leq N$, and it is bounded by $2K$ for all $n$. If $G_{n,m}(y)$,

$G_{n,m}(y')$ are the Green's functions for the dichotomies, then for $y' \in V_y$,

$$\begin{aligned} |P(y) - P(y')| &= \left| \sum_{-\infty}^{\infty} G_{0,k+1}(y)(Df(f^k(y)) - Df(f^k(y')))G_{k,0}(y') \right| \\ &\leq 2M^2\omega_0(\varepsilon)/(1-\theta^2) + 4KM^2\theta^{2N+1}/(1-\theta^2) , \end{aligned}$$

which is small if $\varepsilon$ is small and $N$ is large.

**Remark:** The notation $U, K, \omega_0(\cdot)$ will be used below; they were defined with greater generality than is necessary here. Note that we avoid any hypothesis of *uniform* continuity of $Df$ on an open set since $f$ is merely $C^1$.

**Lemma 2.3** *Suppose $\{T_n\}_{-\infty}^{\infty} \subset \mathcal{L}(X)$ has a discrete dichotomy with constants $M, \theta$, and there exist $g_n : X \to X$ $(n \in \mathbb{Z})$ with $g_n(0) = 0$, $|g_n(x) - g_n(x')| \leq \gamma|x - x'|$ when $|x| \leq r$, $|x'| \leq r$, and $\gamma M < (1-\theta)/(1+\theta)$. Finally suppose $h_n \in X$ for $n \in \mathbb{Z}$ with $\sup_n |h_n| \leq r((1-\theta)/(1+\theta) - M\gamma)/M$. Then there is a unique sequence $\{x_n\}_{-\infty}^{\infty} \subset X$ such that*

$$|x_n| \leq r \text{ and } x_{n+1} = T_n x_n + g_n(x_n) + h_n \text{ for all } n \in \mathbb{Z} .$$

**Proof:** Let $S_r = \{\text{sequences } \{x_n\}_{-\infty}^{\infty} \subset X \mid \sup_n |x_n| \leq r\}$ and define

$$(\Gamma(x))_n = \sum_{-\infty}^{\infty} G_{n,k+1}(h_k + g_k(x_k)) \text{ for } x \in S_r,\ n \in \mathbb{Z} .$$

It is easily verified that $\Gamma(S_r) \subset S_r$ and $\Gamma$ is a contraction for the sup-norm in $S_r$. The fixed point of $\Gamma$ is the desired sequence.

**Theorem 2.4** (The shadowing lemma) *Let $X$ be a Banach space, $V$ an open set in $X$, $f : V \to X$ a $C^1$ map. Assume there is a compact invariant set $C \subset V$ which is hyperbolic and $f|C$ is injective.*

*Then for any $\varepsilon > 0$, sufficiently small, there exists $\delta > 0$ such that, for each sequence $\{y_n\}_{-\infty}^{\infty} \subset C$ with $|y_{n+1} - f(y_n)| \leq \delta$, $\forall n$ (a "$\delta$-pseudo-orbit") there is a unique orbit $\{x_n\}_{-\infty}^{\infty} \subset V$, $x_{n+1} = f(x_n)$ for all $n$, such that $|x_n - y_n| \leq \varepsilon$, $\forall n$.*

**Proof:** The crucial point is to show, for $0 \leq \delta \leq \delta_0$, given any $\delta$-pseudo-orbit $\{y_n\}_{-\infty}^{\infty} \subset C$, $\{Df(y_n)\}_{-\infty}^{\infty}$ has a dichotomy with constants $M' \geq 1$, $\theta' \in (0,1)$, which are independent of the $\delta$-pseudo-orbit considered. Then the result will follow from the last lemma. In fact, $x_n = y_n + z_n$ where we require $|z_n| \leq \varepsilon$ and

$$z_{n+1} - Df(y_n)z_n = g_n(z_n) + h_n$$

with $h_n = f(y_n) - y_{n+1}$, $g_n(z) = f(y_n + z) - f(y_n) - Df(y_n)z$. We have $|h_n| \leq \delta$, $g_n(0) = 0$, and for small $\varepsilon > 0$ so that $B_\varepsilon(C) \subset U$

$$|Dg_n(z)| = |Df(y_n + z) - Df(y_n)| \leq \omega_0(\varepsilon) \text{ for } |z| \leq \varepsilon$$

where $\omega_0, U$ come from the proof of Lemma 2.2. Choose $\varepsilon > 0$ small so that $B_\varepsilon(C) \subset U$ and also $\omega_0(\varepsilon) < (1-\theta')/(M'(1+\theta'))$, and then $\delta > 0$ small so that $\delta \leq \delta_0$ and also $\delta \leq \varepsilon((1-\theta')/(M'(1+\theta')) - \omega_0(\varepsilon))$; then Lemma 2.3 applies.

