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Stability of Droplets for the Three Dimensional Stochastic Ising Model [1]

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Abstract: In this paper we study the evolution of the finite volume stochastic Ising model with magnetic field at very low temperatures starting from regular droplets, namely configurations on which all spins are -1 except those inside a parallelepiped. Eventually the process reaches either $\underline{-1}$, the configuration with all spins equal -1, or $\underline{+1}$, the configuration with all spins equal $+1$. We show that the choice of which one of those configurations is reached first is not random at very low temperatures but depends on a sharp condition on the lengths of the two smallest sides of the parallelepiped defined by the initial configuration. We also find the scale of time needed for this choice (the "relaxation time").

Key words: Stochastic Ising Model, relaxation times, droplets, metastability.

1 Introduction

We study here the stability of droplets for the finite volume three dimensional Stochastic Ising model with magnetic field [Ligg] for very low temperatures. This extends some of the results in [NS1] which considered the problem of stability of droplets and metastability in the two dimensional case. For the motivation on this problem and related questions see [CGOV, KO, MOS, NS1, NS2, Sch1, Sch2] and references therein.

The model is defined in the torus $\Lambda_N = \{1, \ldots, N\}^3$ with N a large but fixed positive integer and with periodic boundary conditions. The Hamiltonian is given by

$$H(\sigma) = -\frac{1}{2} \sum_{<x,y>} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x} \sigma(x)$$

where $\sigma(x) \in \{-1, +1\}$ is the spin at site $x \in \Lambda_N$, the first sum is taken over all pairs of nearest neighbors in Λ_N, the second sum is taken over all sites in Λ_N and h is the magnetic field that we assume positive.

The version of the stochastic Ising model considered here is the process $\{\sigma_t^\eta\}_{t \geq 0}$ on $\{-1, +1\}^{\Lambda_N}$ with $c(x, \eta)$, the rate with which the spin at site x flips when the current configuration is η, given by

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$$c(x,\eta) = \exp -\beta[\Delta_x H(\eta)]^+$$

where $\Delta_x H(\eta) = H(\eta^x) - H(\eta) = \eta(x)[\sum_{y:<x,y>} \eta(y) + h]$ with $\eta^x(y) = \eta(y)$ if $x \neq y$ and $\eta^x(x) = -\eta(x)$, the sum being taken over all nearest neighbors of x and for a real number $x,[x]^+ = \max\{0,x\}$. The process $\{\sigma_t^\eta\}_{t\geq 0}$ is reversible with respect to the Gibbs measure given by

$$\mu(\sigma) = (Z_N)^{-1} \exp(-\beta H(\sigma))$$

$$Z_N = \sum_\sigma \exp(-\beta H(\sigma))$$

One way to construct this process is as follows: at each event of a Poisson point process $\{N(t)\}_{t\geq 0}$ with rate N^3 choose a site x in Λ_N with uniform distribution and flip its spin with probability $c(x,\eta)$.

In $\{-1,+1\}^{\Lambda_N}$ there is a natural partial order given by: $\eta \leq \zeta$ if and only if $\eta(x) \leq \zeta(x)$ for all $x \in \Lambda_N$. The model we consider is *ferromagnetic* or *attractive* in the following sense: if $\eta(x) = \zeta(x) = +1$ and $\eta \leq \zeta$ then $c(x,\eta) \geq c(x,\zeta)$ and if $\eta(x) = \zeta(x) = -1$ and $\eta \leq \zeta$ then $c(x,\eta) \leq c(x,\zeta)$.

Let us say that two configurations η and ζ are nearest neighbors if they differ at a single site, that is, $\eta^x = \zeta$ for some $x \in \Lambda_N$. Denote this by $<\eta,\zeta>$.

For $A \subset \{-1,+1\}^{\Lambda_N}$ define the hitting time of A starting at η

$$T^\eta(A) = \inf\{t \geq 0 : \sigma_t^\eta \in A\}.$$

If $A = \{\zeta\}$, $\zeta \in \{-1,+1\}^{\Lambda_N}$, we write, for simplicity, $T^\eta(\zeta)$ instead of $T^\eta(\{\zeta\})$.

We want to study the evolution of this system as it starts from a configuration where all $+1$ spins are inside a single regular cluster. More precisely we are going to take as initial configurations for the process the set $\mathcal{R}$ of configurations such that all spins are -1 except those inside a parallelepiped with sides $1 \leq P \leq Q \leq R \leq N-2$. The condition on the length of the largest side is to make sure it is not a ring around Λ_N.

Starting in $\eta \in \mathcal{R}$ the process will spend a long time "near" this configuration if the temperature is low but eventually the droplet, that is, the subset of Λ_N where the $+1$ are, will either "shrink" and the process reaches $\underline{-1}$, the configuration with all spins equal -1, or "grow" and the process reaches $\underline{+1}$, the configuration with all spins $+1$. Whether $\underline{-1}$ or $\underline{+1}$ is reached first is not random for very low temperature but is determined by a sharp condition on the sizes of the two smallest sides of the paralellepiped defined by the initial configuration η. Moreover we determine the scale of time needed for this decision, that is, the relaxation time for the process.

In the two dimensional situation ([NS1]) the corresponding question was the stability of rectangles, that is, configurations on which all spins $+1$ are inside a rectangle. There the relaxation time was precisely the time needed for the appearance of a protuberance on the original rectangle, that is, the flip of a -1 spin which is neighbor to the rectangle. In this two dimensional case the protuberance is a one dimensional object . This two dimensional relaxation time, $\exp\beta(2-h)$ for $h < 1$, is essentially the time a one dimensional stochastic Ising model needs to go from $\underline{-1}$ to $\underline{+1}$. Analogously, as we show here, for the three dimensional case the relaxation time is equal to the time a two dimensional process needs to go through the same transition.

Write $\Gamma(h) = 4L - L^2h + Lh - h$ for the cost of energy to produce what was called the "protocritical droplet" in the two dimensional Ising model starting from the configuration with all spins -1. This corresponds to a configuration with all spins -1 except those inside a rectangle with lengths L and $L-1$ together with an additional site adjacent to one of its larger sides. L is defined as the smallest integer not smaller than $2/h$ and is the critical length in the two dimensional case. As was shown in [NS1] this is a configuration through which the process will pass with large probability for low temperature as it moves from $\underline{-1}$ to $\underline{+1}$ and the time it takes for this transition grows like $\exp \beta\Gamma(h)$ as the temperature decreases.

For $\eta \in \mathcal{R}$ write $P(\eta)$,$Q(\eta)$ and $R(\eta)$, with $P(\eta) \leq Q(\eta) \leq R(\eta)$, for the lengths of the three sides of the parallelepiped defined by η. With no loss in generality we may take $\eta \in \mathcal{R}$ to be such that $\eta(x) = +1$ if $x \in \{1, \ldots, P(\eta)\} \times \{1, \ldots Q(\eta)\} \times \{1, \ldots, R(\eta)\}$ and $\eta(x) = -1$ otherwise. Sometimes we write simply P, Q and R when it is clear what element of $\mathcal{R}$ is being considered.

