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On the Inversion of Lagrange-Dirichlet Theorem

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Abstract: We consider a Lagrangian differential system. The celebrated theorem of Lagrange-Dirichlet ensures that a stationary solution of this system is stable, provided that the corresponding critical point of the potential function is a proper (local) maximum. It is also well-known that the statement of this theorem, in general, is not invertible. The Problem of the Inversion of Lagrange-Dirichlet theorem consists in finding criteria which ensure the instability of the equilibrium. Here we give a review of some of these results.

Key words: stability, instability, asymptotic solutions, Mather sets.

Al mio maestro ed amico Luigi Salvadori

§1 Introduction

Let us consider the Lagrange Differential System *(L)* related to the Lagrangian function:

$$L(q,\dot q) = T_{[2]}(q,\dot q) + T_{[1]}(q,\dot q) + U(q) \tag{1.1}$$

where $L \in C^h(B_b \times \mathbf{R}^n, \mathbf{R})$, B_b being a ball of radius b, centered at the origin $q = 0$, h a positive integer to be specified later, and:

$$T_{[2]}(q,\dot q) = \frac{1}{2}\langle S(q)\dot q, \dot q\rangle$$
$$T_{[1]}(q,\dot q) = \langle b(q), \dot q\rangle \equiv \langle A(q)q, \dot q\rangle$$

The function $T_{[2]}(q,\dot q)$ is a *positive-definite* quadratic form, $U(q)$ has a critical point at $q = 0$, $U(0) = 0$. The linear term in the velocity variables $T_{[1]}(q,\dot q)$ contributes to the system *(L)* by the so called *gyroscopic forces*:

$$F_i(q,\dot q) = \Big(\frac{\partial b_i(q)}{\partial q_j} - \frac{\partial b_j(q)}{\partial q_i}\Big)\dot q_j, \quad i = 1,\cdots,n.$$

If the *gyroscopic forces* are absent, or equivalently if the 1-form $\langle b(q), dq\rangle$ is exact, then the Lagrangian function will be said *natural*. If the matrix $A(0)$ is non singular, the *gyroscopic forces* will be said non degenerate.

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In the following we will be concerned with the Lyapunov stability (and the opposite property, the instability) of the stationary solution $q = 0$ of *(L)*.

Let us recall the classical Lagrange–Dirichlet Theorem (L.D.T.):

Theorem 1.1. *Let $h = 2$, and let $q = 0$ be a proper (local) maximum for U; then the stationary solution $q = 0$ of (L) is stable.*

It is also well known that the proof of this theorem is accomplished by considering the Energy function:

$$E(q, \dot{q}) = T_{[2]}(q, \dot{q}) - U(q).$$

Indeed, taking into account the hypothesis, the level surfaces of this function define a family of fundamental neighborhoods of $(q, \dot{q}) = (0, 0)$. Moreover the Energy is a first integral of the system *(L)* so that these neighborhoods are invariant sets.

Of course, if we miss the foresaid maximum property for U the Energy does not define such a family of neighborhoods. However, as it is well-known, the lack of the maximum property for U does'nt imply, in general, the instability of $q = 0$. Indeed, one has just to consider this simple case of potential U (Painlevé's counter example):

$$U(q) = \exp(-\frac{1}{q^2}) \sin(\frac{1}{q}), \qquad q \in \mathbf{R} \setminus \{0\}, \qquad U(0) = 0.$$

It is possible to give a more sophisticated version of the Painlevé 's counter example, just preserving the essential feature of the potential: there exists a sequence $\{h_n\}$ of negative real numbers, $h_n \to 0$ as $n \to \infty$, such that the sets $\Sigma_n = \{q \in B_b : U(q) = h_n\}$ are neighborhoods of the origin $q = 0$. Let us call these sets ***isolating sets***.

It is interesting to observe that in the Painlevé 's counter example the existence of ***isolating sets*** is due to the fact that the potential function U is not a C^ω function. Of course, for one degree of freedom Lagrangian analytic systems, the absence of the maximum property for analytic potential function U becomes a necessary and sufficient condition for the stability. Indeed, neglecting the trivial case $U(q) \equiv 0$, we have:

$$U(q) = aq^k + o(|q|^k), \qquad a \neq 0 \tag{1.3}$$

a being a nonzero constant, k being a positive integer. Therefore, just by inspecting the level curves in the phase space close to $(0, 0)$, one easily realizes that the origin is stable if and only if k is even and a is negative.

Indeed, for one degree of freedom system the hypothesis of analyticity of U can be replaced directly by (1.3). Certainly this equivalence falls for higher degree of freedom; therefore one has several non equivalent possibilities to prevent the existence of the ***isolating sets***.

The aim of this paper is to give an account of several instability results obtained under different hypotheses on U, each of them ensuring a particular condition of non existence of the foresaid *isolating sets*. The reader interested in to have a quite complete picture of the results on the Inversion of Lagrange-Dirichlet Theorem could couple this paper with the comprehensive review given by Salvadori [1] (see also Hagedorn [2], Laloy et al. [3]). We begin just recalling the well known instability result of Liapunov [4] concerning the case in which the absence of the maximum property for the potential at its critical point $q = 0$ can be ascertained by inspecting the quadratic part of U. From now onward our interest will be confined to the so called *critical* cases, i.e. when the stability properties of the stationary solution $q = 0$ cannot be derived from the analysis of the linear part of the Lagrangian vector field (with respect to $(0, 0)$).

We conclude this Section by considering two theorems, the first one due to Çetaev [5], the second to Palamodov [6]: they are a good starting point for introducing the rest of this paper.

Theorem 1.2. *Consider the natural Lagrangian function $L(q, \dot{q}) = T_{[2]}(q, \dot{q}) + U_{[k]}(q)$, $U_{[k]}(q)$ being a homogeneous function of degree k, $k \geq 3$. Let $q = 0$ be not a maximum point for $U_{[k]}$: then it is an unstable stationary solution.*

Let us recall the very simple, but significative proof of the theorem. We consider the Çetaev function:

$$V(q, \dot{q}) = \left\langle q, \frac{\partial T_{[2]}(q, \dot{q})}{\partial \dot{q}} \right\rangle .$$

Taking the time derivative along the solutions of the Lagrangian system, one has:

$$\begin{aligned} \frac{dV}{dt}(q, \dot{q}) &= 2T_{[2]}(q, \dot{q}) + \left\langle q, \frac{\partial}{\partial q}\big(T_{[2]}(q, \dot{q}) + U_{[k]}(q)\big)\right\rangle \\ &= 2T_{[2]}(q, \dot{q}) + \left\langle q, \frac{\partial T_{[2]}(q, \dot{q})}{\partial q}\right\rangle + kU_{[k]}(q). \end{aligned}$$

Let us consider the (nonempty) invariant domain $D \equiv \{(q, \dot{q}) : E(q, \dot{q}) = T_{[2]}(q, \dot{q}) - U_{[k]}(q) < 0\}$. We have: $(0, 0) \in \partial D$. By considering a suitable small sphere B_r, centered at the origin $(0, 0)$, we have:

$$\frac{dV}{dt}(q, \dot{q}) \geq 2\big(T_{[2]}(q, \dot{q}) + U_{[k]}(q)\big).$$

Therefore,

$$\forall (q, \dot{q}) \in B_r \cap D \Rightarrow \frac{dV}{dt}(q, \dot{q}) > 0.$$

So, trajectories starting at the initial point $(q_0, \dot{q}_0) \in B_r \cap D$ must leave in a finite time this set, crossing $\partial B_r \cap D$. $\diamondsuit$

Let us emphasize the role of the homogeneity property of $U_{[k]}(q)$ in Theorem (1.2): the inner product of the vector fields q and $\nabla U_{[k]}(q)$ is strictly positive whenever $U_{[k]}(q)$ is positive. The proof of the Çetaev theorem that we recalled here suggests the idea of analyze similarly the stability problem for more general potentials, by replacing the vector q with suitable vector $v(q)$, such that:

$$\left\langle v(q), \frac{\partial U(q)}{\partial q} \right\rangle > 0$$

whenever $U(q) > 0$. The second theorem indeed is just a significative case of using such a technique.

Theorem 1.3. *Let $L(q, \dot{q})$ be a natural Lagrangian function and assume :*

$$U(q) = U_{[k]}(q) + W(q)$$

where $U_{[k]}(q)$ is a homogeneous form of degree k, $k \geq 3$, having $q = 0$ as an isolated critical point. Moreover $U_{[k]}(q)$ has not a maximum at $q = 0$, $W \in C^k(B_b, \mathbf{R})$ and $W(q) = o(|q|^k)$. Then the equilibrium $q = 0$ is unstable.

