Resenhas IME-USP 1994, Vol. 1, No. 4, 477 - 485.

Statistical Mechanics Of Disordered Models [1]

Henrique von Dreifus

Abstract: We review some rigourous results concerning the properties of the Gibbs measure of Disordered Models. In particular we discuss the region of high temperature in which the models can display a non-analytical behavior (Griffiths' Singularities) and some properties of the critical exponents when the model exihibits phase transitions.

Key words: disordered models, statistical mechanics, Griffiths' singularities, critical exponents.

1 Introduction

The study of the Statistical Mechanics properties of Disodered Models has been one of the main subjects in the investigation efforts of several authors and as a result, in the recent years, various rigourous results were obtained. In this article we review some results and open problems in the field, in particular those that in some extent reflect our own interest in the subject.

Although some of the results that we present here can be extended for a more general class of models, in this article we consider a class whose typical representative is an Ising model in Z^d whose Hamiltonian in a finite volume $\Lambda \subset Z^d$ is defined as

$$H_\Lambda = -\sum_{xy\in\Lambda^*} J_{xy}\sigma_x\sigma_y + B\sum_{x\in\Lambda} h_x\sigma_x + h\sum_{x\in\Lambda}\sigma_x \qquad (1)$$

where $\Lambda^* = \{xy; x, y \in \Lambda\}$. The couplings $J = \{J_{xy}, xy \in Z^{d^*}\}$ and the external fields $\mathrm{h} = \{h_x, x \in Z^d\}$ are independent families of independent identically distributed (within each family) random variables. If $B = 0$, the model may be used to describe a spin glass or a random ferromagnet; if the $J_{xy} = J > 0$, we have the random field Ising model.

When we consider models with disorder, i.e. with random parameters, the first question that arrises is that if the statistiscal mechanics properties differs from the pure system. In this direction, in 1969 Griffiths [1] considered the statistical properties of a random ferromagnetic Ising model, with a Hamiltonian given as above and pointed out that for the site diluted model, i.e., $J_{xy} = J\xi_x\xi_y$, where the independent random variables ξ_x are 1 or 0 with probability p or $1-p$ respectively. In this article Griffiths showed that the quenched magnetization, considered as a function of $z = e^{\beta h}$, displayed a non-analytical behavior at $z = 1$ for values of the inverse temperature β at which the system has neither long-range order nor spontenous magnetization. His argument should apply to a large class

[1] Research partially supported by FAPESP's Projeto Temático "Transição de Fase Dinâmica e Sistemas Evolutivos".

of ferromagnetic models; in particular, if the couplings $J_{xy} > 0$ are independent identically distributed random variables, which may assume with non zero probability arbitrarily large values, these singularities should occur for every value of the temperature. This phenomenon is now recognized to be a regular feature in the statistical mechanics of disordered models and constitutes in a concrete example of the relevance of the disorder for the statistical mechanics properties of disordered systems. Later in 1986, J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer [2] discussed the critical exponents of disordered models and argued that the critical exponent ν, associated to the correlation length of the Diluted Ferromagnetic Ising Model satisfies the bound $\nu \geq \frac{2}{d}$. In those situations the Harris criterion [3] leads one to expect that disorder is relevant for the critical behavior of the system i.e. the fixed point of the renormalization group transformation differs from the fixed point of the pure system.

This article is organized as follows. In section 2 we discuss some general properties of the pure Ising model. In section 3 we review some of the important results concerning Disordered Models. In section 4 we present a brief discussion of the results obtained in collaboration with A. Klein and J. F. Perez [4] on the high temperature/strong field regime of Disordered Models. In section 5, we discuss our results concerning bounds on the critical exponents of diluted ferromagnetic models [5]. Some open problems are mentioned in section 6.

2 Properties of the Pure Ising Models

We start with notation and definitions.