For a positive integer k define

$$Q_c(k) = \begin{cases} \frac{2k}{kh-2} & \text{if } k > 2/h \\ \infty & \text{otherwise} \end{cases} \tag{1}$$

Following [NS1] we are only going to consider the "generic case" for the magnetic field by imposing that it does not assume a countable number of values. Moreover we only consider the more interesting case when $h < 1$ and when the volume of the whole system is large enough. More precisely define $\mathcal{H} = \{h \in (0,1) : h = 2/l + 2/m \text{ for two positive integers } l \text{ and } m\}$. For a given $\eta \in \mathcal{R}$ we say that we are in the *standard case* if $h \in (0,1) \setminus \mathcal{H}$ and $N > R(\eta) + 2$.

The condition in the field is such that $2/h$ is not an integer and neither is $Q_c(P(\eta))$ for any $\eta \in \mathcal{R}$.

Another important property valid in the standard case is that $\Delta_x H(\xi) \neq 0$ for any $\xi \in \{-1,+1\}^{\Lambda_N}$ and $x \in \Lambda_N$ and its value determines the kind of flip it involves.

Our main result here is the following

Theorem 1 *Let $\eta \in \mathcal{R}$ in the standard case. For any $\epsilon > 0$*

a) If $Q(\eta) < Q_c(P(\eta))$

$$\lim_{\beta \to \infty} P(T^\eta(\underline{-1}) < T^\eta(\underline{+1}), T^\eta(\underline{-1}) < \exp \beta(E(h) + \epsilon)) = 1$$

where

$$E(h) = \begin{cases} \Gamma(h) - 2(P(\eta) + Q(\eta)) + P(\eta)Q(\eta)h & \textit{if } P(\eta) > 2/h \\ h(L-2) & \textit{otherwise} \end{cases} \tag{2}$$

b) If $Q(\eta) > Q_c(P(\eta))$

$$\lim_{\beta \to \infty} P(T^{\eta}(\underline{+1}) < T^{\eta}(\underline{-1}), T^{\eta}(\underline{+1}) < \exp \beta(\Gamma(h) + \epsilon)) = 1$$

In the next section we present the basic results necessary for the proof of Theorem 1.

Crucial to this proof are some variational results on the lattice. The problem is basically to find what is the minimum value for the energy that can be attained among the elements of a given subset of $\{-1,+1\}^{\Lambda_N}$. This kind of problem is considered in [N] for the lattice with dimension $d \geq 2$.

Take $\eta \in \mathcal{R}$ with the $+1$ spins defining the parallelepiped $\{1,\ldots,P\} \times \{1,\ldots Q\} \times \{1,\ldots,R\}$ with $L \leq P \leq Q \leq R$. Define

$$\mathcal{I} = \{\zeta \geq \eta : |\zeta| \leq PQR + L(L-1) + 1\}$$

$$\overline{\mathcal{I}} = \{\zeta \geq \eta : |\zeta| = PQR + L(L-1) + 1\}$$

$$\mathcal{J} = \{\zeta \leq \eta : |\zeta| \geq PQ(R-1) + L(L-1) + 1\}$$

$$\overline{\mathcal{J}} = \{\zeta \leq \eta : |\zeta| = PQ(R-1) + L(L-1) + 1\}$$

Let now $\overline{\eta}$ be a configuration obtained from η by flipping to $+1$ all spins inside a two dimensional protocritical droplet on $\{1,\ldots,P\} \times \{1,\ldots,Q\} \times \{R+1\}$ and $\underline{\eta}$ be one obtained from η by flipping to -1 all spins inside the slice $\{1,\ldots,P\} \times \{1,\ldots Q\} \times \{R\}$ except those on a two dimensional protocritical droplet.

The variational results ([N]) that we need here are contained in the following lemma.

Lemma 1 *Let* $\eta \in \mathcal{R}$ *in the standard case*
a) $H(\overline{\eta}) = \min\{H(\zeta); \zeta \in \overline{\mathcal{I}}\}$
b) $H(\underline{\eta}) = \min\{H(\zeta); \zeta \in \overline{\mathcal{J}}\}$
c) $H(\zeta) - H(\eta) > 0$ *if* $\zeta \in (\mathcal{I} \cup \mathcal{J}) \setminus \{\eta\}$

Using this result and those in section 2 we prove parts a) and b) of the Theorem in sections 3 and 4 , respectively.

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2 Preliminary Results

We present here the basic technical results needed to prove Theorem 1.

Define the set of local minima M as follows

$$M = \{\eta : \Delta_x H(\eta) > 0 \text{ for all } x \in \Lambda_N\}.$$

As the smallest increment in energy possible is h we have that

$$\lim_{\beta\to\infty} P(\sigma_t^\eta = \eta \text{ for all } t \in [0, e^{\beta\epsilon})) = 1$$

if $0 < \epsilon < h$.

For $\eta \in M$ define $B_I(\eta)$ as the collection of all configurations from which the process can reach η through a sequence of spin flips such that each one decreases the energy. It is clear that

$$\{-1,+1\}^{\Lambda_N} = \cup_{\eta\in M} B_I(\eta).$$

For $\zeta \notin M$ and $\epsilon > 0$ let $A_\epsilon(\zeta)$ be the event "starting from ζ the process goes through a sequence of spin flips, each one decreasing the energy, reaching M before time $e^{\beta\epsilon}$".

Then as $|\{-1,+1\}^{\Lambda_N}| < \infty$

Lemma 2 *For all $\zeta \notin M$ and $\epsilon > 0$*

$$\lim_{\beta\to\infty} P(A_\epsilon(\eta)) = 1$$

Thus we may think that $B_I(\eta)$ is the "basin of attraction" of $\eta \in M$. We refer to $B_I(\eta)$, $\eta \in M$, as the *level-I basin of attraction of* η.

The interior and the exterior boundary of a subset $\mathcal{S}$ of $\{-1,+1\}^{\Lambda_N}$, denoted by $\partial_{int}\mathcal{S}$ and $\partial_{ext}\mathcal{S}$, respectively, are defined by

$$\partial_{int}\mathcal{S} = \{\zeta \in \mathcal{S} :< \zeta, \rho > \text{ for some } \rho \notin \mathcal{S}\}$$

$$\partial_{ext}\mathcal{S} = \{\zeta \notin \mathcal{S} :< \zeta, \rho > \text{ for some } \rho \in \mathcal{S}\}.$$

We say that a subset $\mathcal{S}$ of $\{-1,+1\}^{\Lambda_N}$ is *connected* if for any two configurations in $\mathcal{S}$ the process can move from one to the other without leaving $\mathcal{S}$.