The proof is accomplished by considering the following vector field:

$$v(q) = q + \alpha |q|^{(2-k)} \nabla U(q),$$

where α is a positive parameter. Due to the hypothesis, we can find $\delta \in (0, b)$ such that:

$$|\nabla U(q)| \geq A|q|^{(k-1)}, \qquad \forall q \in B_\delta,$$

A being a positive constant. We consider the following Çetaev function $V \in C^{(k-1)}(B_\delta \times \mathbf{R}^n, \mathbf{R})$:

$$V(q, \dot{q}) = \left\langle \frac{\partial T_{[2]}(q, \dot{q})}{\partial \dot{q}}, v(q) \right\rangle.$$

Therefore, with α suitable small we have:

$$\frac{dV}{dt}(q, \dot{q}) \geq \frac{1}{2} T_{[2]}(q, \dot{q}) + \left\langle q, \nabla U(q) \right\rangle + \alpha |q|^{(2-k)} |\nabla U(q)|^2,$$

and therefore, eventually restricting δ:

$$\frac{dV}{dt}(q, \dot{q}) \geq \frac{1}{2} T_{[2]}(q, \dot{q}) + kU(q) + \frac{1}{2} \alpha A |q|^k.$$

Let us now consider initial data $(q_0, \dot{q}_0)$ such that $U_{[k}(q_0) > 0$ and $E = T_{[2]}(q_0, \dot{q}_0) - U(q_0) < 0$. As long as the orbit of the motion $q(\cdot, q_0, \dot{q}_0)$ lies in B_δ we have:

$$\frac{dV}{dt}(q, \dot{q}) \geq \frac{1}{2} T_{[2]}(q, \dot{q}) + \frac{1}{4} \alpha A |q|^k.$$

As an immediate consequence we obtain that the motion $q(\cdot, q_0, \dot{q}_0)$ must leave B_δ in a finite time, and therefore the instability follows. $\diamondsuit$

The dynamical interpretation of both these theorems is quite intuitive: more or less they ensured the condition under which each motion $q(\cdot, q_0, \dot{q}_0)$, having the initial datum $(q_0, \dot{q}_0)$ of positive energy and sufficiently close to $(0,0)$, has a nonzero radial component of the velocity, at least until it stays close to the stationary solution. Therefore, perhaps a quite large part of the kinetic energy can be wasted turning around $q = 0$, but still a sufficient amount remains to move away. Indeed this part is continuosly increasing; consequently $(q(t, q_0, \dot{q}_0), \dot{q}(t, q_0, \dot{q}_0))$ cannot stay indefinitely close to $(0,0)$.

We emphasize that in Palamodov's Theorem the nonexistence of *isolating sets* for the origin can be ascertained just by inspecting the first non zero term $U_{[k]}(q)$ in the Mac-Laurin expansion of the potential function $U(q)$. This is the essential tool for discovering the non vanishing radial velocity. However, this theorem is not a generalization of the Çetaev theorem; recall indeed that here $q = 0$ has to be an isolated critical point for $U_{[k]}(q)$. In the following Sections we will see the amount of job we need for relaxing such hypothesis.

Let us give a plan of the rest of the paper. In Sections $2, 3, 4$, we will consider only *natural* Lagrangian systems. Moreover:

(i) Section 2 is concerned with the problem of the Inversion of Lagrange-Dirichlet Theorem when the absence of *isolating sets* is due to the *analyticity* of the potential function U, which has not a maximum at $q = 0$.

(ii) In Section 3 we consider the case in which the absence of *isolating sets* is realized by assuming $q = 0$ to be a *minimum* point for the potential function U.

(iii) In Section 4 again we consider the potential function U whose extremal properties at the critical point $q = 0$ can be suitably ascertained from the inspection of its k-jet at the origin. In particular the instability of the equilibrium will be a consequence of the existence of motions which tend asymptotically to it.

(iv) The last Section is devoted to the difficult problem of the stability of the stationary solution $q = 0$ of *(L)* system having *gyroscopic forces*, i.e. when the Lagrangian function has a nonzero linear kinetic term $T_{[1]}$. We will consider,in particular, the case of non-degenerate *giroscopic forces*, i. e. the case $A(0)$ non singular. In such a situation we will see that the property of $q = 0$ being a *proper minimum* for the potential function U is no more sufficient for the instability; the so called *gyrostatic stabilization* phenomenon can occur. Then the problem of what happens with the instability property of the equilibrium of a *natural* Lagrangian system when a small non degenerate *gyroscopic force* is switched on is analyzed: we will show that, close to the origin, a new invariant set (a Mather set) arises, which turns out to be an unstable set being the limit set of a class of motions.

§2 The analytic case.

This Section is mainly concerned with the following problem. Consider a n degree of freedom *natural* Lagrangian system and let the potential function $U \in C^\omega(B_b, \mathbf{R})$. Then, recall that in the case $n = 1$ the loss of the proper maximum property of U at $q = 0$ is a necessary and sufficient condition for the Inversion of Lagrange-Dirichlet Theorem. Is the same still true for arbitrary n? Hereafter mainly we deal with the case when $q = 0$ is not a maximum.

A first important contribution to the problem of the instability of the equilibrium of analytic *natural* Lagrangian systems with two degree of freedom, was given by Palamodov [6] (and, later on, but indipendently, by Taliaferro [7]). Precisely we have the following result:

Theorem 2.1. *Let $U \in C^\omega(B_b, \mathbf{R}), B_b \subset \mathbf{R}^2$. Assume $q = 0$ be a critical point for U, which is not a maximum; then it is an unstable equilibrium of the differential system: $\ddot{q} = \nabla U(q)$.*

The Palamodov' s proof is mainly based on the properties of the level sets of analytic functions in the plane. The next-Lemmas allow us to construct a suitable Çetaev function.

First Lemma. *Let U_ρ^+ be the intersection of a ball centered at $q = 0$, having radius ρ suitably small, with a connected component of the set of positive values of the potential U, whose closure contains the origin. Then it is possible to define a continuous vector field u in U_ρ^+ such that:*

$$\left\langle u(q), \frac{\partial U(q)}{\partial q} \right\rangle \geq 0.$$

Moreover, the vector field u is a C^1 vector field in $U_\rho^+ \setminus K$, K being the union of a finite number of C^∞ curves, and there exists a positive constant C such that:

$$\frac{\partial u_i(q)}{\partial q_j} w_i w_j \geq C|w|^2,$$

for any $w \in \mathbf{R}^2$, $q \in U_\rho^+ \setminus K$.

Second Lemma. *Let U be an analytic function in a two-dimension domain containing the origin $q = 0$. Then, for a suitable small ρ, U_ρ^+ cannot contain critical points of U.*

Collecting together the results of these two Lemmas, a Çetaev vector field $v(q) = u(q) + \sigma \nabla U(q)$ can be constructed, choosing σ as a suitable small positive constant. Consider indeed the Çeteav function:

$$V(q, \dot{q}) = \langle v(q), \dot{q} \rangle$$

As long as we consider a motion $q(t)$ avoiding the set K during a time interval (t_1, t_2), we have:

$$\frac{dv}{dt}(q(t), \dot{q}(t)) \geq \frac{1}{2} C |\dot{q}(t)|^2 + \sigma |\nabla U(q(t))|^2,$$

and therefore, due to the Second Lemma:

$$\Delta v \equiv v(q(t_2), \dot{q}(t_2)) - v(q(t_1), \dot{q}(t_1)) \geq \alpha(t_2 - t_1).$$

α being a suitable positive constant. Now, if there exists a solution $q(t) = q(t, q_0, \dot{q}_0), t \geq 0$ which intersects K in a measure zero set S of its existence interval, the proof is easily accomplished. Let us prove indeed that the set C of motions, each one of them crossing K in a non zero measure subset $\tilde{S}$ of $[0, \infty)$, is at most countable. Let us consider anyone of such possible motions $q(\cdot) \in C$ and denote its semi-orbit by γ^+ $(\gamma^+ \equiv \{\bigcup_{t\geq 0} q(t)\})$. First of all we remark that, as $\nabla U(q)$ does'nt vanish, then also $K \bigcap \gamma^+$ is a set of non zero measure (as subset of K). Consequently if the set C is more than countable, certainly it contains at least two motions whose semiorbits intersect each other in a susbset of K of positive measure. So the two orbits coincide. $\diamondsuit$

Remark that in the preceeding theorem a very important role is played by the fact that the configuration space is two dimensional. This limitation is removed in the following theorem due to Kozlov and Palamodov [8], [9].

Theorem 2.2. *Let L be a natural analytic Lagrangian function and assume that there exists a positive integer k, $k \geq 3$, such that the Mac-Laurin expansion of U starts with an homogeneous polynomial $U_{[k]}$ of degree k, having not a maximum at $q = 0$. Then this equilibrium is unstable.*

We will come back again to this result in Section 3. Here we just remark that it is a generalization of Theorem 1.2 and that the authors indeed proved more than the instability of the equilibrium: as matter of fact this property is a consequence of the existence of motions which are asymptotic to the equilibrium. Theorem 2.2 does not depend on the number of degree of freedom but the absence of the maximum property of the potential U at the critical point $q = 0$ has to be again recognized by inspecting the first non zero homogeneous form $U_{[k]}$.

Subsequently in 1992, Palamodov [10] was able to generalize completely the instability result for analytic *natural* Lagrangian of any degree of freedom assuming the absence of the maximum property of the potential $U(q)$ at the critical point $q = 0$. He was able to exploit the Hironaka theorem [11] on the resolution of the singularity of an analytic function (see also [12], pg. 81) in order to construct again a suitable Çetaev vector field. Precisely the following theorem was proved.

Theorem 2.3. *Let $L(q, \dot{q})$ be a natural Lagrangian function defined in $B_b \times \mathbf{R}^n$. Assume the kinetic energy $T_{[2]} \in C^1(B_b \times \mathbf{R}^n)$, the potential $U \in C^\omega(B_b)$ and $\nabla U(0) = 0$. Let $U(0) = 0$ be not a maximum. Then there exists a neighborhood $V_0 \subset B_b$ such that any motion starting in V_0 with arbitrarily small positive energy E leaves V_0 in a finite time $\tau(E)$.*

Of course, the instability result follows at once. A relevant consequence of the Palamodov' s result is the following:

Corollary 2.4. *Let us consider a natural Lagrangian system and assume $U \in C^\infty(B_b, \mathbf{R})$. Moreover let $q = 0$ be a critical point of finite multiplicity $k+1$, and not a maximum for U. Then the origin 0 is an unstable stationary solution.*

The proof of this corollary is based on the Tougeron result on the sufficiency of the jets (see [12]). Precisely, as $q = 0$ is a critical point of multiplicity $k+1$, the k-jet of U is sufficient. It means that we can find a smooth function $\psi(\cdot)$ defined in suitable neighborhhod B_r of the origin, $(B_r \subset B_b)$, such that:

$$U(\psi(Q)) = U_{\{k\}}(Q),$$

$U_{\{k\}}$ being the Mac-Laurin polynomial of degree k of U. Moreover $\psi(q) = q + O(|q|^2)$. Therefore we can consider the Lagrangian function in the new coordinates:

$$\tilde{L}(Q, \dot{Q}) = \frac{1}{2}\langle \dot{Q}, (D\psi(Q))^{-1} S(\psi(Q)) D\psi(Q)\dot{Q}\rangle + U_{\{k\}}(Q),$$

where $D\psi$ denotes the Jacobian matrix of ψ. Finally we can invoke the result of Theorem 2.3 and conclude the proof.$\diamondsuit$

Corollary 2.4 depends of course on the possibility to ascertain the sufficiency of some finite $k-$jet of the potential. To this purpose the following result by Takens [13] is interesting. First of all we will precise the meaning of sufficiency. Let us consider the germ of a C^k function $f : (R^n, 0) \to (R, 0)$. The k-jet of f is said to be C^j-sufficient if for every $g : (R^n, 0) \to (R, 0)$, having the same k-jet as f, there exists a germ of a C^j diffeomorphism $\psi : (R^n, 0) \to (R^n, 0)$, such that:

$$g(\psi(x)) = f(x)$$

Let us define now two natural numbers $r_0(f)$ and $r(f)$, $r_0(f) \le r(f) \le \infty$.

