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Some aspects of the dynamics of the Cahn-Hilliard equation [1]

N.D. Alikakos and G. Fusco

Abstract: We describe some aspects of the dynamics of the Cahn-Hilliard equation. In particular we consider the dynamics of spherical interfaces and discuss a result showing that spherical interfaces either persist for ever or until they reach the boundary. We also discuss the dynamics of a small interface attached to the boundary.

Key words: Phase separation, Dynamics of interfaces, Gradient system, Invariant set.

1 Introduction

The phase diagram (u = concentration, T = temperature) of a binary alloy is schematically represented in Figure 1.

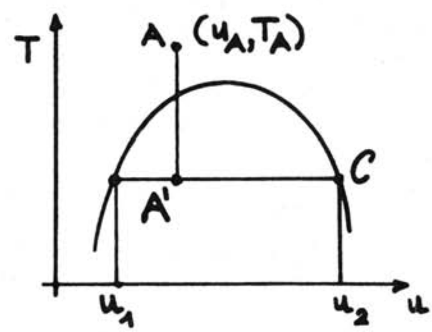

Figure 1: Phase diagram of a binary alloy

If the state of the alloy corresponds to a point $A = (u_A, T_A)$ above the "curve of miscibility" C, then the alloy is in thermodynamical equilibrium. Points below the curve C instead do not correspond to thermodynamical equilibrium. If the alloy is rapidly quenched from A to $A' = (u_{A'}, T_{A'})$ below the curve of miscibility a complicated phenomenon of phase separation begins and the alloy evolves toward a situation where two distinct phases with concentration u_1 and u_2, both corresponding to thermodynamical equilibrium coexist. The concentration u_1, u_2 are determined by the intersection of the curve C with the line $T = T_{A'}$. The amount of each of the two coexisting phases is determined by the condition that

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the average concentration remains constant during separation

$$\int_{\Omega} u \mathrm{d}x = |\Omega| u_A \tag{1}$$

where Ω, a bounded smooth domain of $I\!R^3$, is the region that contains the alloy and $|\Omega|$ is the measure of Ω.

To model the phenomenon of phase separation we can consider the gradient dynamics corresponding to a free energy functional of the type

$$J_\varepsilon(u) = \int_{\Omega} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x \ , \tag{2}$$

where F is a double well potential with two equal minima at u_1, u_2. Assuming conventionally that $u_1 = -1$, $u_2 = 1$ a typical choice for F is $F(u) = \frac{1}{4}(1-u^2)^2$. The contribution of F to J_ε represents the bulk free energy. The other term, by penalizing high gradients of the concentration function u, models the contribution of surface energy. The parameter $\varepsilon > 0$ is a measure of the importance of surface energy versus bulk free energy and it is assumed to be very small: $\varepsilon \ll 1$. When $\varepsilon \ll 1$ it can be expected that global minimizers of (2) constrained by (1), are functions which are near step functions jumping from u_1 to u_2 accross a thin interface of $O(\varepsilon)$ where most of the energy is concentrated. Global minimizers should therefore minimize the measure of the interface which, for $\varepsilon \ll 1$, is expected to be almost proportional to the free energy. This is in fact the case as was proved by Carr, Gurtin, Slemrod [CGS] in the one dimensional case and by Modica [M] and Sternberg [S] in the $n \geq 2$ case.

The gradient dynamics associated to the functional (2) under the constraint (1) depends on the choice of the Hilbert space with respect to which the gradient is computed. If one chooses $L_0^2(\Omega)$, the subspace of square integrable functions with zero average, then one obtains the following modified version of the Allen-Cahn equation

$$\begin{cases} u_t = \varepsilon^2 \Delta u - F'(u) + \frac{1}{|\Omega|} \int_{\Omega} F'(u) \mathrm{d}x & , x \in \Omega \\ \frac{\partial u}{\partial n} = 0 & , x \in \partial\Omega, \end{cases} \tag{3}$$

which sometimes, due to the presence of the integral term, is referred to as the nonlocal Allen-Cahn equation. The presence of this nonlocal term is rather unsatisfactory because, from a physical point of view, only points in a neighborhood of any given point x should influence the time variation of the concentration at x. If, following Fife, the chosen Hilbert space is H_0^{-1}, the subspace of the Sobolev space H^{-1} defined by (1), then one obtains the Cahn-Hilliard equation

$$\begin{cases} u_t = \Delta(-\varepsilon^2 \Delta u + F'(u)) & , x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 & , x \in \partial\Omega \end{cases} \tag{4}$$

which has a local character. We note explicitly that (1) is valid along solutions of (4). This follows by integrating both sides of (4) in Ω and by using

$$\frac{\partial}{\partial n}(-\varepsilon^2 \Delta u + F'(u)) = 0, \quad x \in \partial\Omega$$

which is a consequence of the boundary condition in (4).

