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Coupled Oscillators on a Circle

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Abstract: We consider a continuum of diffusively coupled oscillators on a circle. When each oscillator is of Lienard type, very little is known about the corresponding hyperbolic PDE. When each oscillator is represented by a lossless transmission line, we obtain a partial neutral delay differential equation and give the beginnings of a qualitative theory for the dynamics. In particular, we discuss the properties of the solution map, the existence of the global attractor, behavior near an equilibrium point including the Hopf bifurcation, and some elementary properties near a periodic orbit.

Key words: transmission lines, oscillators, partial differential equations, delay differential equations, attractors, periodic orbits.

1. Introduction. Motivated by problems in physics, physiology and biology, there have been many studies in recent years devoted to the dynamics induced from the ordinary differential equations obtained by coupling large numbers of oscillators on periodic lattices. One important problem is concerned with self excited periodic motions for which each particle oscillates in the same way except for a phase shift (synchronization) (the locking in phenomenon). When the oscillators are subjected to external excitation, it is often the case that patterns involving spatial and temporal chaos are the primary concern.

In general, the qualitative properties of the dynamics depends upon the number of oscillators. On the other hand, it is to be expected that there will be a certain type of stabilization in the dynamics if the number of oscillators is larger than some number N_0 (we refer to this as *spatial stabilization*). One major problem is to show that spatial stabilization occurs; that is, N_0 exists, and to determine value of N_0. Of course, this is a difficult problem and frequently is resolved by numerical techniques.

Assuming that the coupling between the oscillators is of diffusive type, another possible approach and the one that we will advocate here is to replace the lattice by a continuum which has the effect of replacing the large system of ordinary differential equations by a partial differential equation (PDE). The first objective would be to study the dynamical properties of the PDE that are motivated by the original discrete problem. If it is possible to prove that these properties are insensitive to perturbations of a sufficiently general type, then we obtain important information about the original discrete system. Success in such a program at least will show the existence of the number N_0 which characterizes spatial stabilization.

Of course, the type of PDE that is obtained by taking the continuum limit will depend upon the types of oscillators that are considered as well as the struc-

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ture of the coupling between the oscillators. We indicate some of these types to bring out the fact that there are some interesting new difficult problems in PDE. Unfortunately, at this time, we are able to make only a small contribution to one of these classes and leave the others as topics for future research.

In the modeling of the synchronization problem, the basic oscillators used at each point very often are taken to be described by an ordinary differential equation of the van der Pol type or Lienard type; that is, the differential equation has a unique equilibrium point and a periodic orbit which attracts all other orbits. Assuming that the oscillators are on a uniform periodic lattice on the real line and that each oscillator interacts only with its nearest neighbor, we arrive at a system of ordinary differential equations

$$\partial_t^2 u_k - \kappa(u_{k+1} - 2u_k + u_{k-1}) + f(u_k)\partial_t u_k + g(u_k) = 0, \quad k = 1, \ldots, N, \tag{1.1}$$

where $\kappa > 0$ is a constant and ∂_t denotes the derivative with respect to t.

We describe briefly an approach taken to discuss system (1.1) following the papers of Koppel and Ermentrout (1982),(1986) (see also the references therein). For $\kappa = 0$ system (1.1) has an invariant torus T_0^N corresponding to the N-product of the periodic orbit γ_0 of the scalar equation

$$\partial_t^2 v + f(v)\partial_t v + g(v) = 0. \tag{1.2}$$

Since γ_0 is hyperbolically stable, the N-torus T_0^N also will be hyperbolic and exponentially stable. For $\kappa > 0$ and small, system (1.1) will have an invariant torus T_κ^N which is hyperbolic and exponentially stable. The flow on T_κ^N is described by a system of differential equations involving N-angles θ_j, $j = 1, 2, \ldots, N$. Synchonization occurs when $\theta_j(t) - \theta_k(t) \to$ a constant as $t \to \infty$.

Koppel and Ermentrout (1986) proved the existence of a synchronized solution in a very interesting way by making use of a continuum limit. The limit was taken on the ordinary differential equations describing the differences of the angles to obtain a parabolic equation with a dispersive term. Thus, using the continuum limit to obtain information about a discrete problem is not unprecedented.

Let us now consider a continuous system of such oscillators on a circle by taking the parameter κ in (1.1) as K/h^2, where h is the spacing between the oscillators and $K > 0$ is a constant independent of N which represents the diffusive interaction with neighboring oscillators. If we assume that κ has this form, then the limit as $h \to 0$ in (1.1) leads to the hyperbolic PDE on the circle S^1:

$$\partial_t^2 u - K\partial_x^2 u + f(u)\partial_t u + g(u) = 0, \tag{1.3}$$

where ∂_x denotes the derivative with respect to x and the solution of (1.3) is to be considered in $H^1(S^1) \times L^2(S^1)$.

Equation (1.3) has the spatially homogeneous periodic orbit $\gamma_0 = \{p(t), t \in \mathbb{R}\}$, where $p(t)$ is a nontrivial periodic solution of the ODE (1.2). This solution is stable hyperbolic when we consider perturbations in the space of spatially homogeneous functions. If this orbit were known to be stable hyperbolic

in $H^1(S^1) \times L^2(S^1)$ and if we could show that this periodic orbit persists under spatial discretization in space, then we will obtain synchronized periodic orbits due to the symmetry in the discretization. This approach would show that spatial stabilization does occur. At the present time, we have no idea how to prove or disprove any of these remarks.

In the derivation of (1.3), the quantity $\kappa = K/h^2$ approaches ∞ as $h \to 0$. Therefore, the model (1.3) must be considered as completely different from the one of Koppel and Ermentrout.

Endo and Mori (1978) have considered systems of coupled van der Pol oscillators on a circle which (after scalings of the variables) is equivalent to the system

$$\partial_t^2 u_k - \epsilon(1 - u_k^2)\partial_t u_k + u_k + \kappa(u_{k+1} - 2u_k + u_{k-1}) = 0, \quad k = 1, 2, \ldots, N, \tag{1.4}$$

where ϵ is a parameter that depends only upon the characteristics of the van der Pol circuit and $\kappa = L/L_0$, where L is the inductance in the van der Pol circuit and L_0 is the mutual inductive interaction between the oscillators. Using ϵ as a small perturbation parameter, Endo and Mori (1978) have considered the stability and instability of various types of periodic solutions. If we assume that $\kappa = K/h^2$ and take the continuum limit, we obtain the equation

$$\partial_t^2 u - \epsilon((1 - u^2)\partial_t u + u - \partial_x^2 u = 0 \quad \text{on } S^1. \tag{1.5}$$

For $\epsilon = 0$, we obtain the undamped linear wave equation on S^1 for which the corresponding group has the radius of the essential spectrum equal to one. To give rigorous results about such a perturbed problem would be very interesting but, at the same time, seems to be extremely difficult.