To see we have a dichotomy along any $\delta$-pseudo-orbit $\{y_n\}_{-\infty}^{\infty} \subset C$, first choose the integer $N$ so $M\theta^N \leq \frac{1}{2}$; show $\{Df^N(y_{Ni})\}_{-\infty}^{\infty}$ has a dichotomy with constants $2M, 3/4$ (by Theorem 1.7); then if $V_{Ni+k} = Df(f^k(Y_{Ni}))$ for $0 \leq k < N$, $i \in \mathbb{Z}$, $Df^N(y_{Ni}) = V_{Ni+N,Ni} = V_{Ni+N-1} \circ \cdots \circ V_{Ni+1} \circ V_{Ni}$ and we may "interpolate" a dichotomy for $\{V_j\}_{-\infty}^{\infty}$, by Theorem 1.3, with constants $M_1$, $\theta_1 = (3/4)^{1/N}$; and finally, since $\sup_n |Df(y_n) - V_n| \to 0$ as $\delta \to 0$, Theorem 1.5 gives the dichotomy with constants $M' > M_1$, $\theta' \in (\theta_1, 1)$, for $0 < \delta \leq \delta_0$, if $\delta_0$ is small.

By induction, $|y_{n+m} - f^m(y_n)| \leq \delta(1 + K + \cdots + K^{m-1})$ for $m \geq 1$, $n \in \mathbb{Z}$. $[|y_{n+m} - f^m(y_n)| \leq |y_{n+m} - f(y_{n+m-1})| + |f(y_{n+m-1}) - f(f^{m-1}(y_n))| \leq \delta + K|y_{n+m-1} - f^{m-1}(y_n)|.]$ Let $K^* = 1 + K + \cdots + K^{N-1}$; then $|y_{Ni+N} - f^N(y_{Ni})| \leq K^*\delta$ for all $i \in \mathbb{Z}$. Now $P : C \to \mathcal{L}(X)$ is uniformly continuous and has modulus of continuity $\omega_p(\cdot)$ so

$$|P(f^N(y_{Ni})) - P(y_{Ni+N})| \leq \omega_p(K^*\delta) .$$

We apply Theorem 1.7 with $T_i = Df^N(y_{Ni})$, $P_i = P(y_{Ni})$, $\widetilde{P}_{i+1} = P(f^N(y_{Ni}))$, so $T_iP_i = \widetilde{P}_{i+1}T_i$. We have $|\widetilde{P}_{i+1} - P_{i+1}| \leq \omega_p(K^*\delta)$, which is uniformly small for small $\delta$, so we have a dichotomy for $\{T_i\} = \{Df^N(y_{Ni})\}_{-\infty}^{\infty}$ with constants $2M$ and $3/4$. (It suffices that $32M(1 + MK^N)\omega_p(K^*\delta_0) \leq 1$.)

Define $V_j$ ($j \in \mathbb{Z}$) as above, $V_{Ni} = Df^N(y_{Ni})$, and let $\widetilde{T}(t,s) = V_{n,m}$ when $t \geq s$, $t \in [n, n+1)$, $s \in [m, m+1)$. By Theorem 1.3, $\{\widetilde{T}(t,s), t \geq s\}$ has a dichotomy so also $\{V_i\}_{-\infty}^{\infty}$ has a dichotomy with constants $M_1, \theta_1 = (3/4)^{1/N}$ (We may use $M_1 = \max(2MK^{2N}(\frac{3}{4})^{-2}, \frac{4}{3}K^N + 4M^2K^{2N})$.)

Finally, if $n = Ni + k$ ($0 \leq k < N$, $i \in \mathbb{Z}$)

$$|Df(y_n) - V_n| = |Df(y_{Ni+k}) - Df(f^k(y_{Ni}))| \leq \omega_0(K^*\delta) .$$

Given $M' > M_1$, $1 > \theta' > \theta_1$, for $0 < \delta \leq \delta_0$ (and small $\delta_0$) $\{Df(y_n)\}_{-\infty}^{\infty}$ has a dichotomy with constants $M', \theta'$, for any $\delta$-pseudo-orbit $\{y_n\}_{-\infty}^{\infty} \subset C$. (It suffices that $\omega_0(K^*\delta_0) \leq \frac{\theta' - \theta_1}{1 + \theta'\theta_1}(\frac{1}{M_1} - \frac{1}{M'})$.)