For each point $x \in Z^d$ we consider a random variable $\sigma(x)$ which takes values in $\{-1, +1\}$. Given, for each $x \in \Omega \subset Z^d$, a choice of $\sigma(x)$, we denote by $\tilde{\sigma} = \{\sigma(x)\}_{x \in Z^d}$ a configuration of the system and denote by Ω_Λ the set of configurations $\tilde{\sigma}$. On Ω_Λ we define a energy function H_Λ (Hamiltonian), which takes values in $\mathbf{R}$, as

$$H_\Lambda(\tilde{\sigma}) = -\sum_{|x-y|=1} J_{xy}\sigma(x)\sigma(y) + \sum_{x\in\Lambda} h(x)\sigma(x) \tag{2}$$

where $J_{xy} \in \mathbf{R}$; $h(x) \in \mathbf{R}$. This definition corresponds to the case in which we are considering Dirichlet (free) boundary conditions. Other boundary conditions can be considered restricting the values of $\tilde{\sigma}$ when $x \in \partial\Lambda$, with $\partial\Lambda$ defined as follows

$$\partial\Lambda = \{x \in \Lambda | \exists y \notin \Lambda, |x-y| = 1\} \tag{3}$$

For example we can consider the "+" boundary condition which consists to the restriction of the set of configurations to $\tilde{\sigma}$'s such the $\sigma(x) = +1$ for all $x \in \partial\Lambda$.

Given the Hamiltonian we can define the Canonical Gibbs measure in Ω_Λ as follows

$$\mu(\tilde{\sigma}) = \frac{e^{-\beta H_\Lambda(\tilde{\sigma})}}{Z_\Lambda} \tag{4}$$

where

$$Z_\Lambda = \sum_{\tilde{\sigma}} e^{-\beta H_\Lambda(\tilde{\sigma})} \tag{5}$$

For a function F defined on Ω_Λ, the expectation of F, denoted $\langle F \rangle$ is given by

$$\langle F \rangle_\Lambda = Z_\Lambda^{-1} \int F(\tilde{\sigma}) e^{-\beta H_\Lambda(\tilde{\sigma})} \prod_{x \in \Lambda} d\mu(\sigma(x)) \tag{6}$$

where $d\mu(\sigma(x)) = \delta(\sigma^2(x) - 1) d\sigma(x)$ for the Ising model. The free energy of the model in a finite volume Λ is defined as

$$f_\Lambda(\beta, J, h) = -\frac{1}{\beta|\Lambda|} \ln Z_\Lambda \tag{7}$$

and its thermodynamic limit is defined as

$$f(\beta, J, h) = \lim_{\Lambda \to Z^d} f_\Lambda(\beta, J, h) \tag{8}$$

In a finite volume Λ, the free energy is an analytic function of the parameters β, J, and h. However in the thermodynamic limit this is not necessarily true and in fact the free energy has a non-anlalytical behavior for some values of the parameters. This feature is in general refered as *Phase Transition*. Some derivatives of the free energy are of special interest in the study of the properties of the Ising model. Those are

- Magnetization

$$m(\beta, J, h) = -\frac{\partial f}{\partial h}(\beta, J, h) \tag{9}$$

- Internal Energy

$$U(\beta, J, h) = \frac{\partial f}{\partial J}(\beta, J, h) \tag{10}$$

- Magnetic Susceptibility

$$\chi(\beta, J, h) = \frac{\partial^2 f}{\partial^2 h}(\beta, J, h) \tag{11}$$

- Specific Heat

$$c_h(\beta, J, h) = \frac{\partial^2 f}{\partial^2 J}(\beta, J, H) \tag{12}$$

When the external magnetic field $\{h(x)\}_{x \in Z^d}$ is uniform, i.e. $h(x) = h$, and $J_{xy} \geq 0$, a general theorem by Lee and Yang [6] states that if $\Re(h) \neq 0$ then the free energy is an analytic function of h for all values of β and J. In the case when $h = 0$ and $d \geq 2$, it is well known that exists a critical value $\beta_c(J)$ such that if $\beta > \beta_c(J)$ then the magnetization as a function of h has a discontinuity at $h = 0$ and that the susceptibility χ diverges for $\beta = \beta_c(J)$. For $\beta < \beta_c(J)$ the correlation length defined as

$$\xi^{-1}(\beta) = -\frac{1}{|x|} \ln(\langle\sigma(0)\sigma(x)\rangle) \tag{13}$$

is finite for $\beta < \beta_c$ and diverges as $\beta \nearrow \beta_c$.