The phenomenon of phase separation is rather complicated and can follow different routes including spinodal decomposition and/or nucleation, depending on the value of u_A. It is remarkable that the Cahn-Hilliard equation has a rich variety of solutions that can model at least from a qualitative point of view all this highly nonlinear behavior. There is now an ample literature on several aspects of the Cahn-Hilliard equation. [C1], [C2], [G1] deal with spinodal decomposition, [BF] with nucleation, [E], [EF] with the general existence theory and numerical studies, [ABF], [BrH], [G2], [BX1], [BX2] study dynamics of layers in one dimension, [P], [ABCh], [AF2], [St] study dynamics of layers in higher space dimension. Our discussion here focus on the dynamic of (4) for $\varepsilon \ll 1$ and t large. In Section 2 we give a qualitative description of the ultimate dynamics of (4) and in Section 3 we present two theorems that substantiate this description.

2 The ultimate dynamics of the Cahn-Hilliard equation

A basic concept for understanding the asymptotic behavior of dissipative systems is the concept of a "Global Attractor": a compact, connect set which is invariant and attracts all orbits of the system [H]. The gradient structure of the Cahn-Hilliard equation implies the existence of a global attractor $\mathcal{A}_\varepsilon$ in a suitable Sobolev space H. As we have anticipated in the introduction, we are interested in the dynamics of (4) for $0 < \varepsilon \ll 1$. In particular on those aspects of the dynamics that are meaningful in the limit $\varepsilon \to 0$ and therefore are expected to have a regular behavior in the limit $\varepsilon \to 0$. From this point of view the global attractor $\mathcal{A}_\varepsilon$ is probably not the right object to take under consideration. In fact $\mathcal{A}_\varepsilon$ has a very singular bahavior for $\varepsilon \to 0$. For instance it is possible to show that

$$\lim_{\varepsilon \to 0} \dim \mathcal{A}_\varepsilon = \infty \tag{5}$$

and

$$\lim_{\varepsilon \to 0} \text{ number of stationary solutions of (4) } = \infty\,. \tag{6}$$

The stationary solutions that appear in the attractor when $\varepsilon \to 0$ are expected to be more and more unstable in the sense of having a higher and higher dimensional unstable manifold. Therefore they should not be of particular importance for understanding the ultimate dynamics of (4). To this end one should instead concentrate the attention on subsets of the attractor of fixed dimension and try to understand their behavior in the limit $\varepsilon \to 0$. A natural object to consider is then the maximal compact invariant set K_ε contained in the set

$$H_c = \{\phi \mid J_\varepsilon(\phi) \leq c\} \tag{7}$$

(for suitable chosen values of c) and it is also natural to conjecture that, for generic F and Ω, for each $0 < \varepsilon \ll 1$, there is a sequence $c_0, c_1, \ldots, c_{N_\varepsilon}$ such that

$$N_\varepsilon = \dim \mathcal{A}_\varepsilon \tag{8}$$

and

$$\dim K_c = i \ , \quad \text{for } c_i \leq c < c_{i+1} \ . \tag{9}$$

When one adopts this point of view, the relevant question becomes the description of the set K_c for $c_i \leq c < c_{i+1}$ and i a fixed small integer. In particular the description of the set $K^i = K_{c_i}$. These sets in fact, unlike the global attractor $\mathcal{A}_\varepsilon$, are expected to be of fixed complexity and to have a well defined limit for $\varepsilon \to 0$. As was mentioned above the set K^0, which is the set of global minimizers (under the assumption that global minimizers are isolated), is made of layered functions which in the limit $\varepsilon \to 0$ approach step functions with values u_1 and u_2. For fixed i and $\varepsilon \to 0$ the same should be true for the set K^i. Then it becomes relevant to understand the evolution of layered functions and therefore the dynamics of interfaces or fronts. Pego [P] derived by the method of matched asymptotic expansion the following law of evolution of an interface Γ in the singular limit $\varepsilon \to 0$.