Kurzweil (1963) (1967) and Hall (1968) have given some very interesting results about the limits of periodic solutions of (1.7) as $\epsilon \to 0$. On the other hand, there seems to be no information about the global dynamics of (1.7).

Wu and Xia (1993) considered another type of oscillator on a linear periodic lattice. At each point, the dynamics of the oscillator was described by the telegraph equations of a lossless transmission line with a tunnel diode between the lines. As is well known (see, for example, Abolina and Mishkis (1960), Nagumor and Shimura (1961), Brayton (1966), Cooke and Krumme (1968)), such a system is equivalent to a scalar neurtal differential delay equation. Assuming that the oscillators react through resistive coupling, it is shown by Wu and Xia (1993) that the resulting system is given by

$$\partial_t Du_{k,t} - \kappa(Du_{k+1,t} - 2Du_{k,t} + Du_{k-1,t}) = f(u_{k,t}), \quad k = 1, \ldots, N, \tag{1.6}$$

where $u_{k,t}(\theta) = u_k(t + \theta), -r \le \theta \le 0$, $r > 0$ is a positive constant, $Du_{k,t} = u_k(t) - qu_k(t - r)$, and q is a physical parameter, $0 \le q < 1$. The fact that $|q| < 1$ implies that the solutions of the difference equation

$$Dw_t \equiv w(t) - qw(t - 1) = 0$$

has zero as a solution which is exponentially stable. The operator D may be more complicated depending upon the circuity across the transmission line. For example, it could involve many delays which are not necessarily rationally independent. However, the corresponding difference equation is exponentially stable under reasonable physical assumptions.

With $\kappa = K/h^2$ as above, the continuous version of this system is the partial neutral functional differential system (PNFDE) on S^1:

$$\partial_t Du_t - K\partial_x^2 Du_t = f(u_t), \tag{1.7}$$

where $u_t(x,\theta) = u_t(x,t+\theta), -r \le \theta \le 0$, and $Du_t = u(x,t) - qu(x,t-r)$, $0 \le q < 1$, $x \in S^1$. The space of initial data is chosen to be $X \equiv C([-r,0], H^1(S^1))$.

If $K = 0$ in (1.7), we have a neutral delay differential equation which carries the hyperbolic structure from the transmission line describing the original oscillators. On the other hand, this hyperbolic structure is controlled by the fact that the group defined by the difference equation has essential spectral radius < 1 for each $t > 0$. It is to be expected that the diffusive term ($K \neq 0$) will not have too much impact on the essential spectral radius. Therefore, a reasonable theory of dynamics should exist for (1.7), and, in the next sections, we will present the beginnings of such a theory. More specifically, in Section 2, we define the class of equations to be considered, indicate proofs of existence, uniqueness, etc., and give conditions which imply that the solution operator is an α-contraction in the sense of Kuratowski. We present also a result on the existence of a global attractor as well as a regularity result on elements of the attractor. This regularity result should be useful in discussing spatial discretizations, but the details have not been supplied at this time. A type of result that might be expected is the following. Suppose that (1.7) has a hyperbolic periodic orbit (this concept is defined in Section 4) that is stable. Then under spatial discretization, there should be a periodic orbit of (1.6). If the spacing is uniform, then the symmetry under certain rotations should imply that the orbits are synchronized. For a specific type of nonlinearity in (1.6) and certain values of the parameters, it was proved by Wu and Xia (1993) that there is a periodic orbit that is synchronized. Thus, the mentioned expectation is reasonable. In Section 3, we give a generic Hopf bifurcation theorem for large diffusions K. In Section 4, we begin a theory for the behavior of solutions near periodic orbits and the preservation of these orbits under perturbations. It is only a beginning and much remains to be done. We remark that some of the above results were announced in Hale (1993).

In the applications, it is important to consider more general lattices that linear ones; for example, periodic lattices in the plane (or three space) which correspond to the two dimensional (or three dimensional) torus in the continuous case. Many of the remarks made below for (1.7) should hold in this more general setting with ∂_x^2 replaced by the Laplacian.

It should be noted also that the interactive forces between the particles as well as basic circuitry on the transmission line that induces the oscillation for each particle may depend upon the position of the particle. In such a case, the PDE

depends explicitly upon the spatial variable. For example, (1.7) would be replaced by the PNFDE

$$\partial_t D(x, u_t) - K\partial_x(a(x)\partial_x D(x, u_t)) = f(x, u_t), \tag{1.8}$$

Similar remarks hold for the other models.

We end this introduction with a few remarks about a different type of problem. If we consider a periodic lattice of pendulum type equations with linear damping and periodic forcing, with diffusive coupling, and take the continuum limit, then we obtain the following hyperbolic equation on S^1:

$$\partial_t^2 u + \beta\partial_t u - K\partial_x^2 u + f(u) = \lambda p(\omega t), \tag{1.9}$$

where $p(t)$ represents a periodic external forcing and $\beta > 0, \lambda, \omega$ are real parameters. Under either a dissipative condition on the function f or the case where f is periodic and bounded, it is possible to show that the Poincaré map has a global attractor (see, for example, Babin and Vishik (1989), Hale (1988), Temam (1988)). A basic problem is to determine properties of the flow on the attractor. For the case in which $f(u) = \sin u$ (the linearly damped and periodically forced Sine-Gordon equation), considerable research has been devoted to possible flows on the attractor from both the theoretical and numerical point of view (see, for example, see Birnir and Grauer (1994) and the references therein). The Sine-Gordon equation also is a basic model for describing the phase difference between two superconducting layers in a Josephson junction (see Pedersen (1982)). For β and λ of order 10^{-2} or less, Birnir and Grauer (1994) have given a very detailed description of the global attractor and the different bifurcations that occur by varying β and λ. There can be both regular and chaotic dynamics in this range.

