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A Study of the Bogdanov-Takens Bifurcation

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Abstract: A nilpotent singular point for a planar vector field, i.e. a singular point with linear part equivalent to $y\frac{\partial}{\partial x}$, is generically of codimension 2. A two parameter versal unfolding for generic nilpotent singular point was studied independently by Takens and Bogdanov and so one now calls it : the Bogdanov-Takens bifurcation. Historically, it was the last codimension 2 singularity to be treated. The reason is the difficulty one has to prove existence and unicity of the limit cycle which appears by bifurcation. Here, we present a complete and simplified proof of the versality of the Bogdanov-Takens unfolding.

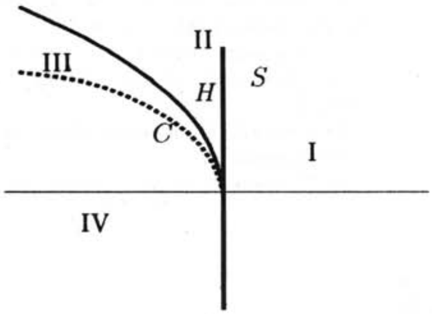
Key words: Bifurcation, planar vector fields, limit cycle, abelian integral.

AMS classification : 34C05, 58F14, 34C23.

Contents

Introduction

Historically the Bogdanov-Takens bifurcation was the last codimension two bifurcation of vector fields in the plane to be treated. This paper presents as main result the versal unfolding of a vector field whose linear part is of the form $y\frac{\partial}{\partial x}$. Though the result is not new, the presentation of the proof is simplified.
The following bifurcation diagram is one of the two that will be obtained, the other being similar:



In the region I there are no singularities present in the flow. The curve S is a curve of generic saddle-node bifurcations, and thus there are a saddle and a repelling fixed point present in II. From II to III we pass by the curve H, which denotes a line of generic Hopf bifurcations. Consequently in III there are a saddle fixed point, an attracting fixed point and a repelling limit cycle around the latter. The limit cycle disappears in a (global) saddle loop bifurcation as we pass from III to IV by the curve C. Finally the attracting and the saddle fixed point in IV coalesce in a saddle node bifurcation as we pass back to I via S.
The curves H and C have quadratic tangency at 0 to the vertical axis, which is equal to the curve S.

1 Preliminaries

Briefly recall the definitions of a germ and a jet.

Definitions

Consider the following equivalence relation of functions: two C^∞-functions $f, g : R^2 \to R$ are equivalent to each other at the point 0, $f \sim g$, if there exists a neighbourhood $U \subset R^2$ of 0 such that $f \equiv g$ on U. The equivalence classes of $\sim$ are called **germs** *of functions at 0.*

Similarly consider the following equivalence relation: two C^∞-functions $f, g : R^2 \to R$ are equivalent to each other at the point 0, $f \overset{k}{\sim} g$, if their difference *is k-flat at zero, that is if $\varphi = f - g$ has all derivatives up to order $k+1$ equal to zero at 0:*

$$D^\alpha \varphi(0,0) = 0\,, \qquad |\alpha| = \alpha_1 + \alpha_2 \leq k+1$$

Here the equivalence classes of $\overset{k}{\sim}$ *are called* **jets of order k, or k-jets**. *In each equivalence class there is exactly one polynomial of order k, which is often taken as the representative of that class, the* **Taylor polynomial**.

Furthermore recall the notions of (fibre)-C^0-equivalency, induced family, unfolding and versal unfolding (quoted from [2]).

Definitions

A C^∞ **k-parameter family of vector fields** *(X_μ) in the plane is a vector field of the form*

$$X_\mu(x,y) = X_1(x,y,\mu)\frac{\partial}{\partial x} + X_2(x,y,\mu)\frac{\partial}{\partial y}\,, \qquad \mu \in R^k$$

where the X_i are C^∞ in all their variables.

Two k-parameter families (X_μ) and (X'_μ) in the plane, with μ in the same space of parameters, are called **(fibre-C^0, C^r)-equivalent** *if there exist a map φ in the parameter space, being a homeomorphism if $r=0$ or a C^r-diffeomorphism if $r \geq 1$, and homeomorphisms h_μ such that for each $\mu \in R^k$, the function h_μ is a C^0-equivalence between the vector fields X_μ and $X'_{\varphi(\mu)}$. If φ can be chosen to be the identity, then we call the families* **fibre-C^0-equivalent over the identity**.

Let $\varphi : (R^l,0) \to (R^k,0)$ (φ may only be defined on a neighbourhood of 0) be a continuous mapping, and (X_μ) a family with parameters $\mu \in R^k$. Then the vector field Y_λ is called the **family induced by** *φ, if*

$$Y_\lambda = X_{\varphi(\lambda)}$$

where $\lambda \in R^l$. The field Y_λ is called **C^r-induced by** *φ if φ is C^r.*

An **unfolding of a germ of a vector field** *X is any family X_μ with $X_0 = X$.*

An unfolding X_μ of X_0 is called a **((fibre-) C^0, C^r)-versal unfolding** *of X_0 if all unfoldings of X_0 are (fibre-)C^0-equivalent over the identity to an unfolding C^r-induced from X_μ.*

Here on $\sim_{C^r}$ will denote C^r-equivalence, where $r \in \{0,1,\ldots\} \cup \{\infty\}$. A C^r-conjugacy will be denoted by $\sim_{C^r{}_{conj}}$.

The present object of study are germs of vectorfields whose 2-jets are topologically equivalent to the following

$$j^2X(0) \sim_{C^0} y\frac{\partial}{\partial x} + (x^2 \pm xy)\frac{\partial}{\partial y}.$$

Such a vector field is said to exhibit a *cusp singularity* at 0. This condition defines a singular submanifold $\Sigma^2_{c\pm} \subset J^2_0V$ of codimension 4, where J^2_0V denotes the space of 2-jets of vector fields at 0 in the plane.
In this paper the following theorem will be proved.

Theorem (Takens 1974, Bogdanov 1976)
Any generic 2-parameter unfolding of a cusp singularity is (fibre-C^0, C^∞)-equivalent to

$$\tilde{X}^{\pm}_{(\mu,\nu)} = y\frac{\partial}{\partial x} + (x^2 + \mu + y(\nu \pm x))\frac{\partial}{\partial y} \quad .$$

Moreover, $\tilde{X}^{\pm}_{(\mu,\nu)}$ is a versal unfolding of $X_{(0,0)}$.

2 The normal form of the vector field

Write $X_\lambda = f(x,y,\lambda)\frac{\partial}{\partial x} + g(x,y,\lambda)\frac{\partial}{\partial y}$, where $f, g \in C^\infty$. The form of X_λ can be 'ameliorated', as the following lemma expresses.

Lemma
If X_λ exhibits a cusp singularity at 0 for $\lambda = 0$, then

$$X_\lambda \sim_{C^\infty} y\frac{\partial}{\partial x} + \left(x^2 + \mu(\lambda) + y\left(\nu(\lambda) \pm x + x^2h(x,\lambda)\right) + y^2Q(x,y,\lambda)\right)\frac{\partial}{\partial y}$$

where h and Q are C^∞-functions in their variables.