(i) $r(f)$ is the smallest natural positive integer such that there exist two positive real numbers C, δ for which:

$$|\nabla f| \ge C|x|^{(r(f)-1)}, \quad \forall x : |x| \le \delta$$

(ii) $r_0(f)$ is the biggest positive integer such that there are positive real numbers C, δ for which:

$$|f(x)| \ge C|x|^{r_0(f)}, \quad \forall x : |x| \le \delta.$$

If no such constants C, δ can be found in (i) (resp. in (ii)), we set $r(f) = \infty$ (resp. $r_0(f) = \infty$).

Theorem 2.5. *Let assume $r(f) \le k$. Then if:*

$$k - r(f) + 1 \ge j\big(r(f) - r_0(f) + 1\big)$$

for some j, then the k-jet of f is C^j sufficient.

Therefore, to apply Theorem 2.3 to a lagrangian problem with potential of finite regularity C^k it is enough to verify that the following inequality holds:

$$k > 3r(U) - 2r_0(U) + 1.$$

If $U \in C^{\infty}$, and $q = 0$ is an isolated critical point such that:

$$|\nabla U(q)| \geq C|q|^{\beta}$$

in some neighbrhood of $q = 0$, for some finite number β, then certainly the $k-$jet of U is C^2 sufficient, provided that $k \geq 3\beta - 2m + 2$, m being the first non zero derivative of the function evaluated at $q = 0$. ◇

The problem of the inversion of the Lagrange-Dirichlet Theorem for *natural* Lagrangian systems with analytic potential is reduced, due to Theorem 2.3, to the problem of considering potential not having a *proper maximum* at the critical point $q = 0$. This problem was solved in the case of *two degree of freedom* by Laloy et al. [14]. The general case of any degree of freedom is still open.

In the following Sections we will consider the Inversion problem in a wider class of Lagrangian functions, allowing weaker regularity, $C^h, h \leq \infty$. Of course, we will consider a Lagrangian function whose potential cannot be reduced to an analytic potential, like in the case of Corollary 2.4. However, as we will see, other conditions will be added on the potential. In particular, the works analyzed in the next Section deal with potential of very low regularity, i.e.$h \in \{1,2\}$, but the presence of *isolating sets* is forbidden by assuming the very strong condition: $q = 0$ has to be a minimum for the potential.

§3 Variational methods

This Section is devoted to illustrate the Inversion of Lagrange-Dirichlet Theorem when the potential function U has a *minimum* at the critical point $q = 0$. At a first glance it could seem quite obvious that in similar conditions the instability holds. However one can easily realize that this ansatz strictly depends on the picture one has for one degree of freedom systems; in principle, if the dimension n of the configuration space is greater than 1, the points with low speed could permanently rotate, remaining close to the equilibrium. Indeed, to prove the instability of the equilibrium is not a quite simple job. Until now the results we will discuss in the following are based heavely on the variational nature of the Lagrange differential systems: the techniques employed come from variational calculus, or from Riemannian geometry. No other techniques have been found, up to now, to handle the Inversion problem in the foresaid hypothesis.

We recall that the first instability result obtained by variational techniques is due to Hagedorn ([2]) who considered a C^2 *natural* Lagrangian function, and assumed $q = 0$ to be a *proper* minimum for U.

Due to its relevance, we give hereafter a short account of Hagedorn' s work.

To this purpose let us remind that the solutions of *(L)* can be put in a one-to-one correspondence with the solutions of the differential system related to the Maupertuis Least Action principle, the Maupertuis Action being defined as:

$$M(y) = \int_{s_0}^{s_1} \sqrt{(E + U(y(s))S_{k,j}(y(s))y'_k(s)y'_j(s)}ds.$$

Here $y(\cdot) \in C^2([s_0, s_1], B_b), y(s_0) = y_0, y(s_1) = y_1, B_b$ is a neighborhood of $y = 0$, such that $U(y) > 0 \forall y \in \overline{B}_b \setminus \{0\}$. Moreover $S(y) \equiv ((S_{k,j}(y)))$ is the positive definite matrix of the kinetic Energy $T_{[2]}$ and finally E is a positive parameter.

We remark that, due to the fact that the function under the integral sign is a homogeneous function of degree 1 with respect to y', the integral does not depend on the parametrization of the curve. A curve $y(\cdot)$ such that the Fréchet derivative of $M(\cdot)$ at $y(\cdot)$ is zero, is called an extremal of the Maupertuis Action. Of course if $y(\cdot) \in C^2$, then it satisfies the corresponding Euler-Lagrange equation, that is we have a *geodesic curve*. Let $y(s)$ be a geodesic. We set:

$$t(s) = \int_{s_0}^{s_1} \sqrt{\frac{S_{kj}(y(s))y'_k(s)y'_j(s)}{2(E + U(y(s)))}}ds,$$
$$q(t) = y(s(t)).$$

Then it easy to verify that $q(t)$ is a solution of the *(L)* system, lying on the energy level E. Finally, we remark that we have a family of Riemannian metrics (the so called Jacobi metrics) whose metric tensor is given by:

$$g_{i,k}(y) = (E + U(y))S_{k,j}(y)$$

where $y \in B_b$.

Hagedorn proved the following theorem:

Theorem 3.1. *Let $L \in C^2(B_b \times \mathbf{R}^n, \mathbf{R})$ be a natural Lagrangian function. Assume $U(q) > 0$ if $q \in B_b \setminus \{0\}$. Then there exists a number $\tilde{r} \in (0, b)$ such that any point $q_0 \in \overline{B}_{\tilde{r}}$ is connected to $q = 0$ by a solution of the Lagrangian system, having an arbitrarily small positive energy E and entirely lying in B_b.*

Of course the instability of the stationary solution $q = 0$ of *(L)* follows at once.

Hagedorn in his paper explained the reason why this result is not a straighforward consequence of Riemannian geometry (with respect to the Jacobi metric), in particular of the celebrated Hopf-Rinow Theorem [15], [16]. In his work he follows a different approach, using a result by Caratheodory [17] in the Calculus of Variations. We refer the reader for more details to the original Hagedorn paper.

Here we want to show how the Hopf-Rinow theorem was recently exploited in the analysis of the instability problem [18]. Indeed, the authors consider a natural

Lagrangian function $L \in C^2(B_b \times \mathbf{R}^n, \mathbf{R})$ and they assume $q = 0$ to be a minimum of the potential function $U(q)$. Then, for any positive value of E the following modified Jacobi metric is considered on B_b:

$$\exp\left(\frac{\nabla k(q) S(q)^{-1} \nabla k(q)}{E + U(q)}\right)((E + U(q))S(q).$$

The function $k(q)$ is choosen in this way. Fix $r \in (0, b)$ and consider a positive function $k \in C^3(B_r, \mathbf{R}^+)$ satisfying:

$$k(q) = 1 \;\forall q \in B_r, \qquad q \to \partial B_b \Rightarrow k(q) \to +\infty.$$

According to a Gordon result [19], the manifold B_b endowed with this metric turns out to be complete: at this point the Hopf-Rinow theorem can be used, to obtain that any couple of points $q_1, q_2 \in B_b$ can be joined by a geodesic. Therefore, just by observing that in fact the modified metric is exactly the Jacobi metric on B_r,we have:

Theorem 3.2. *Let $L \in C^2(B_b \times \mathbf{R}^n, \mathbf{R})$ be a natural Lagrangian and let $q = 0$ be a minimum point for the potential. Then for any positive number $r \in (0, b)$ any positive E and any point $q_0 \in B_r$ there exists a solution $q(\cdot)$ of the Lagrangian system such that:*

(i) $q(0) = q_0$,
(ii) $E = \frac{1}{2}\langle \dot{q}(0), S(q_0)\dot{q}(0)\rangle - U(q_0)$,
(iii) There exists a positive constant $\tau = \tau(r, E)$ such that $q(\tau) \in \partial B_r$, and $q(t) \in B_r$, for $t \in [0, \tau)$.

Let us compare these two Theorems. Theorem 3.2 shows that the instability result still persists for the more general class of potential having a minimum (not necessarily a strict minimum) at $q = 0$.

Moreover, from one side the instability Hagedorn' s result is enhanced, as r can be chosen arbitrarely close to b. On the other side, the Hagedorn theorem allows to connect any point of a suitable small ball $B_{\bar{r}} \subset B_b$ with the origin by means of a geodesic entirely lying in B_b, so that a *boundary value problem* is solved.