To merely indicate some of the types of problems involved and why complicated dynamics is to be expected, let us be a little more specific. Suppose that the ODE

$$\partial_t^2 u + f(u) = 0 \tag{1.10}$$

has the origin as a saddle point with eigenvalues $\pm\mu_0, \mu_0 > 0$, and an orbit $\Gamma = \{p(t), t \in \mathbb{R}\}$ homoclinic to the origin; that is, $p(t) \to 0$ as $t \to \pm\infty$. Also, suppose that $f'(0) + n^2K > 0$, $n = 1, 2, \ldots$, so that the eigenvalues of the the equation

$$\partial_t^2 u - K\partial_x^2 u + f'(0)u = 0 \tag{1.11}$$

are $\pm\mu_0$ and $\pm i(f'(0) + n^2K)^{1/2}$, $n = 1, 2, \ldots$. Thus, the origin is a solution of the equation

$$\partial_t^2 u - K\partial_x^2 u + f(u) = 0 \tag{1.12}$$

with one positive eigenvalue, one negative eigenvalue and the remaining ones lying on the imaginary axis. Equation (1.12) also has a spatially homogeneous orbit

$\Gamma \times S^1$ which is homoclinic to the origin. For $\beta > 0$ small, the origin is a saddle point of

$$\partial_t^2 u + \beta \partial_t u - K\partial_x^2 u + f(u) = 0 \tag{1.13}$$

with index one. In fact, for the linearized equation, $\pm\mu_0$ are eigenvalues and all of the other eigenvalues have real parts equal to $-\beta/2$. There is no orbit homoclinic to the origin for (1.13). For $\lambda > 0$, $\beta > 0$, and small, we can choose ω so that there is a periodic solution $q_{\lambda\beta}$ of (1.9) with $q_{00} = 0$. This periodic solution will be hyperbolic and the fixed point $p_{\lambda\beta} \equiv q_{\lambda\beta}(0)$ of the Poincaré map having unstable manifold of dimension 1. For the ordinary differential equation ($K = 0$ in (1.9)), it is a well known fact that there is a $\lambda_{\mathrm{ODE}}(\beta)$ with the property that, for $\lambda = \lambda_{\mathrm{ODE}}(\beta)$, there is a homoclinic tangency of the stable and unstable manifolds of $p_{\lambda\beta}$ and, for $\lambda > \lambda_{\mathrm{ODE}}(\beta)$, these manifolds intersect travsversally. This implies temporal chaos in (1.9).

On the other hand, something more exciting may have happened before λ reached the value $\lambda_{\mathrm{ODE}}(\beta)$. For β, λ small, the stable manifold of the fixed point $p_{\lambda\beta}$ is very close to the stable manifold of the origin of (1.13). If we choose $\beta < 2\mu_0$, then the strongly stable manifold of the origin of (1.13) is infinite dimensional and is close to the span of the eigenvectors corresponding to the eigenvalues with real parts equal to $-\beta/2$ of the linearization of (1.13) about zero. In such a situation, we expect that the first intersection of the stable and unstable manifolds of $p_{\lambda\beta}$ to occur along the strongly stable manifold. This infinite dimensional manifold is linearly independent of the span of the eigenvector corresponding to the eigenvalue $-\mu_0$ and involves very general spatial dependence. If these remarks can be made precise, then we should obtain spatial temporal chaos and the information should serve as a complement to the work of Birnir and Grauer (1994). It is feasible that one could attack the problem by extending the methods of of Holmes and Marsden (1981), Rodrigues and Silveira (1987) that were used to disucss a similar situation for the beam equation.

2. The solution operator and existence of global attractor. Let $X \equiv C([-r, 0], H^1(S^1))$. If $\varphi \in X$, we write it as $\varphi(\zeta, \theta), \zeta \in S^1, \theta \in [-r, 0]$. For any function $\tilde{f} \in C^{k+1}(C([-r, 0], \mathbb{R}); \mathbb{R})$, $k \geq 1$, we let $f \in C^{k+1}(X, L^2(S^1))$ be defined by $f(\varphi)(\zeta) = \tilde{f}(\varphi(\zeta, \cdot), \zeta \in S^1$. Let $\tilde{D} \in \mathcal{L}(C([-r, 0], \mathbb{R}); \mathbb{R})$ be defined by

$$\tilde{D}\psi = \psi(0) - \tilde{g}(\psi),$$
$$\tilde{g}(\psi) = \int_{-r}^{0} [d_\theta \eta(\theta)]\psi(\theta),$$

where η is of bounded variation and non-atomic at 0; that is, there is a continuous nondecreasing function $\delta : [0, r] \to [0, \infty)$ such that $\delta(0) = 0$

$$\left| \int_{-s}^{0} [d_\theta \eta(\theta)]\psi(\theta) \right| \leq \delta(s)\|\psi\|, \quad s \in [0, r].$$

We define $D \in \mathcal{L}(X, H^1(S^1))$ as

$$D(\varphi)(\zeta) = \tilde{D}(\varphi(\zeta, \cdot)) = \int_{-r}^{0} [d_\theta \eta(\theta)]\varphi(\zeta, \theta), \quad \zeta \in S^1.$$

If $u(\cdot, t) \in H^1(S^1)$ on an interval $[-r, \delta)$, $\delta > 0$, and $u(\cdot, t)$ is continuous in t, then we let $u_t \in X$, $t \in [0, \delta)$ be defined by $u_t(\zeta, \theta) = u(\zeta, t + \theta)$ for $\zeta \in S^1$, $\theta \in [-r, 0]$. If K is a positive constant, we consider the partial neutral functional differential equation (PNFDE)

$$(2.1)_{\tilde{f}} \qquad \partial_t D u_t = K \partial_\zeta^2 D u_t + f(u_t),$$

where K is a positive constant. The initial data for $(2.1)_{\tilde{f}}$ is chosen in the space X.

If we let $A = -K \frac{\partial^2}{\partial \zeta^2}$ with domain $H^2(S^1)$, then e^{-At} is an analytic semigroup on $H^1(S^1)$ and $L^2(S^1)$ and the solution of $(2.1)_{\tilde{f}}$ with initial value $\varphi \in X$ at $t = 0$ is defined to be the solution of the integral equation

$$(2.2)_{\tilde{f}} \qquad D u_t = e^{-At} D\varphi + \int_0^t e^{-A(t-s)} f(u_s) ds, \quad t \geq 0,$$

and $u_0 = \varphi$.