Proof
By hypothesis on X_λ, we have for $(x,y,\lambda) = (0,0,0)$:

$$\mathrm{d}f \wedge \mathrm{d}x \neq 0.$$

Introducing new coordinates

$$(\tilde{x}, \tilde{y}, \tilde{\lambda}) = (x, f(x,y,\lambda), \lambda)$$

yields as new vector field

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{y} \\ \dot{\tilde{y}} &= G(\tilde{x}, \tilde{y}, \tilde{\lambda}) \quad . \end{aligned}$$

Dropping all tildes and rewriting G yields

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= g(x,\lambda) + y\,h(x,\lambda) + y^2 Q(x,y,\lambda)\end{aligned}$$

where g, h and Q are again C^∞ functions. The $g(x,\lambda)$ here is obviously different from the above function $g(x,y,\lambda)$.
From the hypothesis that X_λ exhibits a cusp singularity for $\lambda = 0$, it follows that

$$\begin{aligned}g(x,0) &= ax^2 + \ldots \\ h(x,0) &= \pm bx + \ldots\end{aligned}$$

Here a and b can be chosen to equal 1, if necessary making a C^∞ coordinate transformation. If it could be used here that $g(x,\lambda) = u(x,\lambda)(x^2+\mu(\lambda))$, where $u(x,\lambda)$ is a C^∞ function satisfying $u(0,0) > 0$ (i.e. the division theorem of Malgrange) the lemma would follow easily. However, dividing u out of the $\frac{\partial}{\partial y}$-term of the vector field does introduce it in the $\frac{\partial}{\partial x}$-term, which would become $\frac{y}{u(x,\lambda)}\frac{\partial}{\partial x}$, contrary to our needs. A less straightforward argument is needed, involving Mather's theorem.
In order to apply it, introduce first the standard symplectic form $\Omega = \mathrm{d}x \wedge \mathrm{d}y$, and then the 1-form ω dual to the vector field X:

$$\omega(.) = \Omega(X, .).$$

In our case, the dual form to X_λ is

$$\omega_\lambda = y\,\mathrm{d}y - (g + yh + y^2 Q)\,\mathrm{d}x.$$

Writing

$$g\,\mathrm{d}x = \mathrm{d}V\,,$$

Mather's theorem asserts the existence of a coordinate transformation conjugating $\mathrm{d}V$ to what we want

$$\mathrm{d}V \sim_{C^\infty_{conj}} (x^2 + \mu(\lambda))\,\mathrm{d}x.$$

In the new coordinates, ω takes the form

$$\omega_\lambda = y\,\mathrm{d}y - \Big(x^2 + \mu(\lambda) + y\hat{h}(x,\lambda) + y^2\hat{Q}(x,y,\lambda)\Big)\,\mathrm{d}x.$$

Since the change of coordinates is not canonical, the transformed vector field is only C^∞-equivalent to the original one, not conjugate. The lemma has been proved.

Two remarks: from here on it will be assumed that h and Q depend on a parameter $\lambda \in \Lambda$, where Λ is a compact neighbourhood of 0, but it will not be made explicit in the notation any more. All following estimates are assumed to be uniform in λ.

Secondly remark that we can pass between the two different signs in the normal form by a change of coordinates

$$(x, y, \mu, \nu, t) \mapsto (x, -y, \mu, -\nu, -t).$$

Because of the reversion of time, any attractor changes to a repellor and vice versa. Keeping this in mind, we limit ourselves to the case with the +-sign in the normal form.

3 Singular points and their bifurcations

Let us investigate the singularities of the normal form

$$X_{\mu,\nu} \sim y\frac{\partial}{\partial x} + \left(x^2 + \mu + y\left(\nu + x + x^2h(x)\right) + y^2Q(x,y)\right)\frac{\partial}{\partial y}.$$

It is readily seen that they have to satisfy

$$\begin{cases} y = 0 \\ x^2 + \mu = 0 \end{cases}$$

So there are no singular points for $\mu > 0$, a bifurcation for $\mu = 0$ and two singular points for $\mu < 0$, which will be called $e_\mu = (-\sqrt{-\mu}, 0)$ and $s_\mu = (\sqrt{-\mu}, 0)$. To study their nature, compute the linear part of $X_{\mu,\nu}$,

$$\mathrm{D}X_{\mu,\nu} = \begin{pmatrix} 0 & 1 \\ 2x + yR_1 & \nu + x + x^2h(x) + yR_2 \end{pmatrix}$$

and conclude that s_μ is a saddle and e_μ is either a source or a sink. Precisely, if $\nu > 0$ then $\mu = 0$ is a saddle-node (source) bifurcation point, while it is a saddle-node (sink) bifurcation point for $\nu < 0$.
Furthermore, consider $\mathrm{div}X_{\mu,\nu}$ for $y = 0$:

$$\mathrm{div}X_{\mu,\nu} = \mathrm{trace}\ \mathrm{D}X_{\mu,\nu} = \nu + x + x^2h(x)$$

So at e_μ this trace is zero along a C^∞ curve

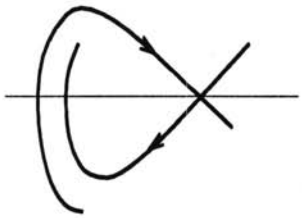
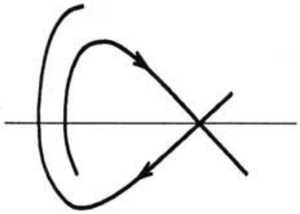
$$H: \quad \nu - \sqrt{-\mu} + |\mu|\, h(-\sqrt{-\mu}) = 0$$

and, with the implicit function theorem,

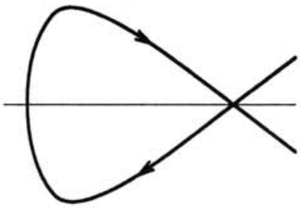
$$H: \mu = -\nu^2 + \mathrm{o}(\nu^2) \quad , \qquad \nu > 0$$

Below we will prove that this curve H is a curve of generic subcritical Hopf bifurcations.

Consider now the vector field for a fixed value $\mu < 0$, and for two values $\nu_1 < 0$, $\nu_2 > 0$, both of large absolute value.

By a rotational argument we will prove that there exists a regular curve C of values $\nu(\mu)$ such that the following situation arises:



Moreover, we will prove that in the region between C and H there exists exactly one repelling invariant cycle, and in the complement of this region there is no invariant cycle whatsoever. We will find also the relative positions of C and H and complete the bifurcation diagram.