In his paper Hagedorn made mention to the fact that the same result he obtained still could be true assuming weaker regularity for the potential. In fact, under the assumption of $U \in C^1$ only the instability result was successively obtained by Taliaferro [20] and later on by Mawhin and Hagedorn himself [21]. The authors employed techniques from the theory of the so-called direct methods in the Calculus of Variations and proved the following theorem:

Theorem 3.3. *Let $L \in C^1(B_b \times \mathbf{R}^n, \mathbf{R})$ be a natural Lagrangian and let $q = 0$ be a minimum point for the potential. Then, for any $\delta \in (0, \frac{b}{2})$, there exists a solution of the Lagrangian system starting from the initial datum $(0, \dot{q}_0)$, $|\dot{q}_0| \leq \delta$, and arriving at a finite time τ on the boundary $\partial B_{b-\delta}$.*

At this point one can wonder if it could be possible to prove the existence of motions which are asymptotic to the fixed point $q = 0$, i.e. solutions $q(t)$ of *(L)* such that $q(t) \to 0$ as $t \to +\infty$. It is quite immediate to realize that the answer in general is not. However, if the point $q = 0$ is a *proper* minimum we have a result which follows at once from a theorem by Bolotin and Kozlov [22]. We will return later on this work; for our purpose here we restate in a weaker form the stronger result they obtained.

Theorem 3.4. *Let M be a compact Riemanniann manifold, TM the tangent bundle and let $L \in C^2(TM, \mathbf{R})$ be a natural Lagrangian whose kinetic energy is defined in term of the metric of M. Let $U(0) = 0$ be a proper minimum for the potential function U. Then for any $q_0 \in M$ there exists a motion $q(t, q_0, \dot{q}_0)$ asymptotic to $q = 0$.*

Let us now exploit this result. We can consider our Lagrangian function $L(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, S(q)\dot{q}\rangle + U(q)$, defined in $B_b \times \mathbf{R}^n$, as a local representation of a Lagrangian function L^M globally defined on a compact manifold M. The kinetic matrix and the potential function of L^M are obtained by means of a C^2 extension on M of the functions S and U, in such a way that M is a compact Riemanniann manifold and $q = 0$ is a strict minimum point for the potential. Now the conditions of Theorem 3.4 occur and we obtain the following corollary.

Corollary 3.5. *Assume $L \in C^2(B_b \times \mathbf{R}^n, \mathbf{R})$ be a natural Lagrangian function and let $q = 0$ be a proper minimum point for the potential. Then, for any $r \in (0, b)$ there exist a point $q_0 \in \partial B_r$ and a motion $q(t, q_0, \dot{q}_0)$ which is asymptotic to $q = 0$.*

We end this Section with the following open problem: is it possible to generalize the previous techniques for analyzing the case in which the potential function does have not a minimum at $q = 0$ but there exists a connected component $\tilde{U} \equiv \{q \in B_r : U(q) > 0\}$ such that $0 \in \partial\tilde{U}$? Of course the Jacobi metric cannot now be considered in all the neighborhood of $q = 0$, being meaningfull only in the domain of possible motions, i.e. in the manifold with boundary: $\{q \in B_r : E + U(q) \geq 0\}$. The paper by Kozlov ([23]) is a very useful reference for interesting dynamical problems involving such kind of domains.

§4 The existence of motions asymptotic to the equilibrium.

At the end of Section 3 we mentioned that up to now there is not the possibility of using variational methods for solving the problem of the Inversion of the Lagrange-Dirichlet theorem in the case when the potential function U has not a maximum at $q = 0$. However, from the results quoted in Section 2 we know that under such a condition the Inversion Problem is solved if $U \in C^\omega$. Therefore we want to consider here the case of a Lagrangian function L with regularity $C^h, h \leq \infty$, and a potential function U having not a maximum at the critical point $q = 0$. Of course, the problem in this general form cannot be solved. Indeed more conditions will be added on U, specifically we will assume that the absence of the maximum can be ascertained in a suitable way. Then the instability of the equilibrium $q = 0$ is obtained by proving the existence of motions asymptotic to the origin. We start with the following result of Kozlov [24]:

Theorem 4.1. *Let L be $C^\infty(B_b \times \mathbf{R}^n)$ be a natural Lagrangian function and let $U(q) = U_{[2]}(q) + U_{[k]}(q) + W(q)$; $U_{[2]}(q)$ is a quadratic negative semidefinite form, $U_{[k]}(q)$ is a homogeneous polynomial of degree k, whose restriction on the subspace $K \equiv \{q : U_{[2]}(q) = 0\}$ has not a maximum at $q = 0$, and $W(q) = o(|q|^k)$. Then there exists a motion asymptotic to $q = 0$.*

As recalled by Kozlov, Koiter [25] already considered the case of analytic potential and $\dim K = n - 1$, proving the instability of the origin by means of a suitable Çetaev' s function.

Of course, Theorem 4.1 generalizes the result of Theorem 2.2. In both these two theorems, the existence of a solution of *(L)* which tends aymptotically to the equilibrium is obtained in two steps. The first one is the construction of a *formal* solution, whose general term decays as $\to +\infty$. However, while in Theorem 2.2, where the Lagrangian function is assumed to be analytic, the proof is completed by showing the convergence of this *formal* solution to a true solution, in the case of Theorem 4.1 this convergence is, in general, impossible. However, by exploiting a very powerful result of Kuznetsov [26] relative to C^∞ differential systems, Kozlov was able to prove that there is a solution of the *(L)* whose asymptotic expansion is exactly given by the foresaid *formal* solution. For its relevance we quote here the Kuznetsov theorem.

Theorem 4.2. *Let $f \in C^\infty(D)$, D being a neighborhood of $(x = 0, \tau = 0)$. Consider the differential system:*

$$\tau^m \frac{dx}{d\tau} = f(x, \tau),$$

and assume that there exists a formal solution of this equation:

$$\overline{x}(\tau) = \sum_{n \geq 0} x_n \tau^n.$$

Then there exist a positive number T and a solution of the differential system $x(\cdot) \in C^\infty(-T, T)$, such that $\overline{x}(\tau)$ is the asymptotic expansion of $x(\tau)$, i.e. for any integer N one has:

$$\tau \to 0 \Rightarrow \tau^{-N}|x(\tau) - \overline{x}(\tau)| \to 0.$$

In fact, Kozlov needed to use a stronger version of the Kuznetsov theorem, i.e. he considered the case of formal solutions whose coefficients x_n are polynomial functions of $\ln \tau$: $x_n = P_{j(n)}(\ln \tau)$, $j(n)$ being the degree of the polynomial suitably related to n ($j(n) \leq n$), and $\tau = t^{-1}$. Palamodov observed that Kuznetsov theorem can be extended to cover also this case.

Let us give just a sketch of the starting point of the iterative procedure employed to construct the formal asymptotic solution. For sake of simplicity in the exposition we consider here just the case in which the quadratic part of U is zero. Later on we will come back to the general case considered in Theorem 4.1. Therefore let us assume:

$$U(q) = U_{[k]}(q) + W(q), \tag{4.1}$$

where $U_{[k]}(q)$ is a homogeneous polynomial of degree $k \geq 3$, having not a maximum at the origin, $W(q) \in C^\infty(B_b), W(q) = O(|q|^{(k+1)})$. Let us assume that the positive maximum of $U_{[k]}(q)$ on the ellipsoid $\{q \in \mathbf{R}^n : \langle q, S(0)q\rangle = 1\}$ is taken at e. We consider the auxiliary differential system, which can be considered as a rather rough simplification of the system *(L)*:

$$A(0)\ddot{q} = \nabla U_{[k]}(q). \tag{4.2}$$

This system has the particular solution $q^{(0)}(t)$:

$$\begin{aligned} q^{(0)}(t) &= z(t)e \\ z(t) &= \left(U_{[k]}(e)(k-2)^2\right)^{\frac{1}{(2-k)}} t^{\frac{-2}{(k-2)}}, \quad t > 0. \end{aligned} \tag{4.3}$$

Precisely the function $q^{(0)}(t)$ is considered as the starting point of the iterative procedure in constructing formal solutions.

For further details we refer the interested reader to the original papers of Kozlov [8], Kozlov and Palamodov [9].

At this point it is natural to wonder if the C^∞ regularity is just a technical device, or there is some obstruction to obtain the Inversion of the Lagrange-Dirichlet theorem for *natural* Lagrangian function having a *finite* regularity potential. In fact, it turned out that the existence of motions asymptotic to the equilibrium still persists weakening the regularity assuptions. Of course, the Kuznetsov' s theorem cannot be longer helpful in such circumstances, and a new approach to the problem is needed. For potential function of the type (4.1), having finite regularity, two similar results on the existence of asymptotic motions were indipendently obtained in [27] and [28]. In the first paper, Taliaferro using a fixed point argument in a convenient functional space was able to prove the following theorem:

Theorem 4.3. *Let $k > 0$ and $\varepsilon \in (0,1]$ be two real numbers. Let $U \in C^2(B_b \setminus \{0\}, \mathbf{R}), U_{[k]} \in C^3(B_b \setminus \{0\}, \mathbf{R}), U_{[k]}(sq) = s^k U_{[k]}(q), \forall s > 0, q \in B_b \setminus \{0\}$. Denoting by U^i (resp. $U^i_{[k]}$) the i-derivative of U (resp. of $U_{[k]}$), $i = 0, 1, 2$, we assume:*

$$U^i(q) = U^i_{[k]}(q) + O(|q|^{(k+\varepsilon-i)})$$

Moreover let us suppose there exists a $\tilde{q}$ such that $U_{[k]}(\tilde{q}) > 0$. The kinetic energy $T_{[2]}$ is assumed to be a C^2 function. Then if $k \in (0,2)$ (resp. $k \geq 2$) there exists a *positive real number* γ *and a solution* $q(t), \quad t \in (-\infty, \gamma)$ *(resp.* $q(t), \quad t \in (0, \gamma)$*) of the Lagrangian differential system tending to* 0 *as* $t \to -\infty$ *(resp. as* $t \to 0^+$*).*

The authors of the latter work used a more geometrical approach to the problem. In fact the effort was made to prove that the solution $q^{(0)}(t)$ (given by (4.3)) of the rather rough approximation of *(L)* corresponds to the equilibrium solution of an enlarged first order differential system *(S)*. These two differential systems are related in such a way that any trajectory of *(S)* tending to the equilibrium corresponds to a motion of *(L)* tending to $q^{(0)}(t)$. Finally the proof of the instability was completed by discovering that the equilibrium solution of *(S)* has a nonempty stable manifold. The precise result is in:

Theorem 4.4. *Let* L *be a natural Lagrangian function,* $L \in C^h(B_b \times \mathbf{R}^n, \mathbf{R}), h > k > 2, \ h, k \in N$. *Let* U *be the potential function:*

$$U(q) = U_{[k]}(q) + W(q).$$

Here $U_{[k]}$ *is a homogeneous polynomial of degree* k *having not a maximum at* $q = 0, W(q) = o(|q|^k)$. *Then the Lagrangian system has a motion asymptotic in the future to the origin.*

Let us sketch the main steps of the proof.