We will need the following result for the existence and regularity of the solutions of $(2.2)_{\tilde{f}}$.

Lemma 2.1. *There are positive constants a, b, c such that, for any $h \in C(\mathbb{R}; H^1(S^1))$, the solution of the equation*

$$(2.3) \qquad D u_t = h(t)$$

satisfies the inequality

$$(2.4) \qquad \|u_t\|_X \leq [a\|u_0\|_X + b \sup_{0 \leq s \leq t} \|h(s)\|_{H^1(S^1)}] e^{ct}, \quad t \geq 0.$$

Proof. The idea of the proof comes from Hale and Meyer (1967). Since $H^1(S^1) \subset C(S^1)$ with continuous embedding, we can use the same techniques as in NFDE on $\mathbb{R}$ (see, for example, Hale (1977), Hale and Verduyn-Lunel (1993)), to show the existence of a solution of (2.3) in $C([-r, 0], C(S^1))$. We now estimate this solution.

Let $K > 1$ be such that $|\tilde{g}(\psi)| \leq K\|\psi\|$ and choose $A \in [0, r]$ such that $1 - \delta(A) > 0$. Let $a = K/[1 - \delta(A)]$, $b = 1/[1 - \delta(A)]$, and choose $c > 0$ such that $ae^{-cA} \leq 1$. For any $t \in [0, A]$ and fixed $x \in S^1$, let $v(t) = u(x, t)$, $\tilde{h}(t) = h(x, t)$. We have

$$v(t) = \int_{-r}^{-A} [d_\theta \eta(\theta)] v(t + \theta) + \int_{-A}^{0} [d_\theta \eta(\theta)] v(t + \theta) + h(t).$$

Therefore,

$$\begin{aligned}|v(t)| &\le K\|v_0\| + \delta(A)\|v_t\| + |\tilde{h}(t)| \\ &\le K\|v_0\| + \delta(A)\|v_t\| + \sup_{0\le s\le t}|\tilde{h}(s)|\end{aligned}$$

for $0 \le t \le A$. Since $K > 1$, the right hand side is a bound for $\|v_t\|$. Inverting the resulting inequality, we have

$$\|v_t\| \le a\|v_0\| + b \sup_{0\le s\le t}|\tilde{h}(s)|, \quad t \in [0, A]. \tag{2.5}$$

Inequality (2.5) and the fact that $c > 0$ implies that we have

$$\|v_t\| \le [a\|v_0\| + b \sup_{0\le s\le t}|h(s)|]e^{ct} \tag{2.6}$$

satisfied on the interval $[0, A]$. Let us assume that (2.6) is satisfied on $[0, kA]$ for some interger k and deduce by induction that (2.6) is satisfied for all $t \ge 0$. For $t \in [kA, (k+1)A]$, we have from (2.5) that

$$\|v_t\| \le a\|v_{t-A}\| + b \sup_{t-A\le s\le t}|\tilde{h}(s)|.$$

¿From our induction hypothesis and the fact that $ae^{-cA} \le 1$ and $c > 0$, we deduce from this last inequality that

$$\begin{aligned}\|v_t\| &\le a\|[a\|v_0\| + b \sup_{0\le s\le t}|h(s)|]e^{c(t-A)} + b \sup_{t-A\le s\le t}|\tilde{h}(s)| \\ &\le a\|[a\|v_0\| + b \sup_{0\le s\le t}|h(s)|]e^{c(t-A)} + b \sup_{t-A\le s\le t}|\tilde{h}(s)|e^{c(t-A)},\end{aligned}$$

which gives inequality (2.6) for $t \in 0, (k+1)A]$.

Since $u(x,t)$ satisfies inequality (2.6) for all $t \ge 0$ and all $x \in S^1$, we obtain the estimate (2.4) with X replaced by $C([-r, 0]; C(S^1))$.

The previous estimate used only the fact that the initial data $\varphi(x,t)$ and the function $h(x,t)$ were continuous. If we now use $\varphi \in X$ and $h \in C(\mathbb{R}; H^1(S^1))$, then we can use arguments similar to the above and Hölder's inequality to obtain the conclusion in the lemma.

Using Lemma 2.1, the theory of analytic semigroups and the same techniques as in Hale (1977), Hale and Verduyn-Lunel (1993) for the existence theory of ordinary NFDE, it is possible to prove the following result.

Theorem 2.1. *For any $\varphi \in X$, there exists a $\delta > 0$ such that $(2.2)_{\tilde{f}}$ has a unique solution $u(\cdot, t : \varphi)$ on $[0, \delta)$, which is continuous in (t, φ) and has k continuous derivatives with respect to φ.*

Furthermore, if all solutions are defined on $[0, \infty)$ and if we define $T(t)\varphi = u_t(\cdot; \varphi)$, then $T(t), t \ge 0$, is a C^k-semigroup on X.

For the development of a general qualitative theory of $(2.1)_{\tilde{f}}$, we impose some additional conditions on the operator $\tilde{D}$. Let $C = C([-r, 0]; \mathbb{R})$, $C_0 = \{\psi \in C :$

$\tilde{D}\psi = 0\}$. We say that $\tilde{D}$ is *stable* if there exist positive constants $\bar{\beta}, \bar{\alpha}$ such that the solution of the homogeneous functional equation

$$\tilde{D}v_t = 0, \qquad t \geq 0, \tag{2.7}$$

with $v_0 = \psi \in C_0$, satisfies the inequality

$$\|v_t\|_C \leq \bar{\beta}e^{-\bar{\alpha}t}\|\psi\|_C, \quad t \geq 0, \psi \in C_0, \tag{2.8}$$

We remark that the operator $\tilde{D}u_t = u(\cdot,t) - qu(\cdot,t-r)$ with $0 \leq q < 1$ is stable. It is not stable if $q \geq 1$.