4 The analysis of the curves C and H.

4.1 Rescaling

In order to investigate the relative positions of the curves C and H, and to prove the existence of a unique limit cycle in the region between them, a singular change of coordinates and parameters is performed, a so-called rescaling. This amounts to an investigation of what happens in a part of the (μ, ν)-plane.
The change depends on a parameter ε and has the following form:

$$\begin{cases} x = \varepsilon^2 \bar{x} \\ y = \varepsilon^3 \bar{y} \end{cases} \qquad \begin{cases} \mu = \varepsilon^4 \bar{\mu} \\ \nu = \varepsilon^2 \bar{\nu} \end{cases} \qquad \varepsilon > 0\,.$$

Again instead of the vector field $X_{\mu,\nu}$ its dual $\omega_{\mu,\nu}$ is considered

$$\begin{aligned} \omega_{\mu,\nu} &= \Omega(X_{\mu,\nu}\,,\ .\,) \\ &= y\,\mathrm{d}y - \left(\mu + x^2 + y(\nu + x + x^2 h) + y^2 Q\right)\,\mathrm{d}x \end{aligned}$$

which becomes in the new coordinates

$$\begin{aligned}\bar\omega_{\bar\mu,\bar\nu} &= \bar y\,\mathrm{d}\bar y - \left(\bar\mu + \bar x^2 + \varepsilon\bar y(\bar\nu + \bar x + \varepsilon^2\bar x\bar h) + \varepsilon^2\bar y^2\bar Q\right)\,\mathrm{d}\bar x \\ &= \bar y\,\mathrm{d}\bar y - \left(\bar\mu + \bar x^2 + \varepsilon\bar y(\bar\nu + \bar x)\right)\,\mathrm{d}\bar x + \mathrm{o}(\varepsilon)\end{aligned}$$

where $\bar\omega_{\bar\mu,\bar\nu}$ is defined by $\omega_{\mu,\nu} = \varepsilon^6\bar\omega_{\bar\mu,\bar\nu}$.
Since there are now three parameters $(\bar\mu,\bar\nu,\varepsilon)$ instead of the previous two (μ,ν), one of them can be fixed. Let this be $\bar\mu$. The curves C and H are to be investigated, which lie in the halfplane $\mu < 0$, so $\bar\mu$ is fixed on -1.
Let us specify the domains of the variables and the coordinates. The variables $(\bar x,\bar y)$ can be taken in a fixed compact set $\bar D$, which should be large enough to cover all 'interesting' phenomena. Something like $\bar D = [-10,10]\times[-10,10]$ will do nicely.
The parameter ν will be taken in the interval $[-\nu_0,\nu_0]$, where ν_0 can be taken arbirarily large, but has to be fixed, whereas ε will be taken in an interval $[0,T]$, where $T = T(\nu_0,\bar D)$ will be determined in the course of the investigation.
Here on till we will have done with the above rescaling, we will drop the bars on the variables, but *not* the bars on the parameters. Moreover, $\bar\omega_{\bar\mu,\bar\nu}$ will be called simply ω, or sometimes $\omega_{\varepsilon,\bar\nu}$.

4.2 A perturbation lemma

So we are considering

$$\begin{aligned}\omega &= y\,\mathrm{d}y + (1 - x^2)\,\mathrm{d}x - \varepsilon y(\bar\nu + x)\,\mathrm{d}x + \mathrm{o}(\varepsilon) \\ &= \mathrm{dH} - \varepsilon\omega_D + \mathrm{o}(\varepsilon)\end{aligned}$$

where $\mathrm{H} = \frac{1}{2}y^2 + (x - \frac{x^3}{3})$ and $\omega_D = y(\bar\nu + x)\,\mathrm{d}x$. Thus ω can be seen as a perturbation of a Hamiltonian form dH.
Remark that the positions of the singular points of ω do not depend on the parameters $(\varepsilon,\bar\nu)$.

No perturbation without a perturbation lemma:

Lemma (Melnikov)
Let $\omega = \mathrm{dH} - \varepsilon\omega_D + \mathrm{o}(\varepsilon)$ be as above; let e and s denote the singularities of dH such that $\mathrm{H}(e) = -\frac{2}{3}$ and $\mathrm{H}(s) = \frac{2}{3}$, and let finally $\overline{es}$ denote the straight line joining e and s.

Then for every $\nu_0 > 0$ there exists a $T > 0$ such that the Poincaré map $P_{\varepsilon,\bar\nu}$ of the vector field $\bar X_{\varepsilon,\bar\nu}$, which is the dual of $\omega_{\varepsilon,\bar\nu}$, or the inverse of this map are defined on $\overline{es}$ for all $(\varepsilon,\bar\nu)\in[0,T]\times[-\nu_0,\nu_0]$.
If $\overline{es}$ is parametrised by the values $h\in[\mathrm{H}(e),\mathrm{H}(s)]$, then $P_{\varepsilon,\bar\nu}$ takes the following form:

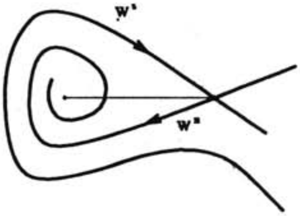
$$P_{\varepsilon,\bar\nu}(h) = h + \int_{\gamma_h}\omega_D + \mathrm{o}(\varepsilon)$$

where γ_h *is the compact component of the set* $\{(x,y) \mid \mathrm{H}(x,y) = h\}$, *oriented clockwise.*

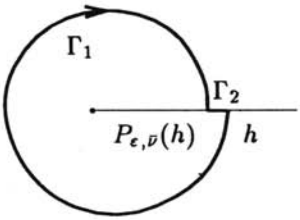
Proof

Let us start with the existence of $P_{\varepsilon,\bar{\nu}}$. We will call the separatrices that lie (partly) in the plane $x < 0$ *the* separatrices, or sometimes W^u and W^s for the outgoing and the incoming separatrix respectively.

Now $P_{0,\bar{\nu}}$ is the identity, and $P_{\varepsilon,\bar{\nu}}$ is a small deformation of it. Either W^u returns to intersect $\overline{es}$, then $P_{\varepsilon,\bar{\nu}}$ is defined everywhere, or W^s intersects $\overline{es}$ somewhere, and then $P_{\varepsilon,\bar{\nu}}^{-1}$ is defined on all of $\overline{es}$ (see the picture).



Proceed by parametrising $\overline{es}$ with $h \in [\mathrm{H}(e), \mathrm{H}(s)]$, and let the value of the parameter h denote (by abuse) the parametrised point on $\overline{es}$. Suppose $P_{\varepsilon,\bar{\nu}}$ is well-defined, and let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a closed curve in phase space as follows. The curve Γ_1 equals the forward orbit of h connecting h with $P_{\varepsilon,\bar{\nu}}(h)$, while Γ_2 is that part of $\overline{es}$ connecting $P_{\varepsilon,\bar{\nu}}(h)$ with h in that order. Then Γ is a closed, clockwise oriented loop. See the following picture.



Consider now the following integral:

$$\int_{\Gamma} \omega = \int_{\Gamma_1} \omega + \int_{\Gamma_2} \omega$$

Since Γ_1 is an orbit of $X_{\varepsilon,\bar{\nu}}$, and since $X_{\varepsilon,\bar{\nu}}$ is dual to ω, we have

$$\int_{\Gamma_1} \omega = 0 \quad .$$

Remains

$$\int_{\Gamma_2} \omega = \int_{\Gamma_2} \mathrm{dH} + \varepsilon \int_{\Gamma_2} (-\omega_D + \varphi(\varepsilon))$$

where $\varphi(\varepsilon)$ is $\mathrm{o}(\varepsilon)$. Firstly

$$\int_{\Gamma_2} \mathrm{dH} = h - P_{\varepsilon,\bar{\nu}}(h)$$

whereas

$$\varepsilon \int_{\Gamma_2} (-\omega_D + \varphi(\varepsilon)) = \mathrm{o}(\varepsilon)$$

since $|\Gamma_2| = |P_{\varepsilon,\bar{\nu}}(h) - h| \to 0$ as $\varepsilon \to 0$. Putting this together, we have

$$\int_{\Gamma} \omega = h - P_{\varepsilon,\bar{\nu}}(h) + \mathrm{o}(\varepsilon)$$

Now let γ_h be the Hamiltonian level curve as in the announcement of the lemma. The closed curve (or one-chain) $\Gamma - \gamma_h$ bounds a two-chain, let us call it σ. Thus $\partial\sigma = \Gamma - \gamma_h$.