All this are related to the fact that for a fixed J exists a value β_c such that the free energy $f(\beta, h) = f(\beta, J, h)$ has a singularity at $(\beta_c, 0)$. Some of the properties of this phase transition are characterized by the divergences of the derivatives of the free energy when $\beta \nearrow \beta_c$. This leads to the definition of the so called "critical exponents" [7]. Examples of those exponents are

- Exponent γ

$$\chi(\beta, 0) \approx (\beta_c - \beta)^\gamma \tag{14}$$

as $\beta \nearrow \beta_c$

- Exponent α

$$c_h(\beta, 0) \approx (\beta_c - \beta)^\alpha \tag{15}$$

as $\beta \nearrow \beta_c$

- Exponent ν

$$\xi(\beta) \approx (\beta_c - \beta)^\nu \tag{16}$$

as $\beta \nearrow \beta_c$

- Exponent η

$$\langle\sigma(0)\sigma(x)\rangle(\beta_c) \sim |x|^{-d+2-\eta} \tag{17}$$

Those exponents are not expected to be independent and relations between them, called scaling relations, can be obtained from a Renormalization Group analysis. One example is the following relation:

$$\nu d = 2 - \alpha \tag{18}$$

For the Ising model in dimension $d = 2$ the values of the exponents can be obtained exactly and the values of α, γ and ν are:

$$\alpha = 0$$
$$\gamma = \tfrac{7}{4}$$
$$\nu = 1$$

For the Ising model in dimension $d = 3$ the values of the exponents can be obtained numericaly and the values are:

$$\alpha = 0.11 \pm 0.02$$
$$\gamma = 1.2417 \pm 0.010$$
$$\nu = 0.630 \pm 0.010$$

3 Models with Disorder

A more realistic thermodynamic description of ferromagnetism should consider the presence of impurities which can be modeled by considering the the parameters $\{J_{xy}\}_{xy\in Z^{d*}}$ and $\{h(x)\}_{x\in Z^d}$ as quenched random variables i.e. for a given function $F : \Omega_{Z^d} \to \mathbf{R}$ and a choice of the parameters J_{xy} and $h(x)$ we consider the Gibbs expectation as defined in (6). Then the thermodynamical properties of the model are obtained by taking the average over the radomness of the parameters J and h. Therefore if we denote by $\mathbf{E}_{J,h}\{\}$ the expectation (average) with respect to J and h the thermodynamic properties are obtained from the quenched expectations defined as

$$\overline{\langle F(\tilde{\sigma})\rangle} = \mathbf{E}_{J,h}\{\langle F(\tilde{\sigma})\rangle(\beta, h)\} \tag{19}$$

As examples of disodered models that are often present in the literature we have the Diluted ferromagnetic models for which $h(x) = 0$; $J_{xy} = 0$ if $|x - y| \neq 1$ and when $|x - y| = 1$, J_{xy} are i.i.d. random variables whose distribution does not have support on the negative real axis. The Ferromagnetic Random Field Ising Model for which $J_{xy} = J > 0$ and $h(x)$ are i.i.d. random varibles with mean zero. The Ferromagnetic Diluted Field Model for which $J_{xy} = J > 0$ and $h(x)$ are i.i.d. random varibles whose distribution does not have support on the negative real axis. The Spin Glass Models in which the J_{xy} are i.i.d. random variables with mean zero. Among the important contributions in the direction of proving rigourous results for disordered models we find the work by Griffiths in 1969 [1]. Later in 1975 Imry and Ma [8] wrote a paper in which they conjecture that for dimension $d > 2$ the Random Field Ising Model (RFIM) exhibit a Phase Transition in the same fashion as the pure model. Although this work by Imry and Ma can not be considered rigourous in the mathematical point of view, the ideas presented there became one of the guide lines of a paper published in 1988 by Bricmont and Kupiainen [9] in which they present a rigourous proof that the RFIM exhibit Spontaneous Magnetization for $d > 2$ when the covariance of the distribution of the magnetic field is small and β is large enough. For $d = 2$ the work of Bricmont and Kupianien were not conclusive and it was only in 1989 that Aizenman and Wehr [10] proved that for $d = 2$ the RFIM does not exhibit a Spontaneous Magnetization. In the small β region the difficult in applying the methods developed in the context of the pure system is due to the existence of the Griffiths'singularities. In 1984 Fröhlich and Imbre [11] developed a high temperature expansion suitable to study this region. Also, as mentioned in the introduction, there are results concerning the behavior or disordered systems near the critical points as for example [2].

In the next sections we give the statements and make comments on some of our contibutions to the field.