If $X_0 = \{\varphi \in X : D(\varphi)(\zeta) = \tilde{D}(\varphi(\zeta,\cdot)) = 0, \zeta \in S^1\}$ and $S(t) : X_0 \to X_0$ is the semigroup defined by the equation $Dw_t = 0$ and $\tilde{D}$ is stable, then the estimate (2.8) implies immediately that

$$\|S(t)\varphi\|_{C([-r,0];C(S^1))} \leq \bar{\beta}e^{-\bar{\alpha}t}\|\varphi\|_{C([-r,0];C(S^1))}, \quad t \geq 0, \varphi \in X.$$

The same type of estimate probably is true in X, but it is not proved at this time. Such an estimate is easily proved for the case where $\tilde{D}(\psi) = \psi(0) - \Sigma_{j=1}^N a_j\psi(-r_j)$, $r_j > 0, j = 1,2,\ldots,N$, $\Sigma_{j=1}^N |a_j| < 1$. This case is interesting, but it does not include all of the applications. In any case, we are going to assume that there are positive constants β, α such that

$$\|S(t)\|_{\mathcal{L}(X_0,X_0)} \leq \beta e^{-\alpha t}, t \geq 0. \tag{2.9}$$

We need also some estimates on the solutions of nonhomogeneous difference equations with the difference operator being stable. The following result is a special case of Lemma 3.4 in Cruz and Hale (1970).

Lemma 2.2. *If $\tilde{D}$ is stable, then there are positive constants a, b, c, d such that, for any $h \in C([0,\infty), \mathbb{R})$, the solution v of the equation*

$$\tilde{D}v_t = h(t),$$

for $t \in [0,\infty)$, satisfies the inequality

$$\|v_t\| \leq e^{-at}[b\|v_0\| + c \sup_{0\leq s\leq t} |h(s)|] + \sup_{[\max(0,t-r)]\leq s\leq t} |h(s)|.$$

The estimate in Lemma 2.2 is particularly interesting because, if $|h(s)|$ is bounded on $[0,\infty)$l, then the ultimate bound on v_t as $t \to \infty$ is determined by the bound on $|h(s)|$ for s in the delay interval $[t-r,t]$ as $t \to \infty$. For instance, this implies that, if $h(t) \to 0$ as $t \to \infty$, then so does v_t.

We must also consider the nonhomogeneous equation

$$Dw_t = h(t), \qquad h \in C([0,\infty); H^1(S^1)). \tag{2.10}$$

We can obtain the corresponding estimate in Lemma 2.2 in $C([-r,0];C(S^1))$ following the same reasoning as above for the homogeneous equation. For special cases, we can obtain this in X, but, at this time, do not have the estimate in the general case. Therefore, we assume that the solution of (2.10) satisfies, for $t \in [0,\infty)$,

$$\begin{aligned}\|w_t\|_X \le & e^{-at}[b\|w_0\|_X + c \sup_{0\le s\le t}\|h(s)\|_{H^1(S^1)}] \\ & + d \sup_{[\max(0,t-r)]\le s\le t}\|h(s)\|_{H^1(S^1)}.\end{aligned} \tag{2.11}$$

We know that there are positive constants $\bar\delta$, $\bar\gamma$ such that

$$\|e^{-At}\varphi\|_{H^1(S^1)} \le \bar\delta e^{-K\bar\gamma t}\|\varphi\|_{H^1(S^1)}, \quad t \ge 0, \ \varphi \in H^1(S^1). \tag{2.12}$$

With these remarks, we can obtain the following respresentation for the semigroup defined by $(2.1)_{\tilde f}$.

Theorem 2.2. *If the solutions of $(2.1)_{\tilde f}$ are defined for all $t \ge 0$, $T(t)$ is a bounded map and (2.9), (2.11) are satisfied, then $T(t)$ is an α-contraction in the sense that*

$$T(t) = \hat S(t) + U(t), \tag{2.13}$$

where $U(t)$ is a compact operator for $t > 0$, $\hat S(t) : X \to X$, $t \ge 0$ is the solution of the equation

$$Dw_t = e^{-At}D\varphi, \qquad w_0 = \varphi \tag{2.14}$$

and there are positive constants $\hat\beta, \hat\gamma$ such that

$$\|\hat S(t)\varphi\|_X \le \hat\beta e^{-\hat\gamma t}\|\varphi\|_X, \qquad t \ge 0. \tag{2.15}$$

Proof. If $\hat S(t)\varphi$ is defined as the solution of (2.14), then (2.11) and (2.12) imply that

$$\|\hat S(t)\varphi\|_X \le [e^{-at}(b + cL\hat\delta) + e^{-K\hat\gamma\max(0,t-r)}]\|\varphi\|_X,$$

where L is a positive constant such that $\|D\varphi\|_{H^1(S^1)} \le L\|\varphi\|_X$ for all $\varphi \in X$. The expression in brackets approaches zero exponentially and so there must exist positive constants $\hat\beta, \hat\gamma$ such that (2.15) is satisfied.

If we let $U(t)\varphi \equiv \tilde w_t = u_t - \hat S(t)\varphi$, then

$$D\tilde w_t = \tilde h(t,\varphi) \equiv \int_0^t e^{-A(t-s)}f(u_s)ds, \quad \tilde w_0 = 0. \tag{2.16}$$

¿From the regularity theory of parabolic equations, the function $\tilde h : [0,\infty)\times X \to H^1(S^1)$ is completely continuous. Let B be a bounded set in X and let $\{\varphi_k\}_{k=1}^\infty$ be a sequence in B. For any $\tau > 0$, there is a subsequence which we label the same

such that the sequence $\{\tilde{h}(t,\varphi_k)\}_{k=1}^{\infty}$ converges in $H^1(S^1)$ as $k \to \infty$ uniformly on $[0,\tau]$ to some function $\tilde{h}(t) \in H^1(S^1)$. Let $\tilde{w}_t^k$ be the solution of (2.16) with $\varphi = \varphi_k$. If $w_t^{kj} = \tilde{w}_t^k - \tilde{w}_t^j$ for integers k, j, then

$$Dw_t^{kj} = \tilde{h}(t,\varphi_k) - \tilde{h}(t,\varphi_j), \quad w_0^{kj} = 0.$$

Relation (2.11) and the fact that $w_0^{kj} = 0$ imply that there is a positive constant C such that

$$\|w_t^{kj}\|_X \leq C \sup_{0\leq s\leq t} \|\tilde{h}(t,\varphi_k) - \tilde{h}(t,\varphi_j)\|_{H^1(S^1)}.$$

This implies that the sequence $\{w_t^k\}_{k=1}^{\infty}$ is a Cauchy sequence, which proves that $U(t)$ is a completely continuous operator and concludes the proof of the theorem.