Applying Stokes' theorem it follows that

$$\begin{aligned} \int_{\Gamma-\gamma_h} \omega &= \int_{\sigma} \mathrm{d}\omega = \int_{\sigma} (\mathrm{d}\,\mathrm{dH} - \varepsilon\, \mathrm{d}\omega_D) + \chi(\varepsilon) \\ &= -\varepsilon \int_{\sigma} \mathrm{d}\omega_D + \chi(\varepsilon) \end{aligned}$$

where $\chi(\varepsilon)$ is another function $\mathrm{o}(\varepsilon)$. Since $\mathrm{area}(\sigma) \to 0$ as $\varepsilon \to 0$, it follows that

$$\varepsilon \int_{\sigma} \mathrm{d}\omega_D + \chi(\varepsilon) = \mathrm{o}(\varepsilon)$$

and thus

$$\int_{\Gamma} \omega = \int_{\gamma_h} \omega + \mathrm{o}(\varepsilon) = -\varepsilon \int_{\gamma_h} \omega_D + \mathrm{o}(\varepsilon)$$

Putting all this together, we get

$$h - P_{\varepsilon,\bar{\nu}}(h) + \mathrm{o}(\varepsilon) = -\varepsilon \int_{\gamma_h} \omega_D + \mathrm{o}(\varepsilon)$$

or

$$P_{\varepsilon,\bar{\nu}}(h) = h + \varepsilon \int_{\gamma_h} \omega_D + \mathrm{o}(\varepsilon)$$

as was to be proved.

4.3 An Abelian integral

Write $\int_{\gamma_h} \omega_D = I(h,\bar\nu) = \bar\nu I_0(h) + I_1(h)$, where

$$I_i(h) = \int_{\gamma_h} y\, x^i \,\mathrm{d}x$$

and introduce

$$G_{\varepsilon,\bar\nu}(h) = \frac{1}{\varepsilon}\left(P_{\varepsilon,\bar\nu}(h) - h\right) = I(h,\bar\nu) + \eta(h,\varepsilon,\bar\nu)$$

where $\eta(h,\varepsilon,\bar\nu) \to 0$ uniformly for $\varepsilon \to 0$.
The (possible) limit cycles of $X_{\varepsilon,\bar\nu}$ intersecting $\overline{es}$ correspond one-to-one to the fixed points of $P_{\varepsilon,\bar\nu}$, which are the zeros of the function $G_{\varepsilon,\bar\nu}$. This function is in turn a perturbation of the function $I(h,\bar\nu)$. Therefore the natural next step is to study the zeros of (the abelian integral) $I(h,\bar\nu)$.

Let us start with two remarks:

1) $I_0(h) > 0$ for $h \in (-\frac{2}{3}, \frac{2}{3}]$

2) $\frac{I_1}{I_0} \to -1$ as $h \to -\frac{2}{3}$

These can be seen as follows. Let σ_h denote the area enclosed by γ_h, then Stokes' theorem yields

$$\begin{aligned} I_i(h) &= -\int_{\partial\sigma_h} y\, x^i \,\mathrm{d}x \\ &= -\int_{\sigma_h} x^i \,\mathrm{d}y \wedge \mathrm{d}x \\ &= \int_{\sigma_h} x^i \,\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

Thus $I_0(h) = \int_{\sigma_h} \mathrm{d}x \wedge \mathrm{d}y = \mathrm{area}(\sigma_h) > 0$ for $h \in (-\frac{2}{3}, \frac{2}{3}]$, and

$$m(h) I_0 \le I_1 \le M(h) I_0$$

for all $h \in (-\frac{2}{3}, \frac{2}{3}]$. The functions $M(h)$ and $m(h)$ denote the largest and the smallest value respectively, which the x-coordinate of γ_h can attain (incidentally these are the points of intersection of γ_h with the x-axis). Thus $\lim_{h\to -\frac{2}{3}} m(h) = \lim_{h\to -\frac{2}{3}} M(h) = -1$.

Consequently the equation $I(h,\bar\nu) = 0$ can be written as follows:

$$0 = \frac{I}{I_0} = \bar\nu + \frac{I_1}{I_0}$$

or

$$\bar{\nu} = B(h) \qquad \text{where} \qquad B(h) = -\frac{I_1}{I_0}$$

and by the above remarks $B(h)$ (the Bogdanov function) is defined for all $h \in [-\frac{2}{3}, \frac{2}{3}]$ and $B(-\frac{2}{3}) = -1$.

Theorem (Bogdanov)
The function B is continuous on $[-\frac{2}{3}, \frac{2}{3}]$, satisfies $B(-\frac{2}{3}) = -1$ and $B(\frac{2}{3}) = \frac{5}{7}$, and its derivative satisfies $\frac{\mathrm{d}B}{\mathrm{d}h} < 0$ for all $h \in [-\frac{2}{3}, \frac{2}{3})$, while $\frac{\mathrm{d}B}{\mathrm{d}h} \to -\infty$ as $h \to \frac{2}{3}$.

In the proof, the following lemma is needed:

Lemma
$B(h)$ satisfies a Ricatti equation:

$$(9h^2 - 4)\frac{\mathrm{d}B}{\mathrm{d}h} = 7h^2 + 3hB - 5$$

Proof of the lemma
This amounts mainly to manipulations with the functions $I_i(h)$. Recall

$$I_i(h) = \int_{\gamma_h} y\, x^i \, \mathrm{d}x$$

The equation for the graph of γ_h is

$$\mathrm{H} = \frac{1}{2}y^2 + x - \frac{x^3}{3} = h$$

This can be solved for y:

$$y_\pm(x) = \pm\sqrt{2\left(\frac{x^3}{3} - x + h\right)}$$

where $(x, y_+(x))$ parametrises the upper half of γ_h, while $(x, y_-(x))$ deals with the lower half.
If, as above, the zeros of $y_\pm(x)$ are denoted by $m(h)$ and $M(h)$ such that $m(h) \leq M(h)$, then we can write I_i, by symmetry, as follows.

$$I_i(h) = 2\int_{m(h)}^{M(h)} y_+(x)x^i \, \mathrm{d}x = 2J_i$$

Defining $R(x, h) = y_+(x) = \sqrt{2\left(\frac{x^3}{3} - x + h\right)}$, the J_i take the form

$$J_i(h) = \int_{m(h)}^{M(h)} w^i R(w, h) \, \mathrm{d}w \quad .$$

In order to obtain relations between the J_i's, differentiate with respect to h:

$$\begin{aligned}\frac{\mathrm{d}J_i}{\mathrm{d}h} &= \int_{m(h)}^{M(h)} \frac{w^i}{R}\,\mathrm{d}w + R(M(h),h)\,M'(h) - R(m(h),h)\,m'(h)\\ &= \int_{m(h)}^{M(h)} \frac{w^i}{R}\,\mathrm{d}w\end{aligned}$$

and rewrite J_i:

$$\begin{aligned}J_i &= \int_{m(h)}^{M(h)} \frac{w^i}{R} R^2\,\mathrm{d}w\\ &= 2h\int_{m(h)}^{M(h)} \frac{w^i}{R}\,\mathrm{d}w - 2\int_{m(h)}^{M(h)} \frac{w^{i+1}}{R}\,\mathrm{d}w + \frac{2}{3}\int_{m(h)}^{M(h)} \frac{w^{i+3}}{R}\,\mathrm{d}w\\ &= 2hJ_i' - 2J_{i+1}' + \frac{2}{3}J_{i+3}'\end{aligned}$$