$$\begin{cases} \Delta\mu = 0 & , \ x \in \Omega \backslash \Gamma \\ \frac{\partial\mu}{\partial n} = 0 & , \ x \in \partial\Omega \\ \mu = \varepsilon\alpha K & , \ x \in \Gamma \ , \end{cases} \tag{10}$$

$$v = \beta \left[\frac{\partial \mu}{\partial n} \right]_\Gamma \ . \tag{11}$$

Here Γ is an orientable imbedded surface non intersecting $\partial\Omega$, $K = K(x)$ is the mean curvature of Γ at x, α, β positive constants, $v = v(x)$ is the velocity of Γ at $x \in \Gamma$ in the direction normal to Γ at x. The sign convention for K is that the curvature of a sphere is positive and the normal velocity of a shrinking interface is taken to be positive. The notation $[\frac{\partial\mu}{\partial n}]_\Gamma$ stands for the jump of the normal derivative of μ accross Γ. The function μ represents the first term in the asymptotic expansion of the expression $-\varepsilon^2\Delta u + F'(u)$ which in physical terms corresponds to the chemical potential. The unknown in equations (10), (11) is a family of manifolds $t \to \Gamma_t$. The mathematical theory for (10), (11) is still incomplete. For the case $\Gamma \subset I\!R^2$, Xinfu Chen [Ch] has proved local existence of a weak solution. Earlier results were given by Duchon and Robert [DR] who considered the case when Γ_0 is a graph. P. Constantin and M. Pugh [CP] studied a closely related problem.

Concerning the rigorous relationship between the Cahn-Hilliard equation and the limit problem (10), (11) Alikakos, Bates and Xinfu Chen [ABCh] proved convergence under the assumption that (10), (11) admit a smooth solution. During the evolution of a layered function according to equation (4), the energy (2) should be mostly concentrated in the interface and therefore, due to the gradient character of (4), in the limit $\varepsilon \to 0$ the measure of the interface should be nonincreasing.

It is easy to show that the limit evolution defined by (10), (11) has this property and that moreover in agreement with the constraint (1) the regions inside and outside Γ have constant measure during the evolution. In fact we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma_t| = -\int_{\Gamma_t} v = -\beta\int_{\Gamma_t}\left[\frac{\partial\mu}{\partial n}\right] = -\beta\int_{\Omega}\Delta\mu = 0\ , \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t| = -\int_{\Gamma_t} Kv = -\frac{\beta}{\varepsilon\alpha}\int_{\Gamma_t}\mu\left[\frac{\partial\mu}{\partial n}\right] = -\frac{\beta}{\varepsilon\alpha}\int_{\Omega}|\nabla\mu|^2 \leq 0\ , \tag{13}$$

where Ω_t is the region enclosed by Γ_t and $|\Gamma_t|$, $|\Omega_t|$ are the measures of Γ_t and Ω_t.

In the following we limit ourselves to the case of layered function with a connected interface homeomorphic to a sphere. On the basis of the above discussion, such an interface, if sufficiently far from the boundary of Ω will first approach a spherical shape. This will require a time interval of $O(1/\varepsilon)$ and is proved in [Ch] in the case $\Omega \subset I\!R^2$ under the assumption that the interface is initially close to a circle. The subsequent evolution cannot be guessed on the basis of the limit problem (10), (11). In fact spherical interfaces are equilibria of this problem as one checks by taking: $\mu = \varepsilon\alpha 1/R$, $R =$ radius of the sphere, $v = 0$. On the other hand, global minimizers should always correspond to interfaces that intersect the boundary $\partial\Omega$. The reason being that, when this is the case, part of the boundary is used together with the interface for enclosing a region of given volume and therefore an interface of smaller measure is needed. Therefore one can expect that once the layer has assumed an approximatly spherical shape it will not, in general, remain in equilibrium inside Ω as suggested by (10), (11) but it will slowly drift to the boundary keeping its spherical "bubble" shape. This is actually the case and we give below, cf. Theorem A, a rigorous result in this direction. Theorem A is taken from [ABrF] where a complete proof can be found. In the following we only sketch the main argument. Another proof which also renders detailed information on the dynamics of spherical interfaces can be found in [AF2] (cf. Theorem 7.2).