If D satisfies (2.9), (2.11), it follows from the representation (2.13) in Theorem 2.2 that the semigroup $T(t)$ has the property that the radius $r_e\sigma(T(1))$ of the essential spectrum of $T(1)$ is less than one. As a consequence, this makes it possible to use classical techniques (see, for example, Hale and Verduyn-Lunel (1993)) to develop, near equilibrium points, the theory of strongly stable, strongly unstable, center-stable, center-unstable and center manifolds. Therefore, we can use the center manifold theorem to prove the Hopf-bifurcation theorem for the situation where there is only one pair of eigenvalues crossing the imaginary axis as a parameter is varied. This method also yields the stability properties of the periodic orbits near the equilibrium point.

We recall that $\mathcal{A}$ is a global attractor for $(2.1)_{\tilde{f}}$ if it is compact, invariant and the ω-limit set of any bounded set is $\mathcal{A}$. It follows from Hale (1988) and Theorem 2.2 that the following result is true.

Theorem 2.3. *If D satisfies (2.9), (2.11), the equation $(2.1)_{\tilde{f}}$ is point dissipative and orbits of bounded sets are bounded, then there exists the global attractor $\mathcal{A}$ for $(2.1)_{\tilde{f}}$.*

The elements in the attractor $\mathcal{A}$ should have more regularity properties with respect to x. In fact, if $u_t \in \mathcal{A}$ for $t \in \mathbb{R}$, then, for any $\sigma \in \mathbb{R}$, we have

$$Du_t = e^{-A(t-\sigma)}Du_\sigma + \int_\sigma^t e^{-A(t-s)}f(u_s)ds, \quad t \geq \sigma.$$

Since u_σ is bounded for $\sigma \in \mathbb{R}$, it follows from (2.12) that

$$Du_t = \int_{-\infty}^t e^{-A(t-s)}f(u_s)ds. \tag{2.17}$$

The function on the right is in $H^2(S^1)$ and, thus, we expect that u_t is in $H^2(S^1)$ and, in fact, the attractor $\mathcal{A}$ should be in a bounded set in $H^2(S^1)$. We do not have a proof of this in the general case, but possible to give the following special case.

Theorem 2.4. *Suppose that*

$$\tilde{D}(\psi) = \psi(0) - \Sigma_{j=1}^{N} a_j \psi(-r_j),$$

where $r_j > 0, j = 1, 2, \ldots, N$, $\Sigma_{j=1}^{N} |a_j| < 1$. *If* $(2.1)_{\tilde{f}}$ *is point dissipative and orbits of bounded sets are bounded, then there exists the global attractor* $\mathcal{A}$ *for* $(2.1)_{\tilde{f}}$ *and* $\mathcal{A}$ *belongs to a bounded set of the space* $C([-r, 0]; H^2(S^1))$.

Proof. We first consider the nonhomogeneous equation $\tilde{D}v_t = \tilde{h}(t)$, where $\tilde{h}(t)$ is a continuous function of t. The varaiation of constants formula implies that, for any $\sigma \in \mathbb{R}$, $t \geq \sigma$,

$$v_t = v_\sigma + \int_\sigma^t [d_s K(t-s)]\tilde{h}(s), \tag{2.18}$$

where $K(t)$ is a function defined for $t \geq -r$, of bounded variation on any compact set and $K(t) = 0$ for $-r \leq t < 0$, $K(0) = 1$ (see Hale and Verduyn-Lunel (1993)). Furthermore, since $\tilde{D}$ is assumed to be stable, there are positive constants a, b such that

$$|d_s K(t)| \leq a e^{-bt}, \quad t \geq -r. \tag{2.19}$$

As a consequence of (2.19) and (2.18), if it is known that v_t exists and is bounded on $\mathbb{R}$, then

$$v_t = \int_{-\infty}^t [d_s K(t-s)]\tilde{h}(s), \quad t \in \mathbb{R}. \tag{2.20}$$

If a solution u of $(2.1)_{\tilde{f}}$ is such that u_t belongs to the attractor $\mathcal{A}$ for $t \in \mathbb{R}$, then u_t satisfies (2.17). Therefore, for any $\zeta \in S^1$, $t \in \mathbb{R}$, it follows from (2.20) that

$$u_t(\zeta, \cdot) = \int_{-\infty}^t [d_s K(t-s)] \int_{-\infty}^s [e^{-A(s-\tau)} f(u_\tau)](\zeta) d\tau.$$

If we now use Hölder's Inequality, (2.19) and the fact that $\int_{-\infty}^s e^{-A(s-\tau)} f(u_\tau) d\tau$ belongs to $H^2(S^1)$, we arrive at the conclusion in the theorem.

3. Local Hopf bifurcation and large diffusion. In this section, we consider the situation in which the function f in (2.1) depends upon a real parameter ν, $\tilde{f}(\psi) = \tilde{f}(\psi, \nu)$, $\tilde{f}(0, \nu) = 0$ for all ν, and the ordinary NFDE

$$\partial_t \tilde{D} v_t = \tilde{f}(v_t, \nu) \tag{3.1}$$

undergoes a Hopf bifurcation at $\nu = 0$. If γ is a periodic orbit of (3.1), then it also is a periodic orbit of (2.1). We want to prove the following result.

Theorem 3.1. *Suppose that $\tilde{D}$ is stable, ν is a real parameter, $\tilde{f}(\psi) = \tilde{f}(\psi, \nu)$, $\tilde{f}(0, \nu) = 0$ for all ν, and the linear variational equation of (3.1) about zero has eigenvalues $\nu \pm i\omega(\nu)$, $\omega(0) > 0$, and all the other eigenvalues have negative real parts. Also, suppose that (3.1) undergoes a generic supercritical Hopf bifurcation at $\nu = 0$ to a periodic orbit $\gamma_\nu = \{v_\nu(t), t \in [0, p_\nu)\}$ of period p_ν. If (2.9), (2.11) are satisfied, then then there exist positive constants ν_0, K_0 such that, for $0 < \nu \leq \nu_0$, $K \geq K_0$, the periodic orbit γ_ν as a solution of $(2.1)_{\tilde{f}}$ is asymptotically orbitally stable with asymptotic phase.*

In the proof of this result, we will obtain more information about the characteristic equation for the linearization about 0 in $(2.1)_{\tilde{f}}$ as a function of K and also about the dependence of center manifolds on K.