Another way of obtaining relations is partial integration:

$$\begin{aligned}J_i &= \frac{1}{i+1} w^{i+1} R(w,h)\Big|_{m(h)}^{M(h)} + \frac{1}{i+1}\int_{m(h)}^{M(h)} \frac{w^{i+1}(1-w^2)}{R}\,\mathrm{d}w\\ &= \frac{1}{i+1}\left(J_{i+1}' - J_{i+3}'\right)\end{aligned}$$

Eliminating from these two relations the term J_{i+3}' yields

$$(2i+5)J_i = -4J_{i+1}' + 6hJ_i'$$

and this reads for $i \in \{0,1\}$

$$\begin{aligned}5J_0 &= -4J_1' + 6hJ_0'\\ 7J_1 &= -4J_2' + 6hJ_1'\end{aligned}$$

Fortunately it can be shown that $J_2 \equiv J_0$. Let ω_i denote $yx^i\,\mathrm{d}x$, the integrand of the I_i, then

$$\begin{aligned}\omega_0 - \omega_2 &= y\,\mathrm{d}x - y\,x^2\,\mathrm{d}x\\ &= y(1-x^2)\,\mathrm{d}x + y\cdot y\,\mathrm{d}y - y^2\,\mathrm{d}y\\ &= y\,\mathrm{dH} - \mathrm{d}\frac{y^3}{3}\end{aligned}$$

Upon integration the first term on the right will yield zero, since γ_h is a level curve of H, and the second term yields zero as well, since γ_h is closed. Thus $I_0 \equiv I_2$. After substitution we arrive at the Picard-Fuchs system:

$$\begin{aligned}5J_0 &= -4J_1' + 6hJ_0'\\ 7J_1 &= -4J_0' + 6hJ_1'\end{aligned}$$

which is equivalent to ($h = \pm\frac{2}{3}$ excepted)

$$\begin{aligned} J_0' &= (9h^2-4)^{-1}\left(\frac{15}{2}hJ_0 + 7J_1\right) \\ J_1' &= (9h^2-4)^{-1}\left(5J_0 + \frac{21}{2}hJ_1\right) \end{aligned}$$

Since

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}h} &= \frac{\mathrm{d}}{\mathrm{d}h}\left(-\frac{J_1}{J_0}\right) \\ &= \frac{J_0'J_1 - J_1'J_0}{J_0^2} \end{aligned}$$

substitution yields

$$(9h^2-4)\frac{\mathrm{d}B}{\mathrm{d}h} = 7B^2 + 3hB - 5$$

as was to be shown.

Proof of the theorem

To say that B satisfies the above Ricatti equation is equivalent to saying that the graph of B is an integral line of the for

$$\Xi = (9h^2-4)\,\mathrm{d}B - (7B^2+3hB-5)\,\mathrm{d}h$$

or, again equivalently, an orbit of the vector field

$$Z = -(9h^2-4)\frac{\partial}{\partial h} - (7B^2+3hB-5)\frac{\partial}{\partial B}$$

dual to Ξ.

The critical points of Z satisfy

$$\left\{\begin{array}{l} 9h^2-4=0 \\ 7B^2+3hB-5=0 \end{array}\right. \qquad \text{that is}$$

$$\left\{\begin{array}{l} h=-\frac{2}{3} \\ B=1 \vee B=-\frac{1}{7} \end{array}\right. \vee \left\{\begin{array}{l} h=\frac{2}{3} \\ B=-1 \vee B=\frac{5}{7} \end{array}\right.$$

Since we know already that $B(-\frac{2}{3}) = 1$, and since on the line $B = 0$ we have $Z = 5\frac{\partial}{\partial B}$, we can restrict our attention to the upper half plane $B \geq 0$.

Two of the four singularities are located in the upper half plane,

$$\alpha_0 = (-\frac{2}{3}, 1) \qquad \text{and} \qquad \alpha_1 = (\frac{2}{3}, \frac{5}{7})\,.$$

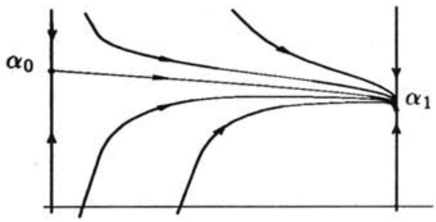
The linear part of Z is

$$\mathrm{D}Z(h,B) = \begin{pmatrix} -18h & 0 \\ -3B & -14B-3h \end{pmatrix}$$

and we conclude that α_0 is a hyperbolic saddle, while α_1 is a (hyperbolic) sink. Consider the compact set K, defined as

$$\mathrm{K} = \left\{ (h,B) \;\Big|\; -\frac{2}{3} \le h \le \frac{2}{3}\,, \quad 0 \le B \le M \right\}$$

Remark that for M large enough, the $\frac{\partial}{\partial B}$-component of Z will be (strictly) negative for $B = M$ in K. Since along the lines $h = -\frac{2}{3}$ and $h = \frac{2}{3}$ the $\frac{\partial}{\partial h}$-component of Z is zero, we conclude that the forward orbits of all points in K remain in K, and in fact we have the following phase portrait (see figure).



The important fact here is that there is a unique orbit having α_0 and α_1 as its limit points: the part of the unstable manifold of α_0 lying in K. Remark that the phase portrait is correct. Limit cycles cannot occur, since the $\frac{\partial}{\partial h}$-component of Z is positive in K, and α_1 is a global attractor of the interior of K.
The point $\alpha_0 = (-\frac{2}{3}, 1)$, as well as at least one interior point of K are points of the graph $(h, B(h))$, and therefore α_1 as well. So the graph of B equals the outgoing separatrix of α_0. Note that this implies that $B(h) \to \frac{5}{7}$ as $h \to \frac{2}{3}$.
Consider the equation

$$7B^2 + 3hB - 5 = 0\,,$$

the equation of all points in phase space where the $\frac{\partial}{\partial B}$-component of Z is zero. The equation defines a hyperbola. An arc of this hyperbola joins α_0 to α_1; call this arc σ. Along this arc, the $\frac{\partial}{\partial h}$-component of Z is positive.
Now at α_0 the slope of σ equals $-\frac{1}{4}$, while the inclination of the unstable manifold of α_0 is $-\frac{1}{8}$. That implies that near α_0 the unstable manifold lies above the hyperbola, and it follows that the unstable manifold is everywhere above σ. But above σ we have that the $\frac{\partial}{\partial B}$-component of Z is negative. We conclude that $\frac{dB}{dh} < 0$ for $h \in [-\frac{2}{3}, \frac{2}{3})$.