Define δ_η as the height of the energy barrier to be overcomed from η to leave $B_I(\eta)$

$$\delta_\eta = \min\{H(\zeta) - H(\eta) : \zeta \in \partial_{int} B_I(\eta)\}$$

and write $\overline{B_I(\eta)} = \{\zeta \in B_I(\eta) : H(\zeta) = H(\eta) + \delta_\eta\}$.

We are going to use several times two kinds of couplings between processes which may be restricted to some connected subset of $\{-1,+1\}^{\Lambda_N}$ and, as in [NS1], we are going to call them *couplings A and B*. By a process restricted to $\mathcal{S}$ we mean a process with the same rates as the original one except for those corresponding to transitions between a configuration in $\mathcal{S}$ and one outside it which are set to zero. Write $\{\tilde{\sigma}_t^\eta\}_{t\geq 0}$ for the restricted process.

Coupling A: Both processes, $\{\sigma_t^\eta\}_{t\geq 0}$ and $\{\tilde{\sigma}_t^\eta\}_{t\geq 0}$, start in the same configuration in $\mathcal{S}$ and evolve together until the first one leaves $\mathcal{S}$; at this moment the second process stays still and they evolve independently afterwards.

Coupling B: Both processes, $\{\tilde{\sigma}_t^\zeta\}$ and $\{\tilde{\sigma}_t^\eta\}_{t\geq 0}$, are restricted to $\mathcal{S}$ but start from different configurations in $\mathcal{S}$; they evolve independently until they meet and evolve together afterwards.

The following result is well known ([NS1]) and will be used several times

Lemma 3 *Let $\mathcal{S}$ be a connected set and $\eta \in \mathcal{S}$ such that $H(\eta) < H(\zeta)$ for all $\zeta \in \mathcal{S} \setminus \{\eta\}$. Then for all $\zeta \in \mathcal{S}$ and $\epsilon > 0$*

$$\lim_{\beta\to\infty} P(\tilde{T}(\zeta) < \exp\beta(H(\zeta) - H(\eta) - \epsilon)) = 0$$

where $\tilde{T}(\zeta) = \inf\{t \geq 0; \tilde{\sigma}_t^\eta = \zeta\}$.

Its first application is to evaluate the time needed for the process $\{\sigma_t^\eta\}_{t\geq 0}$, $\eta \in M$, to reach another minimum.

Lemma 4 *If $\eta \in M$*

$$\lim_{\beta\to\infty} \frac{1}{\beta}\log T^\eta(M \setminus \{\eta\}) = \delta_\eta$$

in probability.

Proof: By lemma 2 once outside $B_I(\eta)$ the process quickly reaches $M \setminus \{\eta\}$ so that it is enough to verify that

$$\lim_{\beta\to\infty} \frac{1}{\beta}\log T^\eta(\partial_{ext} B_I(\eta)) = \delta_\eta.$$

Let $\{\tilde{\sigma}_t^\eta\}_{t\geq 0}$ be the process restricted to $B_I(\eta)$ coupled with $\{\sigma_t^\eta\}_{t\geq 0}$ by coupling A. Then

$$P(T^\eta(\partial_{ext} B_I(\eta)) > \tilde{T}^\eta(\partial_{int} B_I(\eta))) = 1$$

where $\tilde{T}^{\eta}(\partial_{int}B_I(\eta)))$ is the hitting time for the restricted process. Therefore

$$P(T^{\eta}(\partial_{ext}B_I(\eta)) < e^{\beta(\delta_\eta-\epsilon)}) \leq P(T^{\eta}(\partial_{int}B_I(\eta)) < e^{\beta(\delta_\eta-\epsilon)}) \leq \sum_{\rho\in\partial_{int}B_I(\eta)} P(\tilde{T}^{\eta}(\rho) < e^{\beta(\delta_\eta-\epsilon)}). \tag{3}$$

which goes to zero as β increases.

To obtain the upper bound for the exit time of $B_I(\eta)$ we verify that for all $\zeta \in B_I(\eta)$

$$P(\tilde{T}^{\zeta}(\overline{B_I(\eta)}) < 1) \geq ce^{-\beta\delta_\eta} \tag{4}$$

for some constant $c > 0$.

Let $\{\zeta_i\}_{i=1}^{k+s}$ be a sequence of configurations in $B_I(\eta)$ with $< \zeta_i, \zeta_{i+1} >$, for $i = 0, \ldots, k+s-1$, starting at $\zeta = \zeta_0$ and reaching $\eta = \zeta_k$ after k steps, $H(\zeta_i) > H(\zeta_{i+1})$, $i = 0, \ldots, k-1$ and then going to $\zeta_{k+s} \in \overline{B_I(\eta)}$ after s steps with $H(\zeta_i) < H(\zeta_{i+1})$, $i = k, \ldots, k+s-1$. Denote by B the event that "the process evolves along this sequence reaching $\overline{B_I(\eta)}$ before time 1". Then

$$P(\tilde{T}^{\eta}(\overline{B_I(\eta)}) < 1) \geq P(B) \geq |\Lambda_N|^{-(k+s)} P(N(1) \geq k+s) \Pi_{i=1}^{k+s} c(x_i, \zeta_i)$$

where $\{N(t)\}_{t\geq 0}$ is the Poisson process used for the construction of $\{\sigma_t^{\eta}\}_{t\geq 0}$ as given in the introduction and x_i, $i = 1, \ldots, k+s$, are such that $\zeta_{i+1} = \zeta_i^{x_i}$. Now $P(N(1) \geq k+s)$ does not depend on β and $\Pi_{i=1}^{k+s} c(x_i, \zeta_i) = e^{-\beta\delta_\eta}$ so that (4) follows.

(4) implies that

$$P(\tilde{T}^{\zeta}(\overline{B_I(\eta)}) > e^{\beta(\delta_\eta+\epsilon/2)}) \leq (1 - ce^{-\beta\delta_\eta})^{e^{\beta(\delta_\eta+\epsilon/2)}} \tag{5}$$

which goes to zero as β increases.

Before time $T^{\eta}(\partial_{ext}B_I(\eta))$ the processes $\{\tilde{\sigma}_t^{\eta}\}_{t\geq 0}$ and $\{\sigma_t^{\eta}\}_{t\geq 0}$ evolve together. Whenever they are in $\overline{B_I(\eta)}$ the next spin flip can take σ_t^{η} out of $B_I(\eta)$ (thus separating both processes) with probability bounded from below by $|\Lambda_N|^{-3}$. Moreover by Lemma 2 this takes place before time $e^{\beta\epsilon/2}$ with large probability as β increases. Therefore (5) implies that

$$\lim_{\beta\to\infty} P(T^{\eta}(\partial_{ext}B_I(\eta)) > e^{\beta(\delta_\eta+\epsilon)}) = 0 \tag{6}$$

□

In fact the upper bound on the exit time of $B_I(\eta)$, (6), with Lemma 3, implies that the exit from $B_I(\eta)$ will very likely occur by passing through $\overline{B_I(\eta)}$. More precisely let C be the event that "immediately before time $T^{\eta}(\partial_{ext}B_I(\eta))$ the process was in $\overline{B_I(\eta)}$". Then

Lemma 5

$$\lim_{\beta\to\infty} P(C) = 1$$

For $\eta \in M$ let $\{\tau_n\}_{n\geq 0}$ be the following sequence of stopping times:
$\tau_0^\eta = 0$ and $\tau_n^\eta = \inf\{t \geq \tau_{n-1}^\eta : \sigma_t^\eta \in M \setminus \{\sigma^\eta_{\tau^\eta_{n-1}}\}\}$ for $n \geq 1$.
Define now the process $\{X_t\}_{t\geq 0}$ in M by
$X_t^\eta = \sigma^\eta_{\tau_n}$ for $t \in [\tau_n^\eta, \tau_{n+1}^\eta)$.