(i) The following change of coordinates is performed:

$$q = z(t)[e + Q]$$

obtaining a non autonomous differential system. (Here $z(t), e$ have the same meaning as in (4.3)).

(ii) The function $z = z(t)$ is inverted, taking therefore z has the independent variable, and the system *(L)* is rewritten in the form of a convenient first order system (Σ) of $2n$ variables, whose right hand side explicitely depends on the variable z.

(iii) The system (Σ) is enlarged by considering a new independent variable ϕ and adding the equation:

$$\frac{dz}{d\phi} = -z.$$

In this way it is obtained the foresaid differential system *(S)*, which is autonomous and has the origin as a fixed point. The proof of the theorem is accomplished by the analysis of the hyperbolic properties of the origin. Further details can be found in the quoted paper. ◇

Later on in [29] an analougous of Theorem 4.1 under a weaker regularity assumption on the Lagrangian was proved by exploiting the techniques introduced in [28]. Precisely the existence of an asymptotic motion to $q = 0$ was proved in the case $L \in C^h$ ($h \geq k + m(k) + 3$, $m(k)$ being the integral part of $\frac{k-3}{2}$), assuming on the k-jet of U the same conditions as to the ones in Kozlov's paper [24] (see Theorem 4.1).

By introducing a suitable change of coordinate, $q \to (x, y)$, $x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n$, we can represent the Lagrangian function L in the following form:

$$L = \frac{1}{2}\langle \dot{x}, G^{(1)}(x,y)\dot{x}\rangle + \langle \dot{x}, G^{(c)}(x,y)\dot{y}\rangle + \frac{1}{2}\langle \dot{y}, G^{(2)}(x,y)\dot{y}\rangle + V(x,y),$$

with $G^{(1)}, G^{(c)}, G^{(2)}$ respectively $n_1 \times n_1, n_1 \times n_2, n_2 \times n_2$ matrices satisfying the conditions:

$$\begin{aligned} G^{(1)}_{\alpha,\beta}(0,0) &= \delta_{\alpha,\beta}, \\ G^{(c)}_{\alpha,j}(0,y) &= O(\|y\|^{k-2}), \\ G^{(2)}_{i,j}(0,0) &= \delta_{i,j}, \end{aligned}$$

and $V(x,y)$ defined by:

$$V(x,y) = \frac{1}{2}\langle x, l(x,y)x\rangle + V_{[k]}(y) + W(y),$$

with:

$$\begin{aligned} &l \in C^{h-1}(\mathbf{R}^n, L(\mathbf{R}^{n_1}, \mathbf{R}^{n_1})), \quad l_{\alpha,\beta}(0,0) = -\omega_\alpha^2 \delta_{\alpha,\beta}, \quad \omega_\alpha \neq 0, \\ &V_{[k]}(y) = U_{[k]}(0,y), \quad W(y) = O(\|y\|^{k+1}). \end{aligned}$$

The kinetic coefficients of the transformed Lagrangian function will be C^{h-2} and the potential V will be C^{h-1}.

Then we have the following theorem:

Theorem 4.5. *Under the previous assumptions, the Lagrangian system admits a solution which tends to the fixed point* 0 *in the future.*

We give a sketch of the proof of the theorem. The starting point is of course the asymptotic solution of the system:

$$G^{(2)}(0,0)\ddot{y} = \nabla V_{[k]}(y),$$

passing through the maximum point e of the potential function $V_{[k]}(\cdot)$ restricted to the ellipsoid $:\{y \in \mathbf{R}^{n_2} : \langle y, G^{(2)}(0,0)y\rangle = 1\}$. Let us denote such solution by:

$$y_{(0)}(t) = z(t)e,$$

where $z(\cdot)$ looks like in (4.3). The goal is to prove that the Lagrangian system admits a solution whose leading term , for $t \to 0^+$, looks like $z(t)$. The main problem is to control the x-variables, the potential having a shielding effect in this direction. Therefore we introduce the Banach space $(\mathbf{X}, \| * \|_1)$, where

$$\mathbf{X} = \{\Phi \in C^2([t_0,\infty), \mathbf{R}^{n_2}) : \sup\{\|\frac{d^j\Phi(t)}{dt^j}\| t^{\frac{2k}{k-2}}, t \in [t_0, +\infty), j = 0,1,2\} < +\infty\}$$

and
$$\|\Phi\|_1 = \sup\{\|\frac{d^j\Phi(t)}{dt^j}\| t^{\frac{2k}{k-2}},\ \ t \in [t_0, +\infty),\ \ j = 0, 1, 2\}.$$

For $\Phi \in \mathbf{X}$, by putting in the y-part of the equations of motion $(\Phi(t), \dot{\Phi}(t))$ in place of $(x, \dot{x})$, we obtain the system which will be called in the following the *reduced system*. This *reduced system* is analyzed following the ideas already exposed in proving Theorem 4.4. In particular, there is a solution $y(t|\Phi)$, suitably related to $z(t)$, which tends to $y = 0$ as $t \to +\infty$. Precisely:

$$y(t|\Phi) = y^*(z(t)) + \hat{y}(t|\Phi), \tag{4.4}$$

where $y^*(z)$ is a polynomial of degree $m(k) + 1$, and $\hat{y}(t|\Phi) = O(z(t)^{m(k)+\frac{3}{2}})$ as $t \to +\infty$. Moreover the following Lipschitz estimates hold:

$$\begin{aligned} \|y(t|\Phi_1) - y(t|\Phi_2)\| &\le L(a) z(t)^{m(k)+\frac{3}{2}} \|\Phi_1 - \Phi_2\|_1, \\ \|\dot{y}(t|\Phi_1) - \dot{y}(t|\Phi_2)\| &\le L(a) z(t)^{\frac{k+1}{2}} \|\Phi_1 - \Phi_2\|_1, \end{aligned} \tag{4.5}$$

where $L(a) > 0, L(a) \to 0$ as $a \to 0^+$.

These estimates are obtained by a careful analysis of the stable manifold for the fixed point of the autonomous differential system *(S)* related to the *reduced system*; their role is essential in showing that a particular $\tilde{\Phi} \in \mathbf{X}$ can be found, so that the couple $(\tilde{\Phi}(t), y(t|\tilde{\Phi}))$ solves the Lagrangian system *(L)*. This last step is performed by introducing an operator on the $\mathbf{X}$ space. In doing so one has to proceed carefully in order to masterize the effect of oscillating terms arising from the quadratic part of the potential. Avoiding details let us briefly give an account of the key ideas to overcome such a difficulty. We replace the x-part of the Lagrangian differential system with a *forced* system, obtained by substituing in the right hand side $(y(\cdot|\Phi), \dot{y}(\cdot|\Phi))$ in place of $(y, \dot{y})$. Such a system can be represented in the form:

$$\ddot{x}_\alpha = M_{\alpha,\beta}(z(t))x_\beta + P_\alpha(z(t), x, \dot{x}, y(t|\Phi), \dot{y}(t, |\Phi)),$$

where $\big((M_{\alpha,\beta}(z(t)))\big)$ is a symmetric matrix, whose entries are *polynomial* functions of degree $[m(k) + 1]$ with respect to the z variable, and

$$\begin{aligned} P_\alpha(z(t), x, \dot{x}, y(t|\Phi), \dot{y}(t|\Phi)) = & O\big(|z(t)|^{(m(k)+2)}|x|, |x|^2\big) + \\ & O\big(|\dot{x}|^2 + |\dot{y}(t|\Phi)|^2\big) + \\ & O\big(|y(t|\Phi)|^{(k-1)}|\big). \end{aligned}$$

Then, by standard results in the theory of analytic perturbation of symmetric operators a suitable basis of eigenvectors of $\big((M_{\alpha,\beta}(z(t)))\big)$ can be found such that the linear part of the forced system takes, in the new normal coordinates, the diagonal canonical form. At this point we are ready for introducing the operator on $\mathbf{X}$ we mentioned before. Indeed, by means of this last normal system a good operator from $\mathbf{X}$ to itself is defined, and due to all the mentioned properties of $y(t|\Phi)$, it is possible to prove the existence of a fixed point, concluding the proof of the theorem. Further details can be found in the original paper. $\diamond$

With theorem 4.5 it is finished our review of results on the Inversion of Lagrange-Dirichlet Theorem in the case of *natural* Lagrangian System. Of course, several questions are still unanswered. Reconsidering all the results, perhaps the most interesting open problem is the following.