# 3 A transverse homoclinic orbit

**Theorem 3.1** *Assume $V \subset X$ is open, $f : V \to X$ is $C^1$ and also:*

*(i) $x_0 \in V$ is a hyperbolic fixed point of $f$ ($f(x_0) = x_0$, $\sigma(Df(x_0)) \cap S^1 = \emptyset$) with finite rank, i.e. $r_{ess}(Df(x_0)) < 1$ or $\sigma(Df(x_0)) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$ consists only of isolated eigenvalues of finite multiplicity;*

*(ii) there exists a homoclinic orbit $\{y_n\}_{-\infty}^{\infty} \subset V \backslash \{x_0\}, y_{n+1} = f(y_n)$ for all $n$, and $y_n \to x_0$ as $n \to \pm\infty$;*

*(iii) the homoclinic orbit is transverse, i.e.*

$$\eta_{n+1} = Df(y_n)\eta_n \text{ for all } n, \quad \sup_n |\eta_n| < \infty \text{ imply all } \eta_n = 0 .$$

*Then $C = \{x_0; y_n \ (n \in \mathbb{Z})\}$ is a compact invariant hyperbolic set, $f|C$ is injective, and for $\varepsilon > 0$ sufficiently small, there is $N_\varepsilon$ such that, for any positive integer $N \geq N_\varepsilon$ and for each $\sigma \in S = \{0,1\}^{\mathbb{Z}}$, there is a unique orbit $x^\sigma$ $[x^\sigma_{n+1} = f(x^\sigma_n), \forall n]$ such that $|x^\sigma_n - \eta^\sigma_n| \leq \varepsilon$, $\forall n$, where*

$$\eta^\sigma_n = \begin{cases} x_0 & \text{if } \sigma(i) = 0 \\ y_j & \text{if } \sigma(i) = 1 \end{cases} , \qquad n = (2N+1)i + j, \ -N \leq j \leq N ,$$

*and $\sigma \mapsto x^\sigma$ is injective from $S$ to $\ell_\infty(\mathbb{Z}, X)$.*

*Define $\mathrm{d}(\sigma, \sigma') = \sum_1^\infty 2^{-n} \Delta \ (\max_{|k| \leq n} |\sigma(k) - \sigma'(k)|)$ with $\Delta(t) = t/(1+t)$ and $\sigma$, $\sigma' \in S$. Then $(S, d)$ is a compact metric space, homeomorphic to the "middle thirds" Cantor set, and $\sigma \to x^\sigma_n : S \to X$ is continuous, for each $n \in \mathbb{Z}$.*

*The zero sequence $\sigma = 0$ gives $x^\sigma_n = x_0$ for all $n$.*

*If $\sigma \in S \backslash \{0\}$ has finite support ($\sigma(i) = 0$ for all large $|i|$) then $x^\sigma$ is a homoclinic orbit, $x^\sigma_n \to x_0$ as $n \to \pm\infty$.*

*If $\sigma$ is periodic with period $p$, $n \mapsto x^\sigma_n$ is periodic with period $(2N+1)p$. If $\sigma \neq 0$, $\sigma$ has least period $p$ if and only if $x^\sigma$ has least period $(2N+1)p$.*

*(iv) Assume also that $f|V$ is injective.*

*Then $\sigma \mapsto x^\sigma_0$ is injective with compact image $K$; $K$ is a topological Cantor set and the restriction $\varphi$ of $\sigma \mapsto x^\sigma_0 : S \to K$ is a homeomorphism; $f^{2N+1}(K) = K$, and $f^{2N+1}|K \to K$ is $\varphi \circ \beta \circ \varphi^{-1}$ where $\beta : S \to S$ is the Bernoulli shift, $\beta(\sigma)(n) = \sigma(n+1)$ for $n \in \mathbb{Z}$, $\sigma \in S$.*

*Suppose, finally, that (i), (ii) and (iv) hold but perhaps not (iii) and also:*

*(v) $Df(x) \in \mathcal{L}(X)$ is injective with dense range $[Df(x)^*$ injective$]$ for each $x \in V$.*

*Then the global stable and unstable manifolds $W^s(x_0)$, $W^u(x_0)$ are $C^1$ immersed submanifolds of $V$, $\dim W^u(x_0) = \operatorname{codim} W^s(x_0) < \infty$, $y_n \in W^s(x_0) \cap W^u(x_0)$, and (iii) is equivalent to saying $W^s(x_0) \pitchfork_{y_n} W^u(x_0)$ for some (or every) $n \in \mathbb{Z}$.*

**Proof:** It is clear that $C$ is compact and invariant. For any $n$, $f(y_n) = y_{n+1} \neq x_0 = f(x_0)$, and if $f(y_m) = f(y_p)$ for some $p > m$, $n \mapsto y_n$ is periodic for $n \geq m$ and does not tend to $x_0$ as $n \to +\infty$; thus $f|C$ is injective.