We now consider the relaxation time of the process X_t^η and verify that it moves towards configurations which are more stable, that is, with larger δ_η. A configuration ζ with $\delta_\zeta < 2-h$, which implies $\delta_\zeta \leq h(L-2)$ by the definition of L, will first shrink and reach a smaller (in the partial order defined in $\{-1,+1\}^{\Lambda_N}$) and more stable one.

Let $\mathcal{A} = \{\zeta \in M : \delta_\zeta \leq h(L-2)\} = \{\zeta \in M : \delta_\zeta < 2-h\}$.

Lemma 6 *For $\eta \in M$ in the standard case and $\epsilon > 0$*

a) $\lim_{\beta\to\infty} P(T_X^\eta(\mathcal{A}^c) < e^{\beta(h(L-2)+\epsilon)}) = 1$

where $T_X^\eta(\mathcal{A}^c) = \inf\{t \geq 0 : X_t^\eta \notin \mathcal{A}\}$

b) $\lim_{\beta\to\infty} P(X^\eta_{T_X^\eta(\mathcal{A}^c)} \leq \eta) = 1$

Proof: By Lemma 4 each transition of the process X_t^η, $\eta \in \mathcal{A}$, occurs before time $\exp\beta(h(L-2)+\delta)$, for $\delta > 0$, with probability that goes to one as β increases. We prove the lemma by showing that X_t^η will very likely as β increases leave $\mathcal{A}$ before a number of steps which does not depend on β.

For $\eta \in \mathcal{A}$ consider a sequence $\eta = \zeta_0, \zeta_1, \ldots, \zeta_k = \rho$, where $\zeta_i = \zeta_{i-1}^{x_i}$ for some $x_i \in \Lambda_N$, $1 \leq i \leq k$, going to $\rho \in \overline{B_I(\eta)}$ only through transitions which increase the energy. Then

$$0 < H(\zeta_{i+1}) - H(\zeta_i) \leq \delta_\eta \leq h(L-2) < 2-h$$

which implies $H(\zeta_{i+1}) - h(\zeta_i) = h$ and $k \leq L-2$. Therefore each transition corresponds to a flip of $+1$ spin with exactly three neighbors which are -1 and $\rho \leq \eta$.

Let $x \in \Lambda_N$ be such that $\rho^x \notin B_I(\eta)$. As this flip decreases the energy

$$0 < H(\rho) - H(\rho^x) = -2\rho(x)[\sum_{y:<x,y>} \rho(y) + h]$$

We identify the value of $\rho(x)$ by verifying that it can not be -1. In fact if $\rho(x) = -1$ the previous inequality would imply

$$\sum_{y:<x,y>} \rho(y) \geq 0$$

as $h < 1$ and therefore

$$\sum_{y:<x,y>} \eta(y) \geq 0$$

as $\eta \geq \rho$.

But this last inequality can only hold together with $\eta \in M$ if $\eta(x) = +1$. In this case the spin at site x was flipped along the sequence $\{\zeta_i\}_{i=1}^{k}$ since it started with value $+1$ in η and ended up with value -1 in ρ. Therefore ρ^x can only differ from η at sites $\{x_1, \ldots, x_k\} \setminus \{x\}$ where the first configuration is -1 and the second $+1$.

As $\rho^x \geq \rho$ any -1 spin flipping with decrease of energy in ρ does the same in ρ^x. One can proceed this argument flipping successively the spins in ρ^x at sites $x_k, x_{k-1}, \ldots x_1$, whenever they are -1 and doing nothing otherwise, arriving at η only with flips which decrease the energy contradicting the hypothesis that $\rho^x \notin B_I(\eta)$.

The conclusion is that $\rho(x) = +1$. In this case the smallest possible decrease in energy is $2 - h > h(L-2)$.

If $\rho(x) \notin M$ the process reaches M quickly and only through transitions which decrease the energy with probability going to one as β increases by Lemma 2. Therefore

$$\lim_{\beta \to \infty} P(H(\rho) - H(\sigma^{\eta}_{\tau_1^{\eta}}) \geq 2 - h) = 1 \tag{7}$$

If $\sigma^{\eta}_{\tau_1^{\eta}} \in \mathcal{A}$ we repeat the argument. By (7)

$$\overline{B_I(\sigma^{\eta}_{\tau_1^{\eta}})} \cap B_I(\eta) = \emptyset$$

and Lemma 3 implies that the next configuration in M will not be η with large probability for low temperatures.

Therefore the sequence $\{\sigma^{\eta}_{\tau_1^{\eta}}\}$ will have, with probability going to one as β increase, decreasing energies leaving $\mathcal{A}$ before $|\mathcal{A}| < \infty$ steps.

□

We now identify a subset $\mathcal{M} \subset M$ containing the configurations on which the system spends most of the time. It is given by

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$$

where

$$\mathcal{M}_1 = \{\eta \in M : \delta_\eta > 2 - h\}$$

and

$$\mathcal{M}_2 = \{\eta \in M : \delta_\eta = 2 - h \text{ and } \overline{B_I(\eta)} \cap B_I(\zeta) = \emptyset \text{ for all } \zeta \notin \mathcal{A}\}$$

What we prove now is that very likely for low temperatures from any starting configuration the process reaches $\mathcal{M}$ before time $\exp \beta(2 - h + \epsilon)$, $\epsilon > 0$. Each element of $\mathcal{M}$ defines a *Level-II basin of attraction* which includes many elements of M and their Level-I basins of attraction. Before giving a precise definition we need the following lemma which is related to the stability of configurations in $\mathcal{M}_2$.

Lemma 7 *In the standard case let $\eta \in M$ with $\delta_\eta = 2 - h$ and $\xi \in M$ with $H(\xi) \geq H(\eta)$ and $\overline{B_I(\eta)} \cap B_I(\xi) \neq \emptyset$. Then if $\zeta \in M \setminus \{\xi, \eta\}$ we have*

$$\overline{B_I(\xi)} \cap B_I(\zeta) = \emptyset$$

Remark: This lemma states that after going from η to ξ by crossing the energy barrier of $2 - h$ the process will very likely return to η since the energy barrier between ξ and η is smaller than that needed to be overcomed in order to reach any other minima.