Problem (P). *Consider the case of a polynomial potential function of degree k, $U(q) = U_{\{k\}}(q)$ and let $q = 0$ be not a maximum point. We know, (see Theorem 2.3), that the equilibrium is unstable. Let us now perturb this potential, and consider the Lagrangian function with a potential U given by:*

$$U(q) = U_{\{k\}}(q) + W(q),$$

$W(q) = o(|q|^k)$.

Then we ask what kind of restrictions have to satisfies W to ensure the origin is still an unstable equilibrium for the perturbed Lagrangian system.

In other words, we arrived to the problem of finding conditions under which the instability property turns out to be structural (in the sense of Krasovskii [30]). For the sake of commodity of the reader we recall here briefly the definition of structural property. Let us consider a differential system:

$$\dot{x} = f(x, t),$$

$f \in C^1(D \times \mathbf{R}, \mathbf{R}^n)$, D being a neighborhood of $x = 0$ in $\mathbf{R}^n$. A property of the system is said to be structural if a continuous function $\eta : D\setminus\{0\} \to \mathbf{R}^+, \quad \eta(0) = 0$, can be found such that the property still holds for the perturbed system:

$$\dot{x} = f(x, t) + R(x, t)$$

where $R \in C^1(D \times \mathbf{R}, \mathbf{R}^n)$, and:

$$|R(x, t)| \leq \eta(x).$$

Let us now consider the following property for the umperturbed system: $x = 0$ is an unstable equilibrium. As was proved by Krasovskii this property is not structural; the reader is referred to the system (19.4), pag 83 in the quoted book [30]. Let us just remark that neither this system nor the perturbed one are Lagrangian, so that we can consider unsolved the problem if the instability is a structural property, relatively to the class of Lagrangian differential systems. Let us now restrict the unperturbed system considering $f \in C^1(D, \mathbf{R}^n)$. In this autonomous case Krasovskii proved that the following property is structural: $x = 0$ is an unstable equilibrium of the umpertubed system *and* there exists a solution of the umperturbed system $x(t, x_0)$ tending to $x = 0$ as $t \to \infty$. Consequently, due to the reversibility of the solutions of a *natural* Lagrangian system we have the theorem:

Theorem 4.6. *Consider the class of natural Lagrangian system. The property: "there exists a solution of the umperturbed Lagrangian systems which is asymptotic in the future to $q = 0$" is structural.*

Consider again the Problem (P) and assume the existence of asymptotic motions to the solution $q = 0$ of the Lagrangian system with potential $U_{\{k\}}$. Now we know the Problem (P) has a solution but how can we find the function $\eta(q, \dot{q})$? Up to now we are not able to answer to such a question. We think that at this point the relevance of the existence of asymptotic motions is evident. The reader interested in to have more informations on this subject is referred to [30], [31].

§5 Gyroscopic forces.

Let us now spend some more time to analyze the case in which $T_{[1]}$ is not zero, i.e. when in the system *(L)* are present the so called *gyroscopic forces*. In this case, the analysis of the stability problem of the equilibrium becomes even harder. Indeed, as it is well known, the *gyroscopic forces* can have a "stabilizing effect". For instance, they can contribute to the spectrum of the linearized system, so that only purely imaginary eigenvalues arise. In this case, for the discussion of the problem of the stability it could be necessary to invoke tools from the K.A.M. theory [32] (and this could be sufficient only for systems with 2 degree of freedom). In any case, we want to emphasize that also for linear Lagrangian system with *gyroscopic forces* the problem of the stability of the equilibrium is still unsolved.

We begin our review with the classical result of Salvadori [33], giving a criterion of instability for a linear system *(L)*. Let us consider the the quadratic lagrangian function:

$$L_{[2]}(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, S\dot{q}\rangle + \langle Aq, \dot{q}\rangle + \frac{1}{2}\langle Uq, q\rangle.$$

$S, U,$ are symmetric positive definite matrices, A an antisymmetric matrix. Let us consider the Çetaev function:

$$C(q, \dot{q}) = \langle \dot{q}, Sq\rangle,$$

and its derivative along the motion:

$$\frac{dC}{dt}(q, \dot{q}) = |\sqrt{S}\dot{q} + \sqrt{S^{-1}}Aq|^2 + \langle (U + AS^{-1}A)q, q\rangle.$$

Therefore we have the following theorem.

Theorem 5.1. *Assume the matrix $U + AS^{-1}A$ has purely positive eigenvalues, then the stationary solution $q = 0$ is unstable.*

Let us remark that a kind of "effective potential", the function $U_{eff}(q) = \langle (C - AS^{-1}A)q, q\rangle$, and not only the potential U, is required to have a *minimum* at $q = 0$. Indeed, as we know by the Kelvin classical result, in the case of quadratic Lagrangian function having a configuration space of even dimension, a minimum of the potential could be stabilized by sufficiently strong *gyroscopic forces* (see for instance [34]). If one consider two degree of freedom system it is easy to give explicit and sharp condition on the strenght of the gyroscopic matrix. Moreover, comparing this estimate with the condition in Salvadori's theorem, one easily recognize that the latter is not optimal. Of course it would be very interesting to give necessary and sufficient condition for the stability in the general case of n-dimensional linear Lagrangian system. Bolotin and Negrini very recently proved that if the potential function U_{eff} has a *maximum* at $q = 0$, then the *gyroscopic stabilization* occurs. Also the paper [36], recently appeared, contains relevant results on the problem of the *linear gyroscopic stabilization*.

A request similar to that in Theorem 5.1 appears in a theorem of Hagedorn [37] dealing with a non linear Lagrangian system. Let us consider indeed the Lagrangian function:

$$L(q, \dot{q}) = \frac{1}{2}\langle S(q)\dot{q}, \dot{q}\rangle + \langle b(q), \dot{q}\rangle + U(q).$$

Hagedorn assumed L to be a C^2 Lagrangian and proved the following theorem:

Theorem 5.2. *Let the function*

$$U_{eff}(q) = U(q) - \frac{1}{2}\langle b(q), S^{-1}(q)b(q)\rangle \tag{5.1}$$

have a proper local minimum at $q = 0$. Then $q = 0$ is unstable.

The theorem was proved considering the following Jacobi Variational Principle:

$$J = \int_{s_0}^{s_1} \left(\sqrt{2(E + U(q))\langle \dot{q}, S(q)\dot{q}\rangle} + \langle b(q), \dot{q}\rangle\right) ds$$

Of course, a geodesic corresponds, by means the reparametrization $s = s(t)$ already considered in Section 2, to a solution of the system *(L)* lying on the energy level E.

In fact Hagedorn proved more than the instability; indeed, he was able to prove again the same result as he proved for a *natural* Lagrangian system in Theorem 3.1. Indeed let us consider the function:

$$\sqrt{2(E + U(q))\langle u, S(q)u\rangle} + \langle b(q), u\rangle \tag{5.2}$$

for any positive value of E, $(q, u) \in B_b \times \mathbf{R}^n \equiv TB_b$. It is easy to check, exploiting the property of U_{eff} being a positive definite function in B_b, that a non degenerate metric on the tangent bundle TB_b is defined by means of (5.2). Therefore the proof of the theorem can be accomplished like in Theorem 3.1.

Later on Bolotin and Kozlov [22] considered the potential function and the kinetic energy depending also on time. Precisely, the Lagrangian function is assumed to be in $C^2(TM \times \mathbf{R}, \mathbf{R})$, TM being the tangent bundle of a compact Riemannian manifold M. Moreover:

$$\begin{aligned} c_1|\dot{q}|^2 - c_2 &\leq L(q,\dot{q},t) \\ c_3|v|^2 &\leq \left\langle v, \frac{\partial^2}{\partial\dot{q}\partial\dot{q}} L(q,\dot{q},t)v \right\rangle \leq c_4|v|^2, \end{aligned} \tag{5.3}$$

where $|\cdot|$ is the norm induced by the Riemannian metric. Then one can consider the corresponding Hamiltonian function $H \in C^2(T^*M \times \mathbf{R}, \mathbf{R})$, by means of :

$$p = \frac{\partial L}{\partial q}(q,\dot{q},t).$$

Let consider now $H(p,q,t)|_{p=0} \equiv H_0(q,t)$ (the reader can easily compare this function with the function U_{eff} when the Lagrangian looks like (1.1)). If this function is negative definite with respect to $q = 0$, i. e. there exists a function $K(q)$ having a strict maximum at $q = 0$, and

$$H_0(q,t) \leq K(q),$$

then there exist trajectories of the Lagrangian system which are asymptotic to $q = 0$. The existence of such asymptotic motions is proved by means of a variational approach to the problem, directly working on the Hamilton Functional:

$$J(\gamma) = \int_\tau^\infty L(\gamma(t), \dot{\gamma}(t), t)dt$$

Moreover the authors considered also the case of $H_0(q,t)$ to be a negative semidefinite function, and proved the instability of the origin. The results are recasted in a more precise form in the following theorem.

Theorem 5.3. *Let L satisfies* (5.3), *let $H_0(q,t)$ be a negative definite function, and assume that the solutions of the Lagrangian system exist in the future. Then from any $q_0 \in M$ starts a motion tending to $q = 0$ in the future. If $H_0(q,t)$ is just a negative semidefinite function, then any $q_0 \in M$ can be connected to 0 by means of a motion starting at any initial time τ_0, with arbitrary small initial velocity $\dot{q}_{\tau_0}$ in a finite time τ (depending on $q_0, \tau_0, |\dot{q}_{\tau_0}|$).*

Of course one can consider the particular case of time independent Lagrangian having a compact configuration manifold, and easily obtain Theorem 3.4 as a corollary of Theorem 5.3.

In the frame of analytic Lagrangian function, Furta [38] considered the special case of a potential function U:

$$U(q) = U_{[k]}(q) + W(q) \tag{5.4}$$

where the homogeneous form $U_{[k]}(q), k \geq 3$ has not a maximum at $q = 0$, and $W(q) = O(|q|^{(k+1)})$. The condition on the gyroscopic terms is:

$$b(q) = o(|q|^{\frac{(k+2)}{2}}).$$

In [28] the authors generalized this result to the case of a Lagrangian function of finite regularity.