Proceeding to the last claim, that $B'(h) \to -\infty$ as $h \to \frac{2}{3}$, look at the linearisation of Z around α_1:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -12 & 0 \\ -\frac{15}{7} & -12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{where} \qquad \begin{cases} x = h - \frac{2}{3} \\ y = B - \frac{5}{7} \end{cases}$$

For initial conditions x_0, y_0 its solution can be written as

$$\begin{aligned} x(t) &= x_0 e^{-12t} \\ y(t) &= -\frac{15}{7} t\, x_0 e^{-12t} + y_0 e^{-12t} \end{aligned} \quad .$$

Now y can be expressed as a function of x:

$$y(x) = \frac{5}{28} x \ln \frac{x}{x_0} + \frac{y_0}{x_0} x$$

and it is obvious that $\lim_{x \uparrow 0} y'(x) = -\infty$. Here we need a result from [3]. It asserts that a C^2 vector field in the plane that has a hyperbolic singularity at a point p is C^1-linearisable at p. So there exists a local C^1 diffeomorphism of the form $\mathrm{id} + \varphi$ (where φ and its first derivatives are zero at p) mapping the solutions of the linear equation to the solutions of the original equation. Conclude that in the original equation B can be expressed as a function of h, locally around $(h, B) = (\frac{2}{3}, \frac{5}{7})$, and

$$B(h) = \frac{5}{28} \left(h - \frac{2}{3} \right) \ln \left| h - \frac{2}{3} \right| + \psi(h) \qquad\qquad h \leq \frac{2}{3}$$

where $\psi(\frac{2}{3}) = 0$ and ψ is continuously differentiable. The claim follows.

Returning to the equation

$$G_{\varepsilon,\bar{\nu}}(h) = \frac{1}{\varepsilon} \left(P_{\varepsilon,\bar{\nu}}(h) - h \right) = I(h, \bar{\nu}) + \eta(h, \varepsilon, \bar{\nu})$$

remark that it is equivalent to

$$F(h, \varepsilon, \bar{\nu}) = \frac{G_{\varepsilon,\bar{\nu}}(h)}{I_0(h)} = \bar{\nu} - B(h) + \chi(h, \varepsilon, \bar{\nu})$$

with $\chi(h, \varepsilon, \bar{\nu}) \to 0$ uniformly as $\varepsilon \to 0$, since the functions G, I_0 and η are all differentiable at $h = -\frac{2}{3}$, and all equal zero there.

4.4 The Hopf bifurcation

As $\bar{\nu}$ decreases from ν_0, the point e features in a Hopf bifurcation, as expressed in the following

Proposition
For any $\beta \in (-\frac{2}{3}, \frac{2}{3})$ *there exists a constant* $T = T_H(\nu_0, \beta)$ *such that for every* $(h, \varepsilon) \in [-\frac{2}{3}, \beta] \times [0, T]$ *the equation*

$$F(h, \varepsilon, \bar{\nu}) = 0$$

has a unique solution, differentiable in h *and* ε,

$$\bar{\nu} = L_H(h, \varepsilon)$$

such that

$$\frac{\partial}{\partial h} L_H(h, \varepsilon) < 0 \qquad \text{for all} \quad h \in [-\frac{2}{3}, \beta]$$

and

$$L_H(h, 0) = B(h)$$

Remark that the curve $\bar{\nu}_H(\varepsilon) = L_H(-\frac{2}{3}, \varepsilon)$ defines a curve of generic subcritical Hopf bifurcations in the parameter space, subcritical because $\frac{\partial}{\partial h} F(-\frac{2}{3}, \varepsilon, L_H(-\frac{2}{3}, \varepsilon)) > 0$.

Proof
On the interval $[-\frac{2}{3}, \beta]$ the function F is differentiable, and $\frac{\partial}{\partial h}\chi \to 0$ uniformly for $\varepsilon \to 0$.
Since $\frac{\partial}{\partial \bar{\nu}} F(h, 0, \bar{\nu}) = 1$, there exists a constant $T = T_H(\nu_0, \beta)$ such that for all $\varepsilon \in [0, T]$

$$\frac{\partial}{\partial \bar{\nu}} F(h, \varepsilon, \bar{\nu}) > 0$$

The implicit function theorem now yields the existence of a unique function $L(h, \varepsilon)$ with all the required properties.

4.5 The saddle loop bifurcation

As in the case of the Hopf bifurcation, the result is expressed in a

Proposition
There exist constants $T = T_C(\nu_0)$, M, *and* $\beta_1 = \beta_1(M)$, *all greater than zero, such that for every* $h \in [\beta_1, \frac{2}{3}]$ *and* $\varepsilon \in [0, T]$, *the equation*

$$F(h, \varepsilon, \bar{\nu}) = 0$$

has a unique solution

$$\bar{\nu} = L_C(h, \varepsilon)$$

strictly decreasing in h, *and*

$$L_C(\frac{2}{3}, 0) = \frac{5}{7}.$$

Remark that the curve $\bar{\nu}_C(\varepsilon) = L_C(\frac{2}{3}, \varepsilon)$ defines a curve of saddle loop bifurcations in the parameter space.

Proof

The main difficulty compared with the case of the Hopf bifurcation is the fact that $B'(\frac{2}{3}) = -\infty$, so the implicit function theorem does not work here. However it is possible to expand the function F near $h = \frac{2}{3}$ in the spirit of [4], whereof we will quote two results below.

First, introduce an expansion coordinate $u = \frac{2}{3} - h$, and denote, by abuse of notation, the functions in the new coordinates by their old names (e.g. $F(\frac{2}{3} - u)$ is not called $\tilde{F}(u)$ as it should, but $F(u)$).

Definitions

A function $f(u)$ admits an **expansion of type ω of order k at** $u = 0$, *if*

$$f(u) = a_0 + a_1[u\omega + \ldots] + a_2[u^2\omega + \ldots] + \ldots + \varphi_k(u)$$

where $\varphi_k(u)$ is a C^k function satisfying $\varphi_k(0) = \ldots = \varphi_k^{(k-1)}(0) = 0$, the $\ldots$ in $a_1[u\omega + \ldots]$ denote terms of higher order than $u\omega$, and ω denotes the function $\frac{u^{-a_1}-1}{a_1}$ for $a_1 \neq 0$ and $-\ln u$ for $a_1 = 0$.

A function $g(u)$ is said to admit an **expansion of type I of order k at** $u = 0$, *if*

$$g(u) = \sum_{i=0}^{k-1} \left(a_i u^i + b_{i+1} u^{i+1} \ln u\right) + \varphi_k(u)$$

with $\varphi_k(u)$ as above.

A function $h(u)$ is said to admit an **expansion of type II of order k at** $u = 0$, *if*

$$h(u) = \sum_{i=0}^{k-1} \left(a_i u^i + b_{i+1} \sum_{j=1}^{i+1} \gamma_{i+1\,j} u^{i+1} \ln^j u \right) + \varphi_k(u)$$

with $\varphi_k(u)$ as above and $\gamma_{i+1\,i+1} = 1$.

These definitions stem from an analysis of general saddle loop bifurcations. The coefficient a_1 has a privileged position in the definition, because it measures the difference of the ratio of hyperbolicity of the saddle from 1. As $\varepsilon \to 0$ the system approaches a Hamiltonian system where this ratio is a priori equal to 1.

Two results from [4] are needed.

1) ([4] p.72) The functions I_0 and I_1 admit expansions of type I.

2) ([4] p.97-100) The function $\varepsilon\, G_{\varepsilon,\bar{\nu}}$ admits an expansion of type ω, where the expansion coefficients depend differentiably on $(\varepsilon, \bar{\nu})$.