Since $x_0$ is hyperbolic with finite rank, $\{Df(x_0)\}^\infty_{-\infty}$ has a dichotomy with finite rank and constant projection $P_\infty$. By Corollary 1.9, since $Df(y_n) \to Df(x_0)$ as $n \to \pm\infty$, for some positive integer $N$, both $\{Df(y_n)\}_{n \leq -N}$ and $\{Df(y_n)\}_{n \geq N}$

have dichotomies, both with the same rank (since the projections are close to $P_\infty$). Let $\eta_{-N} \in U(-N)$; there is a sequence $\{\eta_n\}_{-\infty}^{\infty}$ with $\eta_{n+1} = Df(y_n)\eta_n$ ($\forall n$) and $\eta_n$ is bounded as $n \to -\infty$. If $T_{N,-N}\eta_{-N} = 0 = \eta_N$ $[T_k = Df(y_k)]$, then $\eta_n = 0$ for all $n \geq N$ so, by hypothesis (iii), all $\eta_n = 0$, $\eta_{-N} = 0$. Thus $T_{N,-N}|U(-N)$ is injective and, by Theorem 1.14, we may extend the dichotomy to $\{Df(y_n)\}_{n\leq N}$, still with the same rank. Hypothesis (iii) and Theorem 1.13 say we have a dichotomy for $\{Df(y_n)\}_{-\infty}^{\infty}$, so $C$ is hyperbolic.

As noted in the proof of the shadowing lemma, for small $\delta_0 > 0$ if $\{\eta_n\}_{-\infty}^{\infty} \subset C$ is any $\delta$-pseudo-orbit with $0 < \delta \leq \delta_0$, $\{Df(\eta_n)\}_{-\infty}^{\infty}$ has a dichotomy with constants (say $M, \theta$) independent of the pseudo-orbit considered. Choose $\varepsilon > 0$ sufficiently small so the shadowing lemma (2.4) applies, $B_\varepsilon(C) \subset U$, $\omega_0(\varepsilon)M < (1-\theta)/4$ [$\omega_0, U$ from Lemma 2.2] and $|x_0 - y_0| > 2\varepsilon$ and $|y_n - y_0| > 2\varepsilon$ for all $n \neq 0$. Choose $\delta$ in $0 < \delta \leq \delta_0$ so we may apply the shadowing lemma and $N_\varepsilon > 0$ so that $|y_{\pm N} - y_0| < \delta/2$ for $N \geq N_\varepsilon$; choose any integer $N \geq N_\varepsilon$. Then for every $\sigma \in S$, $\{\eta_n^\sigma\}_{-\infty}^{\infty}$ is a $\delta$-pseudo-orbit so there is a unique orbit $\{x_n^\sigma\}_{-\infty}^{\infty}$ with $|x_n^\sigma - \eta_n^\sigma| \leq \varepsilon$ for all $n$. If $\sigma \neq \sigma'$ in $S$, there exists $i$ with $\sigma(i) \neq \sigma'(i)$ so, if $n = (2N+1)i$, $|x_n^\sigma - x_n^{\sigma'}| \geq |y_0 - x_0| - 2\varepsilon > 0$, and $\sigma \mapsto x^\sigma$ is injective. It is clear that $\eta_{n+2N+1}^\sigma = \eta_n^{\beta(\sigma)}$ and $|x_n^{\beta(\sigma)} - \eta_n^{\beta(\sigma)}| \leq \varepsilon$, $|f^{2N+1}(x_n^\sigma) - \eta_{n+2N+1}^\sigma| \leq \varepsilon$ for all $n$, so by uniqneness $f^{2N+1}(x_n^\sigma) = x_n^{\beta(\sigma)}$ for all $n$.