Let us remark the common shortcoming of all the results quoted up to here: the *gyroscopic forces* are requested to be "weak", with respect to the conservatives one.

In ([39]) Furta considered the case of C^∞ Lagrangian function where the balance between the *gyroscopic forces* and the potential ones can be completely reversed. Precisely, he proved the following results.

Theorem 5.4. *Let U be a C^∞ function:*

$$U(q) = U_{[k]}(q) + W(q),$$

$k \geq 3, \quad W(q) = 0(|q|^{(k+1)})$. *Consider the Lagrangian system:*

$$\ddot{q} = A(0)q + \nabla U(q),$$

and assume $M_0 \equiv \mathrm{Ker}\, A(0) \neq \{0\}$. Moreover assume that $U_{[k]}(q)$ can take positive values on M_0. Then there exists a solution asymptotic to $q = 0$.

Theorem 5.5. *Consider the C^∞ Lagrangian system:*

$$\ddot{q} = \Gamma(q, \dot{q}) + A(q)\dot{q} + \nabla U(q)$$

where $\Gamma(q, \dot{q})$ is a quadratic function with respect to $\dot{q}$, $A(0)$ is an antsymmetric non degenerate matrix, $U(q) = U_{[k]}(q) + V(q)$, $U_{[k]}$ being a homogeneous polynomial of degree $k \geq 3$, $V(q) = 0(|q|^{(k+1)})$. Moreover, assume one of the two following conditions is satisfied:

(I) *0 is an isolated critical point for $U_{[k]}(q)$ and k is odd.*

(II) $A(0)\frac{\partial^2 U_{[k]}(q)}{\partial q \partial q} + \frac{\partial^2 U_{[k]}(q)}{\partial q \partial q} A(0) \equiv 0.$

Then there exists a solution asymptotic to $q = 0$.

The method employed in proving of these two theorems is based on the construction of *formal* asymptotic solutions of the corresponding differential systems. Then the proofs are accomplished by using the Kuznetsov's theorem. To give the idea of the procedure, let us consider the second theorem where $A(0)$ is a nondegenerate matrix. Let us introduce the following equation:

$$\frac{\partial U_{[k]}(q)}{\partial q} = \lambda A(0) q. \tag{5.5}$$

We are interested in finding non trivial solution $\overline{q}$, for some nonzero λ: indeed this solution is used to perform the first step of the iteration.

Each one of the Hypotheses (I) and (II) in theorem (5.5) is a sufficient condition for the existence of nontrivial solution. We start with the condition (I). By virtue of the assumption:

$$\nabla U_{[k]}(q) = 0 \iff q = 0,$$

we can consider the vector field $u(q) = \frac{\nabla U_{[k]}(q)}{|\nabla U_{[k]}(q)|}$ and the vector fields family:

$$u_s(q) = -sA(0)q + (1-s)u(q), \quad s \in [0,1].$$

Suppose by contradiction that $u_s(q) \neq 0, \quad q \neq 0$. Then $\deg(u(q)) = \deg(-A(0)q) = \det(A(0)) = 1$. On the other hand, if k is odd the vector field $u(q)$ is even, and consequently $\deg(u(q))$ is even. Consequently we have the following result:

Lemma. *If $q = 0$ is an isolated critical point of $\nabla U_{[k]}(q)$ and k is odd, then equation (5.5) admits a nontrivial solution.*

The condition (II) of Theorem 5.5 means that there exists a harmonic function $W_{[k]}(q)$ such that:

$$A(0)\nabla W_{[k]}(q) \equiv \nabla U_{[k]}(q).$$

Therefore the equation (5.5) again admits a solution $\overline{q}$ which is the point on the unit sphere where $W_{[k]}(q)$ takes the maximum (or the minimum).

Let us add some more remarks on the equation (5.5). A necessary condition for the existence of a nontrivial solution for some $\lambda \neq 0$ is:

$$O \equiv \{q \in \mathbf{R} : U_{[k]}(q) = 0\} \neq \{0\}.$$

This condition is a trivial consequence of the homogeneity property of $U_{[k]}$. Therefore, of course, $U_{[k]}$ cannot be a sign definite function.

If $n = 2$, we have also a sufficient condition. Indeed, if there exists a $\overline{q} \in O$, which is a noncritical point of $U_{[k]}$, then $\nabla U_{[k]}(\overline{q})$ and $A(0)\overline{q}$ are non zero vectors, both orthogonal to $\overline{q}$, an therefore collinear each other. If $n \geq 4$, the previous condition, i.e. that the set O does not coincide with the set of the critical points, is of course no more sufficient to ensure the solvability of (5.5). As a simple example, consider the Lagrangian function

$$L(q,\dot{q}) = \frac{1}{2}|\dot{q}|^2 + \frac{1}{2}(q_1\dot{q}_2 - q_2\dot{q}_1) + (q_3\dot{q}_4 - q_4\dot{q}_3) + q_1({q_3}^2 + {q_4}^2).$$

The set O does not coincide with the set of the critical points of $U_{[k]}$. However the system (5.5) has only the trivial solution. It may be interesting to remark that the corresponding Lagrangian system does not admit motions which are asymptotic to the equilibrium $q = 0$.

In [41] another criterion for the existence of non trivial solution of (5.5) was given. Precisely, the matrix $A(0)$ is considered such that:

$$A(0) = JC = CJ$$

where

$$J = J_2 \oplus J_2 \cdots \oplus J_2, \qquad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and C is a symmetric positive definite matrix. Moreover $U_{[k]}(q)$ is assumed to be a pluriharmonic function. Under such hypothesis there exists a function $W_{[k]}(q)$ such that:

$$J\nabla U_{[k]}(q) = \nabla W_{[k]}(q).$$

Therefore the critical points of $W_{[k]}$ on the ellipsoid:

$$\{q \in \mathbf{R}^n : \langle q, Cq \rangle = 1\}$$

are the solutions of the equation (5.5). $\diamondsuit$

The existence of asymptotic motions for C^h Lagrangian systems, $h < \infty$ was analyzed in [40], [41]. Let us consider the contents of these papers in some extent.

We begin by the first paper. Here two natural numbers k, h, $k \geq 3$, $h \geq 3(k-1)$, are considered and the following assumptions are made on the terms of the Lagrangian function (1.1).

- (H_i) $A(\cdot), S(\cdot) \in C^h(\mathbf{R}^n, L(\mathbf{R}^n, \mathbf{R}^n))$, $A(0)$ is an antisymmetric, nonsingular matrix.
- (H_{ii}) $U(q) = U_{[k]}(q) + W(q)$, $W(q) = o(|q|^k)$.
- (H_{iii}) For some $\lambda \in \mathbf{R} \setminus \{0\}$, there exists a nonzero vector $\overline{q}$, solving the equation (5.5)

If $\min\{n,k\} \geq 4$, we need a further hypothesis. Let us denote by $H(\overline{q})$ the Hessian matrix of $U_{[k]}(q)$ at $q = \overline{q}$. We consider the two spaces: $T_{\overline{q}} := \{v \in \mathbf{R}^n, \langle v, A(0)\overline{q} \rangle = 0\}$ and $T^*_{\overline{q}} := \{v \in T_{\overline{q}}, \langle v, \overline{q} \rangle = 0\}$. Then:
(H_{iv}) If $min\{n,k\} \geq 4$, the quadratic form

$$\langle v, H(\overline{q})v \rangle$$

is positive (or negative) definite with respect to all $v \in T^*_{\overline{q}}$.

The main result in [40] is the following Theorem.

Theorem 5.6. *Assume the previous hypotheses. Then the Lagrangian system admits a motion which is asymptotic to the fixed point $q = 0$, i.e. there exists a solution $q(t)$ tending to 0, either as $t \to +\infty$, or as $t \to -\infty$. In particular, the solution $q = 0$ is an unstable equilibrium.*

Let us make a remark on the hypothesis (H_{iv}). It is added in order to avoid to meet in the procedure of the proof some " dangerous resonance" in the spectrum of $A^{-1}(0)H(\overline{q})$. Therefore its role is purely technical: indeed, in the successive paper [41] this hypothesis was removed.

After this preliminary remark, we pass to the description of the procedure followed in proving Theorem 5.6.

The main step is to show that, by means of a change of variables:

$$(q, \dot{q}) \longleftrightarrow (x, y),$$

the system *(L)* becomes equivalent to the following differential system :

$$\begin{aligned} \dot{x} &= \Lambda(y)x + X(x, y), \\ \dot{y} &= B(x, y)x + A^{-1}(0)\frac{\partial U_{[k]}(y)}{\partial y} + Y(y). \end{aligned} \tag{5.6}$$

The right hand side of this system have the following properties:

(i) $\Lambda(y) := \Sigma_1(y)\Sigma_2(y)$, where $\Sigma_1(y)$, $\Sigma_2(y)$ are matrices whose entries are polynomials of degree $k-2$. Moreover: $\Sigma_1^T(y) = -\Sigma_1(y)$, $\Sigma_1(0) = -A^{-1}(0)$, ${\Sigma_2}^T(y) = \Sigma_2(y)$, $\Sigma_2(0) = -A(0)S^{-1}(0)A(0)$.

(ii) $X \in C^{h-1}(\mathbf{R}^{2n}, \mathbf{R}^n)$, $\quad X(x, y) = O(|x|^2 + |x||y|^{k-1} + |y|^{(3k-4)})$.

(iii) $B \in C^{h-1}(\mathbf{R}^{2n}, L(\mathbf{R}^n, \mathbf{R}^n)$, $\quad B(x, y) = O(|x| + |y|)$

(iv) $Y \in C^{h-1}(\mathbf{R}^{2n}, \mathbf{R}^n)$, $\quad Y(y) = O(|y|^k)$.