First remark: since the expansion coefficients of $\varepsilon\, G_{\varepsilon,\bar\nu}$ are all zero when $\varepsilon = 0$, the function $G_{\varepsilon,\bar\nu}$ admits an expansion of type $\tilde\omega$, where $\tilde\omega = \frac{u^{-\varepsilon a_1}-1}{\varepsilon a_1}$. This, and the fact that $I_0(u) \neq 0$ for $u = 0$, it follows that the functions $F = G/I_0$ and $B = I_1/I_0$ admit expansions of type $\tilde\omega$ and II respectively, viz.

$$F(u,\varepsilon,\bar\nu) = a(\varepsilon,\bar\nu) + b(\varepsilon,\bar\nu)\, u\tilde\omega + \varphi(u,\varepsilon,\bar\nu)\,,$$

and

$$B(u) = c + d\, u \ln u + \psi(u)\,.$$

Above (subsection (4.3)) we have computed the coefficients c and d of the expansion of B: $c = \frac{5}{7}$ and $d = -\frac{5}{28}$. Since $F(u,0,\bar\nu) = \bar\nu - B(u)$, and since $\tilde\omega \to -\ln u$ uniformly for $\varepsilon \to 0$, we get

$$a(0,\bar\nu) = \bar\nu - \frac{5}{7} \qquad \text{and} \qquad b(0,\bar\nu) = -\frac{5}{28}$$

(the $u \ln u$ term of B having been calculated above in a different context).
By compactness of $[-\nu_0,\nu_0]$ and differentiability of the functions a, b and φ, there exist constants $T = T_C(\nu_0)$ and $M_1 = M_1(\nu_0)$ such that

$$\frac{\partial}{\partial\bar\nu} a(\varepsilon,\bar\nu) > \frac{1}{2}\,, \qquad b(\varepsilon,\bar\nu) < -\frac{1}{8} \qquad \text{and} \qquad \left|\frac{\partial}{\partial u}\varphi(u,\varepsilon,\bar\nu)\right| < M_1$$

for $\varepsilon \in [0,T]$, $\bar\nu \in [-\nu_0,\nu_0]$ and $u \in [0,\delta]$, where $\delta > 0$ is a universal constant. By possibly choosing T smaller and taking $M = \max(M_1, 8)$ and $\beta_1 = \min(\delta, \mathrm{e}^{-10M})$ we can arrive at

$$\left|\frac{\partial}{\partial\bar\nu} b\right| < |4\beta_1 \ln\beta_1|^{-1} \qquad \text{and} \qquad \left|\frac{\partial}{\partial\bar\nu}\varphi(u,\varepsilon,\bar\nu)\right| < \frac{1}{4}$$

as well.

Since $\frac{\partial}{\partial\bar\nu} a \neq 0$ there exists a differentiable function $\bar\nu_C(\varepsilon)$ such that

$$a\left(\varepsilon, \bar\nu_C(\varepsilon)\right) = 0\,.$$

Moreover, for any $u_0 \in [0,\beta_1]$ the following estimate holds:

$$\begin{aligned} \frac{\partial}{\partial\bar\nu} F(u_0,\varepsilon,\bar\nu) &= \frac{\partial a}{\partial\bar\nu} + \frac{\partial b}{\partial\bar\nu}\, u_0 \ln u_0 + \frac{\partial\varphi}{\partial\bar\nu} \\ &> \frac{1}{2} - \frac{1}{4}\left|\frac{u_0 \ln u_0}{\beta_1 \ln\beta_1}\right| - \frac{1}{4} > 0 \end{aligned}$$

This implies for $u_0 = \beta_1$ the existence of a differentiable function $\bar\nu_{\beta_1}(\varepsilon)$ such that

$$F\left(\beta_1, \varepsilon, \bar\nu_{\beta_1}(\varepsilon)\right) = 0$$

Finally

$$\begin{aligned}\frac{\partial}{\partial u}F(u,\varepsilon,\bar{\nu}) &= -b(1+\ln u)+\frac{\partial}{\partial u}\varphi\\ &< -b(1+\ln\beta_1)+M\\ &< -b(1-10M)+M\\ &< \frac{1}{8}-\frac{2}{8}M<0\end{aligned}$$

Now we can find the function L_C. Fix $(u,\varepsilon)\in[0,\beta_1]\times[0,T]$. We have

$$F(u,\varepsilon,\bar{\nu}_C)<F(0,\varepsilon,\bar{\nu}_C)=0$$

and

$$F(u,\varepsilon,\bar{\nu}_{\beta_1})>F(\beta_1,\varepsilon,\bar{\nu}_{\beta_1})=0\,.$$

By the intermediate value theorem it follows that there exists

$$\bar{\nu}=L_C(u,\varepsilon)$$

such that $\bar{\nu}_C<\bar{\nu}<\bar{\nu}_{\beta_1}$ and, since $\frac{\partial}{\partial\bar{\nu}}F>0$, it is unique.
Moreover, take $0\le u_1<u_2\le\beta_1$, then we have

$$\begin{aligned}F\left(u_2,\varepsilon,L_C(u_2,\varepsilon)\right) &= 0\\ &= F\left(u_1,\varepsilon,L_C(u_1,\varepsilon)\right)\\ &> F\left(u_2,\varepsilon,L_C(u_1,\varepsilon)\right)\end{aligned}$$

and, using $\frac{\partial}{\partial\bar{\nu}}F>0$,

$$L_C(u_2,\varepsilon)>L_C(u_1,\varepsilon)\qquad\text{follows.}$$

This proves the proposition.

4.6 Conclusion

To unite the two propositions, we have to take

$$T_L=\min\left(T_H(\nu_0,\beta_1),T_C(\nu_0)\right)\quad.$$

Then the two curves H and C are given by $\bar{\nu}=L(-\frac{2}{3},\varepsilon)$ and $\bar{\nu}=L(\frac{2}{3},\varepsilon)$, where L equals L_H on $[-\frac{2}{3},\beta_1)$ and L_C on $[\beta_1,\frac{2}{3}]$. In the original coordinates:

$$\begin{aligned}H:\qquad \nu^2 &= \mu\left(L(-\frac{2}{3},|\mu|^{\frac{1}{4}})\right)^2=\mu+o(\mu)\\ C:\qquad \nu^2 &= \mu\left(L(\frac{2}{3},|\mu|^{\frac{1}{4}})\right)^2=\frac{25}{49}\mu+o(\mu)\quad.\end{aligned}$$

This yields the relative position of the two bifurcation curves.
Remark finally that the fact that $L(h,\varepsilon)$ is strictly decreasing as a function of h implies that in the region between H and C there exists a unique (repelling) limit cycle.