Proof: Take $\rho \in \overline{B_I(\eta)} \cap B_I(\xi)$. As ρ is reached through a sequence of spin flips with each one increasing the energy and $H(\rho) - H(\eta) = 2 - h$ it follows in the standard case that $\rho^{x_1} = \eta$ for some $x_1 \in \Lambda_N$ with

$$\Delta_{x_1} H(\eta) = \eta(x_1)[\sum_{y:<x_1,y>} \eta(y) + h] = 2 - h$$

which implies that $\eta(x_1) = -1$ and $\sum_{y:<x_1,y>} \eta(y) = -2$.

Let now $x_2 \in \Lambda_N$ be such that $\rho^{x_2} \in B_I(\xi) \setminus B_I(\eta)$ (which exists since $\rho \in \overline{B_I(\eta)}$). If x_2 is not nearest neighbor of x_1 we have

$$0 > \Delta_{x_2} H(\rho) = \Delta_{x_2} H(\eta)$$

since $\eta(s) = \rho(s)$ if $s \neq x_1$ which is a contradiction with $\eta \in M$. Therefore x_1 and x_2 are nearest neighbors.

By the hypothesis that $H(\xi) \geq H(\eta)$ we have $|\Delta_{x_2} H(\rho)| < 2 - h$ and hence $\Delta_{x_2} H(\rho) = -h$, $\rho(x_2) = -1$ and $\sum_{y:<x_2,y>} \rho(y) = 0$.

If $\rho^{x_2} \notin M$ we find a sequence of sites $x_3, x_4, \ldots, x_{k+1}$ for some $k \leq L - 2$ such that a configuration $\xi \in M$ is reached after the successive flips of the -1 spins at each of these positions and with each spin flip decreasing the energy by h. Write $Z = \{x_1, \ldots, x_{k+1}\}$.

Now $\xi \geq \eta$ and they are different only at Z. Also $\delta_\xi \leq kh < 2 - h$ since $\rho \in B_I(\xi)$ and the inequality relating $H(\xi)$ and $H(\eta)$ in the hypothesis is strict.

Let $\overline{\rho} \in \overline{B_I(\xi)}$ and $\overline{\rho}' \notin B_I(\xi)$ with $< \overline{\rho}', \overline{\rho} >$ (they differ at a single site). Let $\overline{\eta} \in M$ be such that there exists a sequence of spin flips each one decreasing the energy connecting $\overline{\rho}'$ and $\overline{\eta}$. Therefore if $\overline{\rho}' \in M$ take $\overline{\eta} = \overline{\rho}'$, otherwise take $\overline{\eta}$ so that $\overline{\rho}' \in B_I(\overline{\eta})$.

As before, $H(\overline{\rho}) - H(\xi) < H(\rho) - H(\xi) < 2 - h$ implies the existence of a sequence of sites $y_1, y_2, \ldots, y_l$, $l \leq k$ so that one goes from ξ to $\overline{\rho}$ by successively flipping the $+1$ spins at those sites, each flip increasing the energy by h. Also, by attractivity, the flip in going from $\overline{\rho}$ to $\overline{\rho}'$ (and then each flip to $\overline{\eta}$ if $\overline{\rho}' \notin M$) corresponds to a $+1$ spin changing to -1 with $H(\overline{\rho}) - H(\overline{\eta}) \geq 2 - h$ and thus $H(\overline{\eta}) \leq H(\eta)$.

Write S for the collection of the sites flipped in this sequence going from $\overline{\eta}$ to $\overline{\rho}$ and then to ξ. We now show that $S = Z$.

To simplify the exposition we introduce the following notation:
if $\eta \in \{-1, +1\}^{\Lambda_N}$ with $\eta(x) = +1$ ($\eta(x) = -1$) for all $x \in A \subset \Lambda_N$ we write $\eta(A) = +1$ ($\eta(A) = -1$).

Then $\eta(Z) = \overline{\eta}(S) = -1$ since those spins are flipped to $+1$ to reach ξ so that $\xi(Z \cup S) = +1$. Also $\eta(S \setminus Z) = \overline{\eta}(Z \setminus S) = +1$. To check the first statement take $x \in S \setminus Z$ so that $\xi(x) = +1$ and as x is not one of the sites where a -1 spin flipped to $+1$ in going from η to ξ along S we must have $\eta(x) = +1$. The second statement is similar.

It is also clear that η and $\overline{\eta}$ must be equal outside $S \cup Z$.

Define two sequences $\{\eta_i\}_{i=1}^m$, $m = |S|$, and $\{\overline{\eta}_i\}_{i=1}^n$, $n = |Z|$, as follows

$\eta_i(x) = -1$ if $x \in \{s_1, \ldots, s_i\}$ and $\eta_i(x) = \eta(x)$ otherwise for $1 \leq i \leq m$ where $s_1, s_2, \ldots, s_m$ is the sequence of sites where the spins are flipped in going from ξ to $\overline{\rho}$ and then to $\overline{\eta}$. Write also $\eta_0 = \eta$.

$\overline{\eta}_i(x) = -1$ if $x \in \{z_1, \ldots, z_i\}$ and $\overline{\eta}_i(x) = \overline{\eta}(x)$ otherwise for $1 \leq i \leq n$ where $z_1, z_2, \ldots, z_m$ is the sequence of sites where the spins are flipped in going from ξ to ρ and then to η. Write also $\overline{\eta}_0 = \overline{\eta}$.

Note that $\eta_m = \overline{\eta}_n$ since $\eta_m(S \cup Z) = \overline{\eta}_n(S \cup Z) = -1$ and $\eta_m = \overline{\eta}_n = \eta = \overline{\eta}$ outside $S \cup Z$. Write $\zeta = \eta_m = \overline{\eta}_n$.

We verify that $S = Z$ by showing that both sequences are constant, that is $\eta_i = \overline{\eta}_j = \eta$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

If $s_{i+1} \in Z$ for some $i = 0, \ldots, m-1$ we have $\eta_{i+1} = \eta_i$ and $H(\eta_{i+1}) - H(\eta_i) = 0$. If $s_{i+1} \notin Z$ then

$$H(\eta_{i+1}) - H(\eta_i) = \eta_i(s_{i+1})[\sum_{x:<x,s_{i+1}>} \eta_i(x) + h] \leq$$

$$\xi_i(s_{i+1})[\sum_{x:<x,s_{i+1}>} \xi_i(x) + h] = H(\xi_{i+1}) - H(\xi_i) \tag{8}$$

where $\xi_0 = \xi, \xi_1, \ldots, \xi_m = \bar{\eta}$ is the sequence of configurations obtained by successively flipping the spins on the sequence $s_1, \ldots, s_m$ starting at ξ. In (8) we used the fact that $\eta_i \leq \xi_i$ for $i = 0, 1, \ldots, m$.