Again by uniqueness, if $\sigma$ is periodic with period $p$, $n \mapsto x_n^\sigma$ is periodic with period $p(2N+1)$, while $\sigma \equiv 0$ gives $x_n^\sigma = x_0$ for all $n$. If $\sigma \not\equiv 0$ and $n \mapsto x_n^\sigma$ is periodic with period $M$, we show $M$ is a multiple of $2N+1$. Suppose $M = (2N+1)k + m$ for some integers $k \geq 0$ and $m$ with $|m| \leq N$. If $m = 0$, it follows as above that $\beta^k\sigma = \sigma$; if $m \neq 0$, we find a contradiction. Now $|\eta_n^\sigma - \eta_{n+M}^\sigma| \leq 2\varepsilon$ so $|\eta_n^\sigma - \eta_{n+m}^{\beta^k\sigma}| \leq 2\varepsilon$ for all $n$. There exists $i$ with $\sigma(i) = 1$ and we choose $n = (2N+1)i$ so $\eta_n^\sigma = y_0$. If $\beta^k\sigma(i) \neq 1$, $|x_0 - y_0| \leq 2\varepsilon$, which is false; if $\beta^k\sigma(i) = 1$, $|y_0 - y_m| \leq 2\varepsilon$, which is also false unless $m = 0$.

Regarding the symbol space $S$: if $\psi(0) = 0$, $\psi'(t) \geq 0$, $\psi''(t) \leq 0$ and $\psi(t) \not\equiv 0$ for $t > 0$, then $\psi$ is strictly increasing and $\psi(a+b) \leq \psi(a) + \psi(b)$ for $a, b \geq 0$. Since $t \mapsto t/(1+t)$ has these properties, $d$ is a distance (metric) for $S$, and $(S, d)$ is clearly complete. For any integer $N$ let $S_N = \{\sigma \in S \mid \sigma(i) = 0 \text{ if } |i| > N\}$, a finite set; given any $\sigma \in S$ there exists $\sigma_N \in S_N$ with $\sigma(i) = \sigma_N(i)$ for $|i| \leq N$ so $d(\sigma, \sigma_N) \leq 2^{-N}$, so $S$ is totally bounded, hence compact. Define $\theta : S \to \mathbb{R}$ by

$$\theta(\sigma) = 2\sigma(0) + \sum_1^\infty \frac{2\sigma(n)}{3^{2n-1}} + \frac{2\sigma(-n)}{3^{2n}} \ ;$$

$\theta$ is a continuous map whose image is the set of numbers in $[0,3]$ whose ternary expansion contains only 0 and 2, i.e. the "middle thirds" Cantor set $C$. The restriction $\theta|S \to C$ is a continuous bijection, hence a homeomorphism.

Now we show $\sigma \mapsto x_n^\sigma : S \to X$ is continuous for each $n$. Since $M\omega_0(\varepsilon) < (1-\theta)/4$, we may choose $\theta_1$ in $\theta < \theta_1 < 1$ so that $M\omega(\varepsilon)(\frac{1}{\theta_1 - \theta} + \frac{1}{1-\theta\theta_1}) \leq \frac{1}{2}$. Given $\sigma \in S$ and a positive integer $m$, suppose $\overline{\sigma} \in S$ with $d(\overline{\sigma}, \sigma) < 2^{-m}$; then

$\overline{\sigma}(i) = \sigma(i)$ for $|i| \leq m$. Let $m' = m(2N+1)$ so $\eta_k^\sigma = \eta_k^{\overline{\sigma}}$ for $|k| \leq m'$. In notation like that of the shadowing lemma

$$x_n^\sigma = \eta_n^\sigma + \sum_{-\infty}^{\infty} G'^\sigma_{n,k+1}(g(\eta_k^\sigma, x_k^\sigma) + h_k^\sigma)$$

and similarly for $\eta^{\overline{\sigma}}$, $x^{\overline{\sigma}}$. If $|n| \leq m'$,

$$x_n^\sigma - x_n^{\overline{\sigma}} = \sum_{-\infty}^{\infty}(G'^\sigma_{n,k+1} - G'^{\overline{\sigma}}_{n,k+1})(g(\eta_k^\sigma, x_k^\sigma) + h_k^\sigma)$$

$$+\sum_{-\infty}^{\infty} G'^{\overline{\sigma}}_{n,k+1}(g(\eta_k^\sigma, x_k^\sigma) - g(\eta_k^{\overline{\sigma}}, x_k^{\overline{\sigma}}) + h_k^\sigma - h_k^{\overline{\sigma}}) \ .$$