The drifting of the bubble toward the boundary is extremaly slow and it takes a time interval of order $e^{c/\varepsilon}$ ($c > 0$ a constant which depends on the initial position of the bubble but is independent of ε) to approach $\partial\Omega$. When the distance of the bubble from the boundary becomes of the same order of magnitude as the layer thickness, which is $O(\varepsilon)$, a rather drastic and quick transformation takes place resulting in a layered function with an interface intersecting $\partial\Omega$. We assume this interface is still connected as one can expect when the radius of the original bubble is sufficiently small. In this situation, after attaching to the boundary the interface should be close to a half sphere and the region enclosed by it can be regarded as a "drop" on $\partial\Omega$. The subsequent evolution of the interface can again be guessed by assuming that the energy is almost concentrated in the layer and therefore by assuming that along the evolution the measure of the interface is non increasing. Therefore it is natural to expect that the drop, in order to reduce the measure of the interface, will crowl on the boundary toward regions of higher and higher curvature. In Section 3 we present a theorem, Theorem B, which substantiates

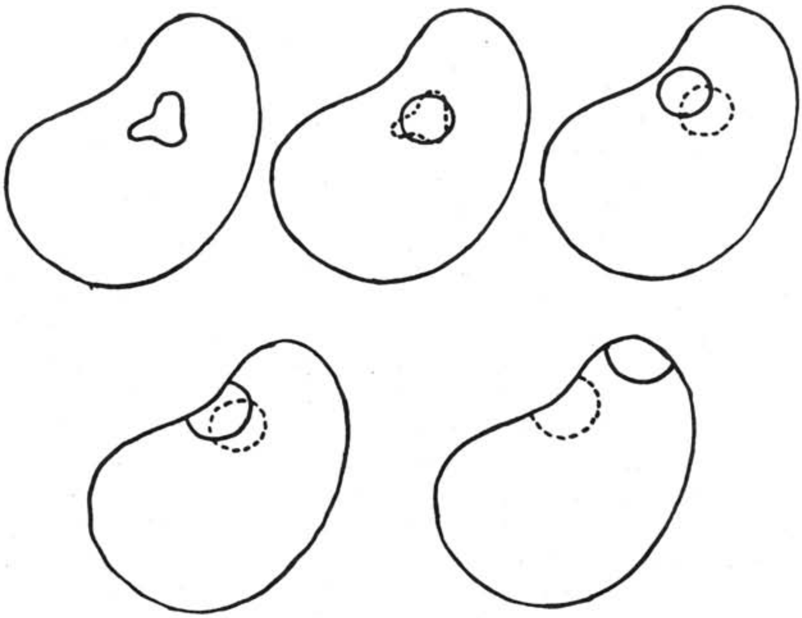

Figure 2: The evolution of a small interface

this conjecture and indicates that, after a time rescaling, the dynamic of a small drop on $\partial\Omega$ is described by the ordinary differential equation

$$\dot{\xi} = \operatorname{grad} K_{\partial\Omega}(\xi) \ , \tag{14}$$

where ξ is the center of the drop and $K_{\partial\Omega}(\xi)$ is the mean curvature of $\partial\Omega$ at $\xi \in \partial\Omega$. Figure 2 summarizes the various phases of the evolution of solutions of (4) with a connected interface enclosing a small region as described above.

Keeping in mind what we have said so far we can guess the structure of the sets K^i in some simple cases. Assuming $\Omega \subset I\!R^2$ and, as before, that the interface is simply connected and encloses a small region, the set K^2 should be homeomorphic to $\overline{\Omega}$. Away from an ε-neighborhood of $\partial\Omega$ the interior of Ω should correspond to the slow motion of a bubble. The boundary $\partial\Omega$ instead should correspond to the

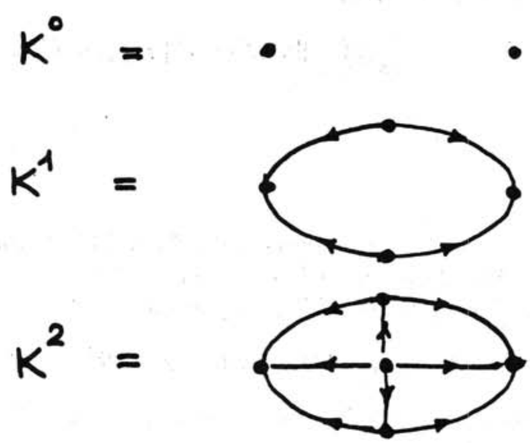

Figure 3: The sets K^i, $i = 0, 1, 2$ when $\Omega =$ ellipse

crowling of a drop on the boundary of Ω and can be identified with the set K^1. Figure 3 describes this situation for the case when Ω is an ellipse.