The right hand side of (8) is equal to h for $i = 0, \ldots, l-1$ and negative for $i \geq l$ where $l \leq k$ is the number of spin flips needed to go from $\overline{\rho}$ to ξ as defined before.

If $H(\eta_{i+1}) - H(\eta_i) < 0$ for some $i = 1, \ldots, l-1$ we must have $H(\eta_{i+1}) - H(\eta_i) \leq h - 2 < 0$ which would imply $\eta_{i+1} \notin B_I(\eta)$ in contradiction with the hypothesis that $\delta_\eta = 2 - h$. Therefore we can assume that $H(\eta_{i+1}) - H(\eta_i) \geq 0$ for all $i = 0, \ldots, l-1$ and hence by (8) that $H(\eta_{i+1}) - H(\eta_i) = h$ whenever $\eta_{i+1} \neq \eta_i$ for $i + 0, \ldots, l-1$. For further reference call this *assumption A*.

Even if assumption A is true and the l first spin flips increased the energy by h if $\eta_{i+1} \neq \eta_i$ for some $i = l, \ldots, m$, (8) implies $\eta_{i+1} \notin B_I(\eta)$ which is a contradiction. Therefore the only way assumption A could be true is if ζ is reached from η by a sequence of at most l spin flips, each one increasing the energy by h: $H(\zeta) - H(\eta) = |S \setminus Z|h \le lh$.

Repeat the same arguments for the sequence $\overline{\eta}_i$, $0 \le i \le n$. ¿From $\overline{\eta}_0 = \overline{\eta}$ one goes to ζ in $|Z \setminus S|$ spin flips. By the analogue of (8) each increase of energy is equal to h: $H(\zeta) - H(\overline{\eta}) = |Z \setminus S|h$.

Let $x \in Z \setminus S$ be the site where the last spin flip in the sequence going from $\overline{\eta}$ to ζ occurs. As $\zeta \le \eta$ and $\zeta(x) = \eta(x) = -1$ we have

$H(\eta^x) - H(\eta) \le H(\zeta^x) - H(\zeta) = -h$ which is in contradiction with $\eta \in M$. Therefore assumption A is false and the lemma is proven

□

We now verify that starting from any configuration the process relaxes to $\mathcal{M}$ in a time of order $\exp \beta(2-h)$.

Lemma 8 *If $\rho \in \{-1,+1\}^{\Lambda_N}$ in the standard case and $\epsilon > 0$*

$$\lim_{\beta\to\infty} P(T^\rho(\mathcal{M}) < e^{\beta(2-h+\epsilon)}) = 1$$

Proof:

We only have to prove that

$$\lim_{\beta\to\infty} P(T^\eta_X(\mathcal{M}) < e^{\beta(2-h+\epsilon)}) = 1$$

for $\eta \in M$ such that $\delta_\eta = 2 - h$ and $\overline{B_I(\eta)} \cap B_I(\xi) \neq \emptyset$ for some $\xi \in \mathcal{A}$.

For η satisfying this condition let ξ be the next configuration in M reached by X^η_t. If $H(\xi) > H(\eta)$, Lemma 7, together with Lemma 5, implies that the process goes back to η with large probability as β increases.

Any configuration in $\overline{B_I(\eta)}$ can be reached with a single spin flip from η with probability larger than $|\Lambda_N|^{-3}$. Therefore, before time $\exp \beta(2 - h + \epsilon/2)$, $\epsilon > 0$, X^η_t reaches $\xi \in M \setminus \mathcal{A}$ such that $\overline{B_I(\eta)} \cap B_I(\xi) \neq \emptyset$ and $H(\eta) > H(\xi)$.

There are four possibilities to consider

Possibility 1: $\delta_\xi > 2 - h$. In this case $\xi \in \mathcal{M}$ and the lemma is proven.

Possibility 2: $\delta_\xi < 2 - h$. Then $\delta_\xi \le h(L-2)$ and Lemma 6 implies that $\xi' \in M \setminus (\mathcal{A} \cup \{\eta\})$, $H(\xi') < H(\xi)$, is reached before time $\exp \beta(h(L-2) + \epsilon/2) << \exp \beta(2 - h + \epsilon)$ with large probability as β increases. This ξ' may either satisfy Possibility 1 or Possibility 3 below.

Possibility 3: $\delta_\xi = 2 - h$ and $\overline{B_I(\xi)} \cap B_I(\zeta) \neq \emptyset$ for some $\zeta \in \mathcal{A}$.

In this case we are back to the condition satisfied by the initial configuration (η) with $H(\xi) < H(\eta)$ and we repeat the arguments as before with ξ replacing η. Since the total number of configurations is finite eventually a configuration in M is reached for which Possibility 3 does not hold.

Possibility 4: $\delta_\xi = 2 - h$ and $\overline{B_I(\xi)} \cap B_I(\zeta) = \emptyset$ for all $\zeta \in \mathcal{A}$. Then $\xi \in \mathcal{M}$ and the lemma is proven.

□

For $\eta \in \mathcal{M}$ we define $B_{II}(\eta)$, the *Level II basin of attraction of* η, as the set of all configurations from which the process can reach η before time $\exp\beta(2-h+\epsilon)$, ϵ positive and small, with probabilities that do not go to zero as β increases.

Define also, for $\eta \in \mathcal{M}$,

$$\Delta_\eta = \min\{H(\xi) - H(\eta); \xi \in \partial_{int} B_{II}(\eta)\}$$

which gives the height of the energy barrier to be crossed to leave $B_{II}(\eta)$ starting from η and

$$\overline{B_{II}(\eta)} = \{\xi \in B_{II}(\eta) : H(\xi) = H(\eta) + \Delta_\eta\}.$$

3 Subcritical Case

We prove here part a) of the Theorem.

We are going to write P, Q and R instead of $P(\eta), Q(\eta)$ and $R(\eta$, respectively.

If $P < L$ then $\delta_\eta \leq h(L-2)$ and $\eta \in \mathcal{A}$. By Lemma 6 a smaller configuration in $M \setminus \mathcal{A}$ is reached before time $\exp\beta(h(L-2)+\epsilon/2)$, $\epsilon > 0$. Repeating the arguments for each new configuration it is easy to see that $\underline{-1}$ is reached before time $\exp\beta(h(L-2)+\epsilon)$ with large probability for low temperatures .

Assume $L \leq P \leq Q \leq R$. Let $\{\sigma_t^{1,\eta}\}_{t\geq 0}$ be the process on which all spins inside the parallelepiped defined by η are frozen except those on one of the external slices with sides with lengths P and Q.