5 The saddle node bifurcation

We return to our original normal form. As said above the two singular points e_μ and s_μ feature in a saddle-node bifurcation as μ goes through zero. To see that this bifurcation is actually generic, we have to use a different rescaling

$$\begin{cases} x = \varepsilon^2 \bar{x} \\ y = \varepsilon^3 \bar{y} \end{cases} \begin{cases} \mu = \varepsilon^4 \bar{\mu} \\ \nu = \pm\varepsilon \end{cases} \qquad \varepsilon > 0$$

posing $X_{\mu,\nu} = \varepsilon \bar{X}_{\varepsilon,\bar{\mu}}$. This yields

$$\bar{X}_{\varepsilon,\bar{\mu}} = \bar{y}\frac{\partial}{\partial \bar{x}} + \left(\bar{\mu} + \bar{x}^2 \pm \bar{y}\right)\frac{\partial}{\partial \bar{y}} + \mathrm{O}(\varepsilon)$$

At the bifurcation point $\varepsilon = 0$, $(\bar{x}, \bar{y}) = (0,0)$, this implies

$$\mathrm{D}\bar{X}_{0,\bar{\mu},\bar{\nu}}(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & \pm 1 \end{pmatrix}.$$

The vector field is partially hyperbolic at the origin, and for $|\bar{\mu}| \le M$ the only thing occuring is a generic saddle node bifurcation. This remains the case if $\varepsilon \in [0, T_S]$, if T_S is chosen small enough.

6 Filling the gap

The previous section describes the behaviour of $X_{\mu,\nu}$ entirely for the region

$$\left\{(\mu,\nu) \;\middle|\; |\nu| \le T_S\,,\, |\mu| \le \nu^4 M\right\}.$$

Since for $\mu > 0$ there are no singularities, there is just paralell flow in that region of the parameter space, and the behaviour in the region

$$\left\{(\mu,\nu) \;\middle|\; -T_L^4 \le \mu \le -\frac{\nu^2}{\nu_0^2}\right\}$$

has been characterised as well. Unfortunately, between the two regions in the half plane $\mu < 0$ there remains a gap, since their boundaries have a different order of contact at $(\mu,\nu) = (0,0)$.

To close this gap, compute the difference between two vector fields differing only in their value of the parameter ν. Here it is important to remember that $\mu = \mu(\lambda)$ and $\nu = \nu(\lambda)$, so we are considering λ_1 and λ_2 such that $\mu(\lambda_1) = \mu(\lambda_2)$ and $\nu(\lambda_1) \ne \nu(\lambda_2)$. Then

$$X_{\lambda_2} - X_{\lambda_1} = y\left(\nu_2 - \nu_1 + x^2(h_2 - h_1) + y\,(Q_2 - Q_1)\right)\frac{\partial}{\partial y}$$

where

$$\begin{aligned} h_2 - h_1 &= h(x, \lambda_2) - h(x, \lambda_1) = (\nu_2 - \nu_1)\, U(x, \lambda_1, \lambda_2) \\ Q_2 - Q_1 &= Q(x, y, \lambda_2) - Q(x, y, \lambda_1) = (\nu_2 - \nu_1)\, V(x, y, \lambda_1, \lambda_2)\,. \end{aligned}$$

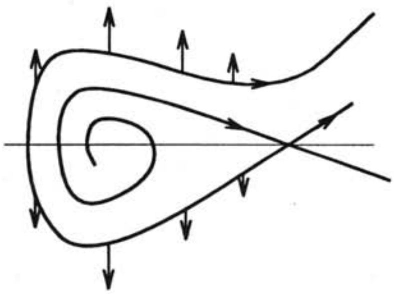
Thus

$$\begin{aligned} X_{\lambda_2} - X_{\lambda_1} &= y\,(\nu_2 - \nu_1)(1 + x^2 U + yV)\frac{\partial}{\partial y} \\ &= y\,(\nu_2 - \nu_1)\,(1 + \zeta(x, y, \lambda_1, \lambda_2))\,\frac{\partial}{\partial y} \end{aligned}$$

where ζ is a C^∞ function, $\zeta = \mathrm{O}(|x^2 + y^2|^{\frac{1}{2}})$. There exists a neighbourhood of $(x, y) = (0, 0)$ in phase space, and another neighbourhood of $(\mu, \nu) = (0, 0)$ in parameter space, such that there $|\zeta| < \frac{1}{2}$. Then

$$X_{\lambda_2} - X_{\lambda_1} = y\,(\nu_2 - \nu_1)\,\chi\frac{\partial}{\partial y}$$

where χ is a strictly positive function of (x, y, μ, ν).
Consider now, for μ fixed, a ν greater than the Hopf-bifurcation value ν_H. We have the following phase portrait, the arrows indicating the direction of change if ν is increased:

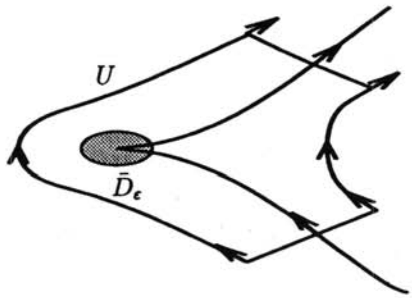


Let $h(\nu)$ denote the y-coordinate of the intersection of the unstable manifold of s_μ with the y-axis. By the above form of the difference of the two vector fields, it follows that $h(\nu)$ is strictly increasing. From there the existence of the α-limit set of the stable separatrix is ensured for all ν.
It remains to show that this α-limit set cannot be a limit cycle.
By contradiction: suppose for ν_2 there is a limit cycle, to be called C, while for a fixed $\nu_1 < \nu_2$ there is none. Since for ν_1 the stable separatrix of the saddle connects e_μ and s_μ, there is a point on this separatrix which is inside C. Consider its orbit under the field X_{λ_2}. By the existence of the limit cycle, this orbit will intersect the stable separatrix of the vector field X_{λ_1}. But that is not possible by

the above property of the difference of the vector fields.

7 Tying up loose ends

The final equivalence result is needed for some fixed neighbourhoods U and V of the origins of phase and parameter space respectively.
However, after rescaling, the phase portrait has been studied for $(\bar{x}, \bar{y})$ in a fixed compact set $\bar{D}$. Transforming back to the original phase space this corresponds with a set $\bar{D}_\varepsilon$ which shrinks down to $\{(0,0)\}$ as $\varepsilon \to 0$.
The difficulty can be overcome as follows. Fix a neighbourhood U of $(0,0)$ as in the following picture



Then we can obtain, possibly by restricting (μ, ν) once more to a still smaller neighbourhood of $(0,0)$, that $X_{\mu,\nu}$ is equivalent to a vector field which equals $X_{0,0}$ near ∂U, and which equals $X_{\mu,\nu}$ in $\bar{D}_\varepsilon$. By the way, $\bar{D}_\varepsilon$ can be chosen to have the same form as U, only smaller (i.e. bounded left and right by orbits, and above and below by transverse sections of the flow). Having done this, and since we know that $U \backslash \bar{D}_\varepsilon$ does not contain singular points, we see that the flow is trivial in $U \backslash \bar{D}_\varepsilon$, by the Poincaré-Bendixon theorem.
For take an outgoing orbit of $\bar{D}_\varepsilon$. Its ω-limit set is, by the Poincaré-Bendixon theorem, either a fixed point or a limit cycle. In $U \backslash \bar{D}_\varepsilon$ there are no fixed points. There are neither any limit cycles, since any limit cycle surrounds at least one fixed point (index argument). So the ω-limit set cannot be inside $U \backslash \bar{D}_\varepsilon$, and the orbit has to leave the set by the only exit possible.

We conclude that
$$X_{\mu,\nu}\Big|_{\bar{D}_\epsilon} \sim X_{\mu,\nu}\Big|_{U}$$

8 Final remarks

We have established the C^0-fibre equivalence of two generic families: in each region of the parameter space outside the bifurcation curves, we have obtained a unique, well-defined phase portrait.
It is even possible to obtain a C^0-equivalence, see [5].

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