3 Two theorems on the dynamics of "bubbles" and "drops"

If $a > 0$ is a given number we let $\Omega_a = \{x \in \Omega \mid \mathrm{d}(x, \partial\Omega) > a\}$. Let $\rho > 0$ be fixed and assume that Ω_ρ is non empty. Let $\delta > 0$ be so small that $\Omega_{\rho+\delta} \neq \emptyset$. For each $\xi \in \overline{\Omega}_{\rho+\delta/2}$ define $s^\xi : \Omega \to I\!R$ by setting

$$s^\xi(x) = \begin{cases} -1 & , |x - \xi| < \rho, \\ 1 & , |x - \xi| > \rho, \end{cases} \tag{15}$$

and let $\mathrm{d}^\xi = \mathrm{d}(\xi, \partial\Omega) - \rho$.

Theorem A.[ABrF] *There exist $\overline{\varepsilon} > 0$ and constants c, $C > 0$, such that, given $\xi_0 \in \Omega_{\rho+\delta}$, there is a solution u^ε, $\varepsilon \in (0, \overline{\varepsilon}]$ of equation (4) with the following properties*

(i) $\|u^\varepsilon(0) - s^{\xi_0}\|_{L^1(\Omega)} < C\varepsilon$,

(ii) one of the following holds

(a) $\inf_{\xi\in\Omega_{\rho+\delta}} \|u^\varepsilon(t)-s^\xi\|_{L^1(\Omega)} < C\varepsilon$, $\forall t\in[0,\infty)$,

(b) there is $T>0$ *such that*

$$\inf_{\xi\in\partial\Omega_{\rho+\delta}} \|u^\varepsilon(T)-s^\xi\|_{L^1(\Omega)} < C\varepsilon \ .$$

(iii) $T > e^{\dfrac{cd^{\xi_0}}{\varepsilon}}$.

Proof: We only indicate the main points and refer to [ABrF] for a complete proof and related results. We also assume $F(u)=\frac{1}{4}(1-u^2)^2$.

The rescaled stationary Cahn-Hilliard equation

$$\Delta(-\Delta u + F'(u)) = 0 \ , \quad x\in I\!R^3 \ , \tag{16}$$

admits a one parameter family of bounded radial solutions $u(\cdot,\rho): I\!R^3 \to I\!R$

$$u(x,\rho) = U^*(|x|,\rho) \ , \tag{17}$$

where the function $U^*(r,\rho)$ is increasing in $r\in(0,\infty)$ and satisfies

$$\begin{cases} U^*(\rho,\rho) = O(\rho^{-1}) \ , \\ 1+U^*(0,\rho) = O(\rho^{-1}) \ . \end{cases} \tag{18}$$

Moreover as $r\to\infty$, $U^*(r,\rho)$ approaches exponentially a constant $\alpha(\rho)<1$

$$\alpha(\rho) - U^*(r,\rho) = 0(e^{-\nu(\rho)(r-\rho)}) \ , \quad r>\rho \ ; \quad \nu(\rho) = F''(\alpha(\rho))^{1/2} \ , \tag{19}$$

and

$$1-\alpha(\rho) = O(\rho^{-1}) \ . \tag{20}$$

These results are proved in [AF2] (cf. Proposition 2.1) in the $I\!R^2$ case but the proof, with obvious modifications, applies to $I\!R^n$. For each $\xi\in\Omega_{\rho+\delta/2}$ define a function $u^\xi:\overline{\Omega}\to I\!R$ by setting

$$u^\xi(x) =: U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho+a^\xi}{\varepsilon}\right) \ , \quad x\in\overline{\Omega} \ . \tag{21}$$

where a^ξ is determined by imposing the condition

$$\int_\Omega u^\xi \mathrm{d}x = \int_\Omega u^{\overline{\xi}} \mathrm{d}x \ , \tag{22}$$

for some fixed $\overline{\xi}\in\Omega_{\rho+\delta}$ and it is assumed $a^{\overline{\xi}}=0$. A standard argument based on the estimate (19) shows (cf. Lemma 3.1 in [AF2]) that