The linearization about 0 in $(2.1)_{\tilde{f}}$ is

$$(3.2)_\nu \qquad \partial_t D w_t - K\partial_\zeta^2 D w_t - L_\nu(w_t) = 0$$

where $L_\nu(\varphi)$ is the derivative of $f(\varphi, \nu)$ with respect to φ evaluated at $\varphi = 0$.

The eigenvalues of $(3.2)_\nu$ are those values of λ for which there is a nontrivial solution of the form $e^{\lambda t}\psi(x)$. This is equivalent to saying that λ is a solution of the characteristic equation

$$D(e^{\lambda \cdot})(\lambda - K\partial_x^2)\psi - L(e^{\lambda \cdot})\psi = 0.$$

If we expand ψ in terms of the eigenfunctions φ_n on S^1 corresponding to the eigenvalues $-n^2$, $n = 0, 1, \ldots$ of $\partial^2/\partial x^2$ (we are taking S^1 as the homeormorphic image of $[0, 2\pi)$) and equate coefficients, we obtain the equations

$$(3.3)_n \qquad \Delta(\lambda, \nu, Kn^2) \equiv \tilde{D}(e^{\lambda \cdot})(\lambda + Kn^2) - \tilde{L}(e^{\lambda \cdot}) = 0.$$

The solutions of $(3.3)_0$ are the eigenvalues of the linearization of (3.1) about zero. We use the following result to understand some of the structure of the solutions of $(3.3)_n$ for $n > 0$.

Lemma 3.1. *If $\Delta(\lambda, \nu, \gamma)$ is defined as in $(3.3)_n$, then*

$$\limsup_{\gamma \to \infty} \sup\{\operatorname{Re}\lambda : \Delta(\lambda, \nu, \gamma) = 0\} = \sup\{\operatorname{Re}\lambda : \tilde{D}(e^{\lambda \cdot}) = 0\}.$$

The proof is not too difficult and is omitted.

Now let us suppose that the hypotheses of Theorem 3.1 are satisfied. For (3.1), there is a positive constant ν_0 such that, for $|\nu| \leq \nu_0$, we can construct a center manifold CM_ν for (3.1). The center manifold CM_ν is exponential attracting orbits near it as long as the solutions stay in a neighborhood of zero. The flow on CM_ν, as ν passes through zero will change from having a stable equilibrium point to a periodic orbit γ_ν which is hyperbolic and stable as a solution of (3.1). The set CM_ν also is a locally invariant set for the partial NFDE $(2.1)_{\tilde{f}}$ for any $K > 0$. From Lemma 3.1, there is a positive constant K_0 such that, for $K \geq K_0$, there is

a $\delta > 0$ such that the solutions of $(3.3)_n$ for $n > 1$ have real parts $\leq -\delta < 0$ since the operator $\tilde{D}$ is stable. Therefore, the set CM_ν is an exponentially attracting center manifold for $(3.2)_\nu$. This implies the result stated in Theorem 3.1.

An example of a lossless transmission line for which there are self-excited oscillations corresponds to the NFDE (see Hale (1977), p.7)

$$(3.4)\quad \partial_t[w(t) - qw(t-r)] = \beta[w(t) - qw(t-r)] + 2w(t) - g(w(t) - qw(t-r)),$$

where $0 < q < 1$ and $\beta > 0$ are constants. The function g is a given nonlinear function which vanishes together with its first derivative at 0. As an example, take $g(x) = x^3$. It was shown in Brayton (1966), Hale and Meyer (1967), that there is a $q_0 \in (0,1)$ such that there is a generic supercritical Hopf bifurcation at $q = q_0$ for the ordinary NFDE. Under some additional conditions on g, the global continuation of this periodic orbit with respect to parameters has been considered by Krawcewicz, Wu and Xia (1993).

¿From Theorem 3.1, we know that the stable periodic orbit of (3.1) that arose from the Hopf bifurcation at $q = q_0$ is also a stable periodic orbit for (2.1) if $K > K_0$.

It would be interesting to see if the dynamics changes as we let the parameter K approach zero. Can the Hopf orbit be destabilized in this way? For retarded functional differential equations, this is possible (see Yoshida (1982), Morita (1984), Memory (1989) for a detailed discussion). The same probably also is true for the above PNFDE.

4. Local behavior near a periodic orbit. The next step in the theory is to develop the local theory near periodic orbits of $(2.1)_{\tilde{f}}$. In ordinary differential equations, the most natural way to do this is to take a transversal section to the periodic orbit and use the corresponding Poincaré map, which is as smooth as the vector field. It is natural to try the same thing here. For a given transversal N to the periodic orbit, the Poincaré map π_N is given by $\pi_N(\varphi) = T(\tau(\varphi))\varphi$ for some continuous function $\tau(\varphi)$. Since $T(t)\varphi$ is not necessarily differentiable with respect to t, the map π_N may not be differentiable. The principal difficulty here is that it is not known if there exists a transversal N to the periodic orbit with the property that the Poincaré map π_N is continuously differentiable. Therefore, it is not obvious a priori how to define the Floquet multipliers. We can follow the procedure for ordinary NFDE in Hale and Verduyn-Lunel (1993, Section 10.3) to prove that each periodic solution of $(2.1)_{\tilde{f}}$ is a C^{k+1}-function of t and, thus, any periodic orbit is a C^{k+1}-manifold. If we assume in addition that f in $(2.1)_{\tilde{f}}$ is analytic, then we can use the ideas in Hale and Scheurle (1985) to show that each periodic solution of $(2.1)_{\tilde{f}}$ is an analytic function of t and, as a consequence, each periodic orbit is an analytic manifold. With this observation, we can define the linear variational equation about any constant periodic solution $p(t)$ of $(2.1)_{\tilde{f}}$ and define the Poincaré map as the value at the period of the solution operator of the linear variational equation. The Floquet multipliers are the elements of the spectrum of the corresponding Poincaré map, excluding, of course, the obvious

multiplier 1 that comes from the fact that $dp(t)/dt$ is a solution of the linear variational equation. We say that γ is *hyperbolic* if no Floquet multiplier of γ has modulus one and *stable hyperbolic* if each multiplier has modulus less than one.

If γ is a hyperbolic periodic orbit, then we can follow the same procedure as in Hale and Verduyn-Lunel (1993, Chapter 10) to prove that the synchronized stable and unstable sets of γ are smooth manifolds. It should be possible to use the ideas in Henry (1981) to introduce a good coordinate system about γ and use exponential dichotomies to prove that these synchonized sets are actually the stable and unstable sets of γ. However, at this time, this has not been done in detail. If this procedure is valid, then we can discuss the effects of perturbations of $(2.1)_{\tilde{f}}$ by a smooth function $\tilde{g}$ which may even depend upon time. The results should be of the same nature as the ones that are well known for ordinary differential equations.