4 High Temperature Phase of Disordered Models

As mentioned in the previous section, Fröhlich and Imbre [11] developed a high temperature expansion for Disordered Models. Their work have two basic ingridients: a Mayer like expansion and a multi-scale approach for the probalistic estimates. In this direction we developed in collaboration with A. Klein and J.F. Perez [4] an alternative high temperature/strong field expansion. Our approach not only is simpler but also allowed us to obtain stronger results concerning the diffirentiability of the quenched free energy. Next we give the statement of our main result.

Consider the Ising Model with a Hamiltonian given as in (1) with $J_{xy} = 0$ if $|x - y| \neq 1$. For a given subset $S \subset \mathbf{R}$, $\mathbf{P}\{S\}$ denotes the probability of $J_{xy} \in S$. We will use $p_c^b(d)$ to denote the critical probability for bond percolation and $p_\infty = \mathbf{P}\{J_{xy} = \infty\}$. For a given local observable A we set $||A|| = \sup_{\tilde{\sigma}} ||A(\tilde{\sigma})||$. If A and B are local observables, we write $d(A, B)$ for the distance between the supports of A and B.

THEOREM (High Temperature Regime) If $p_\infty < p^b{}_c(d)$ there exists $\beta_1 = \beta_1(d) > 0$, such that:

(i) For all $0 < \beta < \beta_1$ we can find $C = C(\beta) < \infty$ and $m = m(\beta) > 0$, such that for any two local observables A and B and any finite $\Lambda \subset Z^d$ containing their supports, we have

$$\mathbf{E}_J(|\langle A; B\rangle_\Lambda|) \leq C|suppA|||A||||B||e^{-md(A,B)}, \tag{20}$$

for all $B \in \mathbf{R}$, $\{h_x\}_{Z^d} \in \mathbf{R}^{Z^d}$, $h \in \mathbf{R}$ and any boundary conditions on Λ.

(ii) There exists a set $\mathcal{J}$ of realizations of the random couplings J_{xy} with $\mathbf{P}\{J \in \mathcal{J}\} = 1$, and for each $0 < \beta < \beta_1$ we can choose $\mu = \mu(\beta) > 0$ with $\lim_{\beta \to 0)} \mu(\beta) = \infty$, such that if $J \in \mathcal{J}$ and $0 < \beta < \beta_1$, then for all $B \in \mathbf{R}$, $\{h_x\}_{Z^d} \in \mathbf{R}^{Z^d}$ and $h \in \mathbf{R}$:

(a) For any two local observables A and B, any finite Λ containing their supports, and any boundary condition on Λ, we have

$$(|\langle A; B\rangle_\Lambda|) \leq D_A||A||||B||e^{-\mu d(A,B)}, \tag{21}$$

for some $D_A = D(suppA, J, \beta) < \infty$.
(b) For every local observable A, the thermodynamical limit

$$\langle A\rangle \equiv \lim_{\Lambda \to Z^d} \langle A\rangle_\Lambda \tag{22}$$

exists and is independent of the boundary conditions used in each finite volume Λ. In particular, there is a unique Gibbs state.

(iii) For all $0 < \beta < \beta_1$, $B \in \mathbf{R}$, $h \in \mathbf{R}^{Z^d}$ and $h \in \mathbf{R}$, the quenched expectation of a local observable A is an infinitely differentiable function of the uniform external field h. In particular, for each $n = 1, 2, \ldots$ there exists a constant $C_n < \infty$, depending only on C, m and n, such that

$$\mathbf{E}(|\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}\rangle|) \leq C_n |suppA| \|A\| exp\{-\frac{2m}{(n+1)} d(A, x_1, \ldots, x_n)\} \tag{23}$$

for all local observables A and $x_1, \ldots, x_n \in Z^d$ and

$$\frac{\partial^n}{\partial h^n} \mathbf{E}(\langle A \rangle) = (-\beta)^n \sum_{x_1, \ldots, x_n \in Z^d} \mathbf{E}\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}\rangle. \tag{24}$$

REMARK Our results require no assumptions on the probability distributions except for the obvious requirement of no percolation of $J_{xy} = \infty$. A similar theorem for the case with a random field $h(x)$ and a large value of the non random parameter B as in the equation (1) is also proved in [4]. The basic ingredient of our proof is a bound for truncated expectations given by a sum of self avoiding random walks. After we obtain this bound the result follows either from a theorem by Kesten [12] or a simple multi scale probabilistic estimate. For the statement and the proof for the strong field case we refer the reader to [4].