$$a^\xi = O(e^{-\frac{c}{\epsilon}}) \ . \tag{23}$$

Here and in the following c, C, ... stand for generic constants independent of ε that can change from line to line. The function u^ξ depends on ε and for $\varepsilon \ll 1$ presents an internal layer in an ε-neighborhood of the sphere $\{x \mid |x-\xi| = \rho\}$. Moreover, as $\varepsilon \to 0$, $u^\xi \to s^\xi$ pointwise and uniformly in compacts non intersecting $\{x \mid |x-\xi| = \rho\}$ and

$$\|u^\xi - s^\xi\|_{L^1(\Omega)} < C\varepsilon \; . \tag{24}$$

The main point in the proof is an analysis of the set $H_{\overline{c}}$ defined in (7) when

$$\overline{c} = \sup_{\xi \in \Omega_{\rho+\delta}} J_\varepsilon(u^\xi) + e^{-\frac{c}{\varepsilon}} \; . \tag{25}$$

Let $\phi \in H_{\overline{c}}$ be a function that satisfies

$$\int_\Omega \phi = \int_\Omega u^{\overline{\xi}} \; , \tag{26}$$

$$\begin{cases} \inf\limits_{\xi \in \Omega_{\rho+\delta/2}} \|\phi - u^\xi\|_{W^{1,2}_\varepsilon} < \eta \varepsilon^5 \; , \\ \inf\limits_{\xi \in \partial\Omega_{\rho+\delta/2}} \|\phi - u^\xi\|_{W^{1,2}_\varepsilon} > \overline{\delta} \; , \end{cases} \tag{27}$$

where $\eta > 0$ is a number to be fixed later, $\overline{\delta} > 0$ is a small fixed number and $\|f\|^2_{W^{1,2}_\varepsilon} = \varepsilon^2 \|\nabla f\|^2_{L^2} + \|f\|^2_{L^2}$.

We shall show that

$$\inf_{\xi \in \Omega_{\rho+\delta}} \|\phi - u^\xi\|_{W^{1,2}_\varepsilon} < e^{-\frac{c}{\varepsilon}} \; . \tag{28}$$

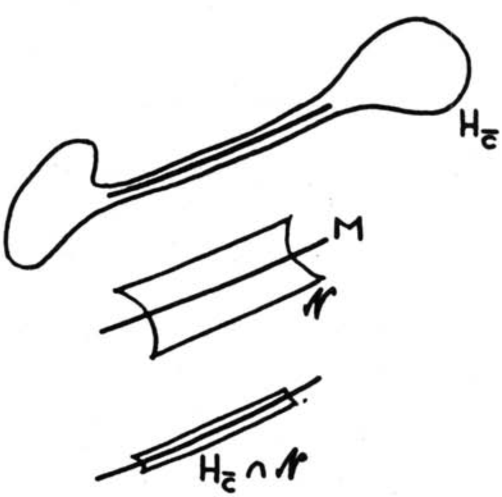

Figure 4: The sets $H_{\overline{c}}$, $\mathcal{N}$ and $H_{\overline{c}} \cap \mathcal{N}$

This estimate can be interpreted by saying that the intersection between $H_{\overline{c}}$ and the set $\mathcal{N}$ defined by conditions (27) is very thin, cf. Figure 4 where $M = \{u^\xi \mid \xi \in \Omega_{\rho+\delta}\}$.

A quite standard argument shows that each $\phi \in \mathcal{N}$, uniquely determines a $\xi \in \Omega_{\rho+\delta/2}$ such that

$$\phi = u^\xi + \psi \tag{29}$$

with ψ orthogonal to the manifold M in the L^2 sense

$$\langle \psi, u^\xi_{\xi_i} \rangle = 0 \ , \quad i = 1,2,3, \tag{30}$$

where $u^\xi_{\xi_i}$ is the derivative of u^ξ with respect to ξ_i. Then we expand $J_\varepsilon(\phi)$ in the form

$$\begin{aligned} J_\varepsilon(\phi) \quad &= \quad J_\varepsilon(u^\xi) + \int_\Omega (\varepsilon^2 \nabla u^\xi \nabla \psi + F'(u^\xi)\psi) \\ &\quad + \frac{1}{2}\int_\Omega (\varepsilon^2 |\nabla \psi|^2 + F''(u^\xi)\psi^2) + \int_\Omega u^\xi \psi^3 + \frac{1}{4}\int_\Omega \psi^4 \ . \end{aligned} \tag{31}$$