To be more precise take η such that its $+1$ spins are in $\{1,\ldots,P\}\times\{1,\ldots,Q\}\times\{1,\ldots,R\}$. Then $\{\sigma_t^{1,\eta}\}_{t\geq 0}$ is the process with rates given by

$c_1(x,\eta) = 0$ if $x \in \{1,\ldots,P\}\times\{1,\ldots,Q\}\times\{1,\ldots,R-1\}$ and $c_1(x,\eta) = c(x,\eta)$ otherwise.

Couple this process with the original one through the basic coupling [Ligg] so that

$$P(\sigma_t^\eta \leq \sigma_t^{1,\eta} \text{ for all } t \geq 0) = 1 \tag{9}$$

The first step is to prove that $\{\sigma_t^{1,\eta}\}_{t\geq 0}$ reaches $\mathcal{M}\setminus\{\eta\}$ before time $\exp\beta(E(h)+\epsilon)$, $\epsilon > 0$.

Lemma 9 *Let $\eta \in \mathcal{R}$ in the standard case with $L \leq P \leq Q \leq R$ and $Q < Q_c(P)$. Then for $\epsilon > 0$*

$$\lim_{\beta\to\infty} P(T_1^\eta(\mathcal{M}\setminus\{\eta\}) < e^{\beta(E(h)+\epsilon)}) = 1$$

where $T_1^\eta(\mathcal{M}\setminus\{\eta\}) = \inf\{t \geq 0 : \sigma_t^{1,\eta} \in \mathcal{M}\setminus\{\eta\}\}$.

To prove this lemma we first obtain a good estimate on the probability that the process leaves the Level-II basin of attraction of η ($B_{II}(\eta)$) during a time of the order of its relaxation time. This idea was used in [NS1] for similar problems.

Denote by $B_{II}^1(\eta)$ the Level II basin of attraction of $\{\sigma_t^{1,\eta}\}_{t\geq 0}$.

Lemma 10 *If $\eta \in \mathcal{R}$ in the standard case with $L \le P \le Q \le R$ and $Q < Q_c(P)$ then for all $\epsilon > 0$ and $\xi \in B^1_{II}(\eta)$ there exists $\beta_0 < \infty$ such that*

$$P(T^\xi_1(B^1_{II}(\eta)^c) \le e^{\beta(2-h+\epsilon)}) \ge \frac{1}{3}e^{-\beta(E(h)-2+h)}$$

for $\beta > \beta_0$.

Proof:
Let $\zeta \in \overline{B^1_{II}(\eta)}$ be one of the configurations that may be obtained from η by flipping all $+1$ spins on the slice $\{1,\dots,P\} \times \{1,\dots,Q\} \times \{R\}$ except those inside a protocritical droplet (a rectangle with sides of length L and $L-1$ with an additional site neighbor to one of the longest sides).

Let also $\underline{\zeta}$ be the configuration obtained from ζ by flipping the (unique) spin $+1$ which flips with rate one. Note that $H(\zeta) - H(\underline{\zeta}) = 2 - h$.

Write $\mathcal{F} = B^1_{II}(\eta) \cup \{\underline{\zeta}\}$ and let $\{\sigma^{2,\eta}_t\}_{t\ge 0}$ be the process restricted to $\mathcal{F}$ coupled with $\{\sigma^{1,\eta}_t\}_{t\ge 0}$ via coupling A and $\mu_{\mathcal{F}}$ be its invariant measure.

Clearly Lemma 10 holds for $\xi \in \partial_{int} B^1_{II}(\eta)$. Assume $\xi \in B^1_{II}(\eta) \setminus \partial_{int} B^1_{II}(\eta)$. In this case Lemma 6 implies that for $\delta > 0$

$$\lim_{\beta\to\infty} P(T^\xi_1(B^1_{II}(\eta)^c) > T^\xi_1(\eta), T^\xi_1(\eta) < \exp\beta(2-h+\epsilon)) = 1 \tag{10}$$

Therefore

$$\begin{aligned} &P(T^\xi_1(B^1_{II}(\eta)^c) < \exp\beta(2-h+\epsilon)) \ge \\ &P(T^\xi_1(\eta) < T^\xi_1(B^1_{II}(\eta)^c), T^\xi_1(\eta) < \exp\beta(2-h+\epsilon/2)). \\ &P(T^\eta_1(B^1_{II}(\eta)^c) \le \exp\beta(2-h+\epsilon/2)) \ge \\ &\frac{1}{2}P(T^\eta_2(B^1_{II}(\eta)^c) \le \exp\beta(2-h+\epsilon/2)) \end{aligned} \tag{11}$$

$$\tag{12}$$

for β large enough. In the first inequality we used the strong Markov property and for the second used (10) and the coupling between the processes.

To obtain a lower bound for (11) we introduce the process $\{\sigma^{2,\mu_{\mathcal{F}}}_t\}_{t\ge 0}$ restricted to $\mathcal{F}$ with the initial configuration chosen according to the invariant measure for the process $\mu_{\mathcal{F}}$. We couple this process with $\{\sigma^{2,\eta}_t\}_{t\ge 0}$ via coupling B.

Now if $s = \exp\beta(2-h+\epsilon)$

$$\begin{aligned} &P(T^\eta_2(B^1_{II}(\eta)^c) > s) \le \\ &P(\sigma^{2,\mu_{\mathcal{F}}}_t \ne \sigma^{2,\eta}_t \text{ for } 0 \le t \le s) + P(\sigma^{2,\mu_{\mathcal{F}}}_s \in B^1_{II}(\eta)) \end{aligned} \tag{13}$$

The first term in the right hand side of (13) goes to zero much faster than the second one.

$$P(\sigma_t^{2,\mu_{\mathcal{F}}} \neq \sigma_t^{2,\eta} \text{ for } 0 \leq t \leq s) \leq [\sup_{\xi,\rho\in\mathcal{F}} P(\sigma_t^{2,\xi} \neq \sigma_t^{2,\rho} \text{ for } 0 \leq t \leq se^{-\beta\epsilon/2})]^{e^{\beta\epsilon/2}} \tag{14}$$

with both processes in the right hand side of the inequality constructed independently.

The probability between brackets in (14) goes to zero by Lemma 8 and thus the term in the r.h.s. goes to zero faster than $2^{-e^{\beta\epsilon/2}}$ for β large enough.

For the second term in (13) we have

$$P(\sigma_t^{2,\mu_{\mathcal{F}}} \in B_{II}^1(\eta)) = 1 - \mu_{\mathcal{F}}(\underline{\zeta}) \leq 1 - \frac{1}{2}e^{-\beta(E(h)-2+h)} \tag{15}$$

for β large enough.