Even without a coordinate system about γ, we can follow the methods of Hale and Verduyn-Lunel (1993, Section 10.3), making use of the synchronized stable and unstable manifolds, to prove the following theorem.

Theorem 4.1. *If (2.9) and (2.11) are satisfied and γ is a periodic orbit of $(2.1)_{\tilde{f}}$ for which there is a Floquet multiplier with modulus larger than 1, then γ is unstable. If γ is stable hyperbolic, then γ is asymptotically stable with asymptotic phase. In this latter case, there is a transversal to the periodic orbit for which the Poincar'e map is C^k.*

We also can prove the following result.

Theorem 4.2. *If (2.9), (2.11) are satisfied and $\gamma_{\tilde{f}}$ is a periodic orbit of $(2.1)_{\tilde{f}}$ which is stable hyperbolic, then there is a neighborhood V of $\tilde{f}$ in $C^2(C([-r,0],\mathbb{R});\mathbb{R})$ and a neighborhood U of $\gamma_{\tilde{f}}$ such that, for any $\tilde{g} \in V$, there is a unique periodic orbit $\gamma_{\tilde{g}}$ of $(2.1)_{\tilde{g}}$ in U, $\gamma_{\tilde{g}}$ is stable hyperbolic, $\gamma_{\tilde{g}} \to \gamma_{\tilde{f}}$ and the periods converge as $\tilde{g} \to \tilde{f}$.*

Proof. We only give an outline of the proof. Take any transversal N to the periodic orbit $\gamma_{\tilde{f}}$ and let $\pi_{\tilde{f}} : \mathcal{D}(\pi_{\tilde{f}}) \to N$ be the corresponding Poincaré map. Then there exists a neighborhood V of $\tilde{f}$ in $C^2(C([-r,0],\mathbb{R});\mathbb{R})$ such that, for any $\tilde{g} \in V$, we can define a Poincaré map $\pi_{\tilde{g}} : \mathcal{D}(\pi_{\tilde{g}}) \subset N \to N$. This map is continuous in $\tilde{g}$. If $p_{\tilde{f}} \in N$ is the point on $\gamma_{\tilde{f}}$, let $B(\delta, p_{\tilde{f}})$ be the ball with center $p_{\tilde{f}}$ and radius δ and choose $\delta > 0$ sufficiently small and the neighborhood V of $\tilde{f}$ so that $N \cap B(\delta, p_{\tilde{f}}) \subset D(\pi_{\tilde{g}}$ for every $\tilde{g} \in V$. The set $N \cap B(\delta, p_{\tilde{f}})$ is bounded open and convex. Since $\pi_{\tilde{f}}$ is an α-contraction and $\gamma_{\tilde{f}}$ is stable hyperbolic, the ω-limit set $\omega_{\tilde{f}}(B)$ is $p_{\tilde{f}}$; that is, $p_{\tilde{f}}$ is a local attractor for $\pi_{\tilde{f}}$ relative to the set $N \cap B(\delta, p_{\tilde{f}})$. Furthermore, we can use the continuous dependence of $\pi_{\tilde{g}}$ on $\tilde{g}$ and choose V so that $\omega_{\tilde{g}}(B)$ is a compact set which approaches $p_{\tilde{f}}$ as $\tilde{g} \to \tilde{f}$; that is, the local attractor for $\pi_{\tilde{g}}$ relative to the set $N \cap B(\delta, p_{\tilde{f}})$ is upper semicontinuous at $\tilde{f}$. Using Hale (1977, Chapter 4, Lemmas 4.2, 4.3) (or Hale and Verduyn-Lunel (1993)), we conclude that $\pi_{\tilde{g}}$ has a fixed point in B. This fixed point corresponds to a periodic orbit $\gamma_{\tilde{g}}$ of $(2.1)_{\tilde{g}}$. The orbit $\gamma_{\tilde{g}}$ is a C^2-manifold and we can define the

linear variational equation relative to the solution of $(2.1)_{\tilde{g}}$ describing $\gamma_{\tilde{g}}$. For $\gamma_{\tilde{g}}$, we can choose a synchronized stable manifold $W_0^s(\gamma_{\tilde{g}})$ as a transversal section to $\gamma_{\tilde{g}}$ (since $\gamma_{\tilde{g}}$ is hyperbolic and stable). If the period of $\gamma_{\tilde{g}}$ is $\omega_{\tilde{g}}$, this means that the Poincaré map $\hat{\pi}_{\tilde{g}}$ relative to the transversal $W_0^s(\gamma_{\tilde{g}})$ is given by $\hat{\pi}_{\tilde{g}}(\varphi) = T_{\tilde{g}}(\omega_{\tilde{g}})\varphi$, where $T_{\tilde{g}}(t)$ is the semigroup for $(2.1)_{\tilde{g}}$. Since the map $\hat{\pi}_{\tilde{g}}$ is obtained from the evaluation of the semigroup at a point $\omega_{\tilde{g}}$ which is independent of the point φ, it follows that $\hat{\pi}_{\tilde{g}}$ is C^2 and has the stable hyperbolic fixed point $p_{\tilde{g}}$. Therefore, there is a neighborhood $U_{\tilde{g}}$ of $p_{\tilde{g}}$ in which the only invariant set of $\hat{\pi}_{\tilde{g}}$ is $p_{\tilde{g}}$. The size of the neighborhood $U_{\tilde{g}}$ is determined by the exponential decay rate of the iterates of the derivative of $\hat{\pi}_{\tilde{g}}$ evaluated at $p_{\tilde{g}}$ and the second derivatives of $\tilde{g}$. Since $\tilde{g}$ is C^2-close to $\tilde{f}j$, this means that all of the neighborhoods $U_{\tilde{g}}$ can be chosen to be balls with diameter ρ independent of $\tilde{g} \in V$. This implies that the original Poincaré map $\pi_{\tilde{g}}$ on $N \cap B(\delta, p_{\tilde{f}})$ has a unique fixed point. This fixed point is a local attractor and the upper semicontinuity of the local attractors at $\tilde{f}$ implies the continuity properties stated in the theorem.

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