We estimate the various terms of (31).

a) $J_\varepsilon(\phi) - J_\varepsilon(u^\xi) < e^{-\frac{c}{\varepsilon}}$.
To show this we note that $\phi \in H_{\overline{c}}$ implies

$$\begin{aligned} J_\varepsilon(\phi) - J_\varepsilon(u^\xi) \quad &< \quad \sup_{\xi \in \Omega_{\rho+\delta}} J_\varepsilon(u^\xi) + e^{-\frac{c}{\varepsilon}} - J_\varepsilon(u^\xi) \\ &\leq \quad e^{-\frac{c}{\varepsilon}} + \sup_{\xi \in \Omega_{\rho+\delta}} J_\varepsilon(u^\xi) - \inf_{\xi \in \Omega_{\rho+\delta}} J_\varepsilon(u^\xi) \ . \end{aligned} \tag{32}$$

On the other hand, if ξ, $\overline{\xi}$ are any two points in $\Omega_{\rho+\delta}$ and B_ξ, $B_{\overline{\xi}}$ are the balls of center ξ, $\overline{\xi}$ and radius $d = \min\{d(\xi, \partial\Omega), d(\overline{\xi}, \partial\Omega)\}$, then

$$\begin{aligned} J_\varepsilon(u^\xi) - J_\varepsilon(u^{\overline{\xi}}) \quad &= \quad \int_{\Omega \setminus B_\xi} \left(\frac{\varepsilon^2}{2} |\nabla u^\xi|^2 + F(u^\xi) \right) \\ &\quad - \int_{\Omega \setminus B_{\overline{\xi}}} \left(\frac{\varepsilon^2}{2} |\nabla u^{\overline{\xi}}|^2 + F(u^{\overline{\xi}}) \right) + O(e^{-\frac{c}{\varepsilon}}) \ . \end{aligned} \tag{33}$$

This is a consequence of the estimate (23) and of the fact that aside from the small correction due to the presence of a^ξ the functions $u^\xi\Big|_{B^\xi}$, $u^{\overline{\xi}}\Big|_{B^{\overline{\xi}}}$ are one a translation of the other. From the definition of u^ξ, U^* and in particular from (19) it follows that $u^\xi(x) - \alpha(\rho/\varepsilon) = O(e^{-\frac{c}{\varepsilon}})$, for $x \in \Omega \setminus B_\xi$ and the same is true for $u^{\overline{\xi}}(x) - \alpha(\rho/\varepsilon)$. Therefore (33) implies

$$|J_\varepsilon(u^\xi) - J_\varepsilon(u^{\overline{\xi}})| = O(e^{-\frac{c}{\varepsilon}}) \tag{34}$$

that together with (32) proves a).

b) $\int_\Omega(\varepsilon^2\nabla u^\xi\nabla\psi+F'(u^\xi)\psi)=\int_\Omega(-\varepsilon^2\Delta u^\xi+F'(u^\xi))\psi+\varepsilon^2\int_{\partial\Omega}\psi\frac{\partial u^\xi}{\partial n}=O(e^{-\frac{c}{\varepsilon}})$.
From equations (22), (26), (29) it follows

$$\int_\Omega\psi=0\ , \tag{35}$$

and equation (16) and the definition of u^ξ imply

$$-\varepsilon^2\Delta u^\xi+F'(u^\xi)=\text{const.} \tag{36}$$

Therefore the first integral on the right hand side of b) vanishes. The other integral is $O(e^{-\frac{c}{\varepsilon}})$. In fact $\frac{\partial u^\xi}{\partial n}=O(e^{-\frac{c}{\varepsilon}})$, because u^ξ approaches exponentially a constant for $\frac{|x-\xi|}{\varepsilon}\to\infty$ and by the trace theorem $\|\psi\|_{L^2(\partial\Omega)}\le C\varepsilon^{-1}\|\psi\|_{W^{1,2}_\varepsilon}$.

c) $\int_\Omega(\varepsilon^2|\nabla\psi|^2+F''(u^\xi)\psi^2)\ge C\varepsilon^2\|\psi\|^2_{W^{1,2}_\varepsilon}$.
This estimate is a rather standard consequence of the estimate

$$\int_\Omega(\varepsilon^2|\nabla\psi|^2+F''(u^\xi)\psi^2)\ge C\varepsilon^2\|\psi\|^2_{L^2}\ , \tag{37}$$

which, under the conditions (30) and (35), is shown to be true in [AF1] (cf. the proof of the estimate 4.13).