□

Now we can go back and prove Lemma 9.

$$\begin{aligned} &P(T_1^\eta(\mathcal{M}\setminus\{\eta\}) > \exp\beta(E(h)+\epsilon)) \leq \\ &P(T_1^\eta(\mathcal{M}\setminus\{\eta\}) > \exp\beta(E(h)+\epsilon), \\ &T_1^\eta(B_{II}^1(\eta)^c) \leq \exp\beta(E(h)+\epsilon) - \exp\beta(2-h+\epsilon/2)) + \\ &P(T_1^\eta(B_{II}^1(\eta)^c) > \exp\beta(E(h)+\epsilon) - \exp\beta(2-h+\epsilon/2)) \end{aligned} \tag{16}$$

The first term in the r.h.s. of (16) goes to zero by Lemma 7. For the second term let

$t_i = i\exp\beta(2-h+\epsilon/2)$, $i = 1, 2, \ldots,$.

Then

$$\begin{aligned} &P(T_1^\eta(B_{II}^1(\eta)^c) > \exp\beta(E(h)+\epsilon) - \exp\beta(2-h+\epsilon/2)) \leq \\ &[\sup_{\xi\in B_{II}^1(\eta)} P(T_1^\xi(B_{II}^1(\eta)^c) > \exp\beta(2-h+\epsilon/2)]^{\exp\beta(E(h)-2+h+\epsilon/2)-2} \end{aligned} \tag{17}$$

which goes to zero by Lemma 9

□

We now verify that the configuration in $\mathcal{M}\setminus\{\eta\}$ reached by $\{\sigma_t^{1,\eta}\}_{t\geq 0}$ is smaller than η.

Write $E = \{\xi \in \mathcal{M}\setminus\{\eta\} : \xi \geq \eta\}$.

Lemma 11 *For $\eta \in \mathcal{R}$ in the standard case with $L \leq P \leq Q \leq R$, $Q \leq Q_c(P)$ and $\epsilon > 0$*

$$\lim_{\beta\to\infty} P(T_1^\eta(E) < e^{\beta(\Gamma(h)-\epsilon)}) = 0$$

Proof: It follows from lemma 3 and parts a) and c) of lemma 1.
□

Since, by lemma 9 the process $\{\sigma_t^{1,\eta}\}_{t\geq 0}$ reaches $\mathcal{M}\setminus\{\eta\}$ before time $e^{\beta(E(h)+\epsilon)}$ $<< e^{\beta(\Gamma(h)-\epsilon)}$ for ϵ small and low temperatures, this lemma implies that it reaches a smaller configuration in $\mathcal{M}$. But there is only one, namely the one on which all spins are -1 except those inside $\{1,\ldots,P\}\times\{1,\ldots,Q\}\times\{1,\ldots,R-1\}$. By (9) the original process also shrinks and the arguments can be repeated until $\underline{-1}$ is reached.

4 Supercritical Case

We prove here part b) of the Theorem.

Define $\{\sigma_t^{3,\eta}\}_{t\geq 0}$ as the process on which all -1 spins are frozen except those over one of the "external slices" of η. More precisely we take its rates as follows $c_3(x,\eta)=c(x,\eta)$ if $x\in\{1,\ldots,P\}\times\{1,\ldots,Q\}\times\{1,\ldots,R+1\}$ and $c_3(x,\eta)=0$ otherwise. Couple $\{\sigma_t^{3,\eta}\}_{t\geq 0}$ with $\{\sigma_t^{\eta}\}_{t\geq 0}$ so that

$$P(\sigma_t^{3,\eta}\leq\sigma_t^{\eta}\text{ for all }t\geq 0)=1$$

Lemma 12 *If $\eta\in\mathcal{R}$ in the standard case with $P\leq Q\leq R$, $Q\geq Q_c$ and $\epsilon>0$*

$$\lim_{\beta\to\infty}P(T_3^{\eta}(\mathcal{M}\setminus\{\eta\})<\exp\beta(\Gamma(h)+\epsilon))=1$$

where $T_3^{\eta}(\mathcal{M}\setminus\{\eta\})=\inf\{t\geq 0:\sigma_t^{3,\eta}\in\mathcal{M}\setminus\{\eta\}\}$.

Proof:

We only sketch the proof as it is quite similar to the one given for Lemma 9.

Let $\zeta\in\overline{B_{II}^3(\eta)}$ be one of the configurations obtained from η by the addiction of a two dimensional protocritical droplet on the slice which is not frozen in the process $\{\sigma_t^{3,\eta}\}_{t\geq 0}$. Let now $\zeta_1,\zeta_2,\ldots,\zeta_{L-1}$ be a sequence of configurations obtained from ζ by flipping successively, in some arbitrary order, each -1 spin on the eroded line of the two dimensional protocritical droplet so that each flip decreases the energy by h. Then ζ_{L-1} is in M and corresponds to η with a square with sides with length L of spins $+1$ attached to the unfrozen external slice of η in the process $\{\sigma_t^{3,\eta}\}_{t\geq 0}$.

Next consider the process $\{\sigma_t^{4,\eta}\}_{t\geq 0}$ restricted to $\mathcal{G}=B_{II}^3(\eta)\cup\mathcal{E}$, with $\mathcal{E}=\{\zeta_1,\ldots,\zeta_{L-1}\}$, coupled with $\{\sigma_t^{3,\eta}\}_{t\geq 0}$ via coupling A.

As before we show that $\sigma_t^{4,\eta}$ leaves $B_{II}^3(\eta)$ before $\exp\beta(\Gamma(h)+\epsilon)$, $\epsilon>0$, with large probability for low temperatures. This is done exploiting the fact that the relaxation time for the process in $\mathcal{G}$ is $\exp\beta(h(L-1)+\delta)$, $\delta>0$ and that during a time of this order the process reaches ζ_{L-1} with a probability proportional to $\mu_{\mathcal{G}}(\zeta_{L-1})$ where $\mu_{\mathcal{G}}$ is the invariant measure of the restricted process.
□

Having proven that $\{\sigma_t^{4,\eta}\}_{t\geq 0}$ reaches $\mathcal{M} \setminus \{\eta\}$ before time $\exp\beta(\Gamma(h)+\epsilon)$ we proceed to show that this final configuration can not be smaller than η.

To do this we introduce the process $\{\sigma_t^{5,\eta}\}_{t\geq 0}$ on which all -1 spins outside η are frozen coupled with $\{\sigma_t^{\eta}\}_{t\geq 0}$ in such a way that

$$P(\sigma_t^{5,\eta} \leq \sigma_t^{4,\eta} \leq \sigma_t^{\eta} \text{ for all } t \geq 0) = 1$$

We use this process to prove

Lemma 13 *If $\eta \in \mathcal{R}$ in the standard case , $P \leq Q \leq R$, $Q > Q_c$ and $\epsilon > 0$ small enough*

$$\lim_{\beta\to\infty} P(T_4^{\eta}(\mathcal{M} \setminus \{\eta\}) < \exp\beta(\Gamma(h)+\epsilon)) = 0$$

Proof: It follows from lemma 3 and parts b) and c) of lemma 1. □ Therefore the process reaches a larger configuration in $\mathcal{M}$ which is larger than η with an extra slice. This new configuration is also supercritical by attractivity. We repeat the arguments until $\underline{+1}$ is reached.

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