Now

$$\begin{aligned}
|G'^\sigma_{ij} - G'^{\overline{\sigma}}_{ij}| &= \left|\sum_{|k|>m'} G'^\sigma_{i,k+1}(Df(\eta_k^\sigma) - Df(\eta_k^{\overline{\sigma}}))G'^{\overline{\sigma}}_{k,j}\right| \\
&\leq 2M^2K \sum_{|k|>m'} \theta^{|i-k-1|}\theta^{|j-k|} \\
&\leq 2M^2K \times \begin{cases} C''\theta_1^{|i-j|} & \text{for all } i,j \\ \dfrac{2\theta}{1-\theta^2}\theta^{2m'-|i+j|} & \text{for } |i|,|j| \leq m' \end{cases}
\end{aligned}$$

where $C'' = \sup_{i,j} \sum_{-\infty}^{\infty} \theta^{|i-k-1|}\theta^{|j-k|}\theta_1^{-|i-j|} < \infty$, by the calculation in Lemma 1.6. Then for $|n| \leq m'$

$$\begin{aligned}
|x_n^\sigma - x_n^{\overline{\sigma}}| \leq\ & \sum_{|k|\leq m'} \frac{2M^2K\theta}{1-\theta^2}\theta^{2m'-|n+k+1|}(\delta + \varepsilon\omega_0(\varepsilon)) \\
&+ \sum_{|k|>m'} 2M^2KC''\theta_1^{|k+1-n|}(\delta + \varepsilon\omega_0(\varepsilon) \\
&+ \sum_{|k|>m'} M\theta^{|n-k-1|}2(\delta + \varepsilon\omega_0(\varepsilon)) + \sum_{|k|\leq m'} M\theta^{|n-k-1|}\omega_0(\varepsilon)|x_k^\sigma - x_k^{\overline{\sigma}}| \ .
\end{aligned}$$

This implies, for a constant $C''' = O(\delta + \varepsilon\omega_0(\varepsilon))$,

$$\max_{|k|\leq m'} |x_k^\sigma - x_k^{\overline{\sigma}}|/(\theta_1^{m'-k} + \theta_1^{m'+k}) \leq C''' \ ;$$

with fixed $n$ and $m \to \infty$ (or $\overline{\sigma} \to \sigma$), we see $x_n^{\overline{\sigma}} \to x_n^\sigma$.

If $\sigma$ has finite support, $d(\beta^i(\sigma),0) \to 0$ as $i \to \pm\infty$, so

$$x_n^\sigma = x_j^{\beta^i(\sigma)} \to x_j^0 = x_0 \quad [n = (2N+1)i + j\ ,\ \ |j| \leq N] \ \text{ as } n \to \pm\infty\ .$$

(iv) Assume also that $f$ is injective. Then the orbit $[x_n^\sigma]_{-\infty}^{\infty}$ is determined by any of its points so $\sigma \mapsto x_0^\sigma : S \to X$ is a continuous injection with compact image $K \subset V$, and the restriction $\varphi$ of $\sigma \mapsto x_0^\sigma : S \to K$ is a homeomorphism. Also $f^{2N+1}(x_0^\sigma) = x_0^{\beta(\sigma)}$ so $f^{2N+1}|K \to K$ is $\varphi \circ \beta \circ \varphi^{-1}$.

Finally assume (i), (ii), (iv) and (v) but not (iii). Theorem 6.1.9 of [7] shows the global manifolds $W^s(x_0)$, $W^u(x_0)$ are immersed $C^1$ submanifolds of $V$ with complementary dimensions, $\dim W^u(x_0) = \text{codim} W^s(x_0) < \infty$, and $y_n \in W^u(x_0) \cap W^s(x_0)$ for all $n$. We have

$$T_{y_n} W^s(x_0) = \{\eta_n \in X \mid \exists \{\eta_k\}_{k \geq n} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k \geq n\}$$

$$T_{y_n} W^u(x_0) = \{\eta_n \in X \mid \exists \{\eta_k\}_{k \leq n} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k < n\}$$

(These are merely interpretations of the difference equations defining the derivatives: see Theorem 5.2.2 of [7] for the case of continuous time).
Then $W^s(x_0) \pitchfork_{y_n} W^u(x_0)$ is equivalent to $T_{y_n} W^s(x_0) \cap T_{y_n} W^u(x_0) = \{0\}$, i.e.

$$(\eta_k)_{-\infty}^{\infty} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k \text{ implies } \eta_n = 0 \text{ (so all } \eta_k = 0)$$

which is (iii).

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