d) $\left|\int_\Omega u^\xi\psi^3\right|\le C\eta\varepsilon^2\|\psi\|^2_{W^{1,2}_\varepsilon}$.
To show this one first proves that $\phi\in\mathcal{N}$ implies

$$\|\psi\|_{W^{1,2}_\varepsilon}<C\eta\varepsilon^5\ . \tag{38}$$

From this and the Sobolev imbedding theorem it follows

$$\left|\int_\Omega u^\xi\psi^3\right|\le C\int_\Omega|\psi|^3\le\frac{C}{\varepsilon^3}\|\psi\|^3_{W^{1,2}_\varepsilon}\le C'\eta\varepsilon^2\|\psi\|^2_{W^{1,2}_\varepsilon}\ . \tag{39}$$

Putting together a), b), c), d) we obtain from equation (31)

$$e^{-\frac{c}{\varepsilon}}\ge(C-C'\eta)\varepsilon^2\|\psi\|^2_{W^{1,2}_\varepsilon} \tag{40}$$

which, assuming $\eta<C/C'$, implies (28).

The estimate (28) is the main step in the proof of the theorem. In fact, for $\varepsilon\ll1$, it results $e^{-\frac{c}{\varepsilon}}<\eta\varepsilon^5$, and therefore, if $u(t)=u^{\xi(t)}+\psi(t)$ is a solution of (4) such that

$$\|\psi(0)\|_{W^{1,2}_\varepsilon}<\eta\varepsilon^5\ ,\ \ \xi(0)\in\Omega_{\rho+\delta}\ , \tag{41}$$

then the only possibility for (27) to be violated is that the second of the conditions (27) is violated. That is, there is $\bar{t} > 0$ such that

$$\inf_{\xi \in \partial\Omega_{\rho+\delta/2}} \|u(\bar{t}) - u^{\xi}\|_{W_{\varepsilon}^{1,2}} = \bar{\delta} \ , \tag{42}$$

that, assuming $\bar{\delta}$ sufficiently small, yields the existence of $0 < T < \bar{t}$ such that $\xi(t) \in \partial\Omega_{\rho+\delta}$.

This essentially proves points (i), (ii) of the theorem. Point (iii) is also a consequence of the estimate (28) and is obtained by estimating with the help of (28) and some improved form of it the right hand side of (4). □

Remark: As we have seen the proof of Theorem A is essentially an analysis of the set $H_{\bar{\varepsilon}}$ and uses the fact that (4) is a gradient dynamics correspoding to the functional J_{ε}. Therefore a statement like the one in Theorem A also applies to any other gradient dynamics that can be associated to J_{ε} and satisfies (1), in particular to the dynamics defined by equation (3).

We end this section by quoting whithout proof a theorem taken from [AChF] and dealing with the dynamics of an infinitesimal "drop" crowling on $\partial\Omega$.

Theorem B: *There exist $\bar{\varepsilon} > 0$ such that, given $\xi_0 \in \partial\Omega$, there is a family u_{ε}, $\varepsilon \in (0, \bar{\varepsilon}]$ of solutions of (4) that satisfies*

$$\begin{cases} \lim\limits_{\varepsilon \to 0} u_{\varepsilon}(x, \varepsilon^{-1}\tau) = 1 & , \ x \in \overline{\Omega} \backslash \{\xi(\tau)\}, \\ \lim\limits_{\varepsilon \to 0} u_{\varepsilon}(\xi(\tau), \varepsilon^{-1}\tau) = -1 & , \end{cases} \tag{43}$$

where $\xi(\cdot) : [0, \infty) \to \partial\Omega$ is the solution of the problem

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \beta \operatorname{grad} K_{\partial\Omega}(\xi) \ , \\ \xi(0) = \xi_0 \ . \end{cases} \tag{44}$$

In equation (44) $K_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with the sign convention that the curvature of a sphere is positive and β is a constant.

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N.D. Alikakos
Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996-1300
USA

G. Fusco
Dipartimento di Matematica
II Universitá di Roma
Via Ricerca Scientifica
00133 – ROMA
e-mail:fusco%40676.hepnet@Csa4.LBL.Gov
Italy