5 Bounds on Critical Exponents of Diluted Ferromagnetic Models

In this section we mention our results concerning the critical behavior of Diluted Ferromagnetic Models. In [5] we give a proof of the bound $\nu \geq \frac{2}{d}$ first discussed by J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer in [2]. Our result is state in the following theorem:

THEOREM Let $\xi(\beta)$ be defined as

$$\xi^{-1}(\beta) = -\lim_{|x| \to \infty} \ln \frac{\overline{\langle \sigma(0)\sigma(x)\rangle}}{|x|} \tag{25}$$

and β_c defined as

$$\beta_c = \sup\{\beta | \overline{\langle \sigma(0)\sigma(x)\rangle} \leq e^{-|x|/\xi} \text{ as } |x| \to \infty, \text{ for some } \xi > 0\} \tag{26}$$

then

$$\xi(\beta) \ln \xi(\beta) \geq C(\beta_c - \beta)^{-2/d} \tag{27}$$

Our proof of this result combines ideas developed in [2] with a generalization of a technique, due to E. Lieb and B. Simon [13] and [14], developed to obtain estimates of correlation length in the context of non random models.

A important consequence of this result concerns to cases in which the critical exponent ν of the corresponding non random model satisfies a bound of the form $\nu < \frac{2}{d}$, as it happens with the Ising Model in dimension 3. In those cases either the phase transition exhibited by the diluted model occurs without a diverging specific heat or the scale relation (18) does not apply, in contrast with the non random model for which the scale relation is belived to be true and the specific heat is known to be divergent.

6 Open Problems

Some problems in the field that we have been interested are:

To obtain an alternative proof for the Bricmont and Kupianien [9] result . Their proof, as it happens in the work of Fröhlich and Imbre [11], makes use of a Mayer like expansion coupled to probability estimates which in the case of [9] are obtained from a sofisticated mechanism based on Renormalization Group ideas. We belive that a considerable simplification may be obtained if one produces an alternative for the use of Mayer like expansions.

Another question refers to the relevance of the Griffiths Singularities for the critical phenomena. Our result in [4] states that in the region of the Griffiths singularities the Free Energy is a C^∞ function although non analytic. The question is if there exists any trace of this non analiticity in some relevant Statistical Mechanics quantity. Our conjecture is that such trace might be found in a careful analysis of the behaviour of the correlation length when the temperature of system approachs the value for which the Griffiths Singularities start to occur.

Other problems are the Sharpeness of the Phase Transition, the Statistical Mechanics of Spin Glass in the low temperature regime and questions related to Sthocastic Dynamics associated to Disordered Models.

References

1. R. B. Griffiths, *Phys. Rev. Lett.*, **23**, 17-19 (1969).

2. J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer, *Phys. Rev. Lett.*, **57**, 2999 (1986).

3. A. B. Harris, *J. Phys. C*, **7** 1671 (1974).

4. H. von Dreifus, A. Klein and J.F. Perez, *Taming Griffiths' Singularities: Infinite Differentiability of Quenched Correlation Functions*, to appear on Commun. on Math Phys.

5. H. von Dreifus, *Ann. Inst. Henri Poincaré* , **55**, 657-669 (1991).

6. T. D. Lee and C. N. Yang, *Phys. Rev. Lett.* , **39** 410-419 (1952).

7. J. Glimm and A. Jaffe, *Quantum Physics*, Springer-Verlag, New York (1981).

8. Y. Imry and S. Ma, *Phys. Rev. Lett.* , **35**, 1399 (1975).

9. J. Bricmont and A. Kupianien, *Commun. Math. Phys.* , **116** 539 (1988).
10. M. Aisenman and J. Wehr, *Commun. Math. Phys.*, **130** 489 (1990).
11. J. Fröhlich and J. Imbre, *Commun. Math. Phys.* , **96** 145-180 (1984).
12. H. Kesten, *Percolation Theory for Mathematicians*, Birkhauser (1982).
13. E. Lieb, *Commun. Math. Phys.*, **77** 127 (1980).
14. B. Simon, *Commun. Math. Phys.*, **77** 111 (1980).

Henrique von Dreifus
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281
e-mail:dreifus@ime.usp.br
BRASIL