From now on, we concentrate our analysis on system (5.6), assuming for sake of definiteness that there exists a real negative number λ, such that (5.5) admits a non trivial solution $\overline{q}$.

Then, we are looking for asymptotic solutions of (5.6) which stay "close" to an asymptotic solution of the following "basic system":

$$\dot{y} = A^{-1}(0)\ \frac{\partial U_{[k]}(y)}{\partial y}(y).$$

One can readily verify that this system admits the solution:

$$y_0(t) = t^{\alpha}\{|\lambda|(k-2\}^{\alpha}\overline{q}, \quad \alpha = -\frac{1}{k-2}.$$

To construct the foresaid asymptotic solution of (5.6), we fix a positive number t_0 (in the following t_0 will be chosen sufficiently large) and we consider the linear space:

$$\mathbf{X} = \{f \in C^1([t_0, \infty), \mathbf{R}^n) : \sup[|\frac{d^j f(t)}{dt^j}|t^{-2(k-1)\alpha}, j = 0, 1, t \in [t_0, \infty)] < \infty\}.$$

We make $\mathbf{X}$ into a Banach space, by introducing the norm:

$$|f|_1 := \sup[|\frac{d^j f(t)}{dt^j}|t^{-2(k-1)\alpha}, j = 0, 1, \quad t \in [t_0, \infty)].$$

We fix $f \in \mathbf{X}$ and we consider the "y-reduced system":

$$\dot{y} = B(f(t), y)f(t) + A^{-1}(0)\frac{\partial U_{[k]}(y)}{\partial y} + Y(y). \tag{5.7}$$

System (5.7) admits at least one solution, denoted by $y(t|f)$, satisfying in particular:

$$y(t|f) = y_*(t^\alpha) + \tilde{y}(t|f),$$

where $y_*(t^\alpha)$ is a polynomial (in the t^α variable) of degree $k-2$, and does not depend on f, whereas $\tilde{y}$ satisfies the inequality:

$$|\tilde{y}(t|f)| \leq K(t_0)t^{\alpha(k-b)},$$

where $b \in (0, 2)$, and $K(t_0) \to 0^+$ as $t_0 \to +\infty$. Using these properties, one can proceed along the same path followed in the proof of Theorem 4.5 in order to define a suitable operator T acting on the space $\mathbf{X}$. Of course any fixed point $\tilde{f}$ of T gives rise to the solution $(\tilde{f}(\cdot), y(\cdot|\tilde{f}))$ of the system *(L)*. Therefore, the proof is accomplished by proving that T is a contraction. We skip further details referring the interested reader to the cited paper. ◇

As already announced, in paper [41] the result of Theorem 5.6 was improved in two aspects:

(i) Hypothesis (iv) is removed.

(ii) The regularity assumption on L is weakened. To be more precise, let us introduce the normalized solution a of the equation (5.5), i. e. let us fix λ:

$$|\lambda| = (k-2)^{(-1)}.$$

We consider the matrix $\Sigma = -A^{-1}(0)\nabla^2 U_{[k]}(a)$ and we call Λ the largest real part of the eigenvalues of Σ; then we take $h \geq h(\Lambda, k) \equiv (k-2)\Lambda + k - 1$. We remark that if Hypothesis (iv) is assumed, then $h(\Lambda, k) = 2k$ and the result of Theorem 5.6 still holds for C^{2k} Lagrangian functions.

Moreover in [41] an effort is made toward the problem of the analysis of systems for which the equation (5.5) cannot be solved. To give an idea of such analysis, let us start again by considering a C^∞ Lagrangian function and suppose that U has a proper minimum at $q = 0$. Let us consider the configuration space to be a compact Riemannian manifold M (the Riemannian metric defining the quadratic kinetic energy); we know that every point of M can be joined by a motion of zero energy to the origin. Now we switch on a small non-degenerate *gyroscopic force*. Is the instability property preserved? We are not able to answer such a question, a part the following particular case:

Theorem 5.7. *Let $n = 2$ be the dimension of the configuration space; moreover let us assume that the potential function U satisfies:*

$$U(q) = U_{[k]}(q) + o(|q|^k),$$

$U_{[k]}$ being a homogeneous form of degree $k, k \geq 3$, having a proper minimum at $q = 0$. Then this stationary solution is stable.

In other words, no matter how small is the intensity of a non-degenerate gyroscopic force, a stabilization phenomenon occurs. The proof of this theorem is obtained by using the K. A. M. theory.

For higher dimension we have a weaker result: the stabilization occurs at least formally, i.e. the instability cannot be discovered by any finite algebraic procedure. More precisely, we have the following theorem.

Theorem 5.8. *Consider the C^∞ Lagrangian function (1.1), with non-degenerate gyroscopic forces. Then the Lagrangian system has a formal n-dimensional invariant manifold $N \subset TM$ passing through the equilibrium position $q = 0$. In local coordinates $(q, \dot{q})$, this formal invariant manifold is represented by means of formal power series:*

$$q = f(x) = x + f_{[2k-3]}(x) + \cdots, \dot{q} = g(x) = g_{[k-1]}(x) + \cdots, x \in \mathbf{R}^n$$

and:

(i) *the restriction of the Lagrangian system to N is:*

$$A(0)\dot{x} + \nabla V(x) = 0,$$

where the $(2k-3)$-jet of V coincides with the $(2k-3)$-jet of U.

(ii) *there exists a non-negative formal integral $F(x) = F_{[2]}(x) + \cdots$ of the Lagrangian system, such that $N \equiv \{x \in \mathbf{R}^n : F(x) = 0\}$.*

(iii) *If the equilibrium $q = 0$ is a strict minimum of U, and moreover this minimum is $(2k-3)$-decidable, i.e.*

$$U(q) \geq |q|^\nu, \quad c > 0, \quad k \leq \nu \leq 2(k-1),$$

then the equilibrium is formally stable.

Now we ask what happens to the family of asymptotic solutions existing in the case of *natural* Lagrangian with potential having a proper absolute minimum on M at O ($O \equiv \{q = 0\}$) when we switch on a small non degenerate *gyroscopic force* $F = \varepsilon A(q)\dot{q}$? It turns out that close to 0 there is an unstable invariant set having a set of semyasymptotic solutions. More precisely, let us assume $S, A \in C^2, U \in C^k$. We have:

Theorem 5.9. *Let $O \in M$ be a point of strict local minimum of the potential U, and suppose that the nondegenerate gyroscopic force is of type $F = \varepsilon A(q)\dot{q}$, where ε is sufficiently small. Then there exists a compact invariant set $\Sigma \subset TM$ such that:*

(i) The projection of Σ to M is injective,

(ii) Σ tends to $(0,0)$ as $\varepsilon \to 0$. More precisely, if the first non zero form $U_{[k]}$ is non degenerate, then:

$$|q| = O(\varepsilon^{\frac{2}{(k-2)}}, \quad |\dot{q}| = O(\varepsilon^{\frac{k}{(k-2)}}), \quad \forall (q,\dot{q}) \in \Sigma;$$

(iii) Σ is an unstable invariant set and every solution in Σ is orbitally unstable. Moreover, there is a ball $B \subset M$, independent of ε, such that for any closed invariant set $C \subset \Sigma$, any $\delta > 0$, and any point $q_0 \in B$, there exists a solution $q(t), t \geq 0$, such that $q(0) = q_0$ and $dist\big((q(t), \dot{q}(t)), C\big) < \delta$ for some $t \geq 0$.

(iv) for any point $q_0 \in B$ there exists a solution $q(t), t \geq 0$ starting from q_0 and whose ω-limit set intersect with Σ. Moreover, every recurrent in Birkhoff' s sense trajectory in the ω-limit set is contained in Σ.

We recall that G. Birkhoff [42] called a solution $q(t)$ as in (iv) semiasymptyotic to the set Σ as $t \to \infty$.

The proof of the Theorem is based on global variational methods; the set Σ is defined, according to Mather [43], as the union of supports of invariant Borel probability measures μ with compact support in TM that minimize the mean action functional:

$$A(\mu) = \int_{TM} L \, d\mu$$

Instability of the set Σ and the existence of solutions satisfying (iv) follow from the results in [44], [45]. Then, to complete the proof of Theorem 5.9 we just need to show that $\mathrm{diam}(\Sigma) \to O$ as $\varepsilon \to 0$; the interested reader is referred to the quoted paper ([41]). $\diamondsuit$

To explain the content of Theorem 5.9 let us consider the following example. Let $n = 2$ and L given by:

$$L(q, \dot{q}, \varepsilon) = \frac{1}{2}|\dot{q}|^2 - \frac{\varepsilon}{2}\langle Jq, \dot{q}\rangle + U_{[k]}(|q|).$$

The corresponding Hamiltonian function is:

$$H(q, p, \varepsilon) = \frac{1}{2}|p + \frac{\varepsilon}{2}Jq|^2 - U_{[k]}(|q|),$$

and $H_0(q,\varepsilon) \equiv H(q, 0, \varepsilon) = \frac{\varepsilon^4}{8}|q|^2 - U_{[k]}(|q|)$.

We remark that: $H_0(q,\varepsilon) = \min\{L(q, \dot{q}, \varepsilon),\ \dot{q} \in R^2\} = L(q, \frac{\varepsilon}{2}Jq, \varepsilon)$.

Let us consider the minimum of $H_0(q,\varepsilon)$; we have

$$\nabla U_{[k]}(q) = \frac{\varepsilon^2}{4} q \iff U_{[k]}(q) = \frac{\varepsilon^2}{4k}|q|^2.$$

Now it turns out from the proof of Theorem 5.9 that the set Σ is exactly the set of minimum points of L, i.e. the closed trajectory γ given by:

$$\dot{q} = \frac{\varepsilon}{2} J q$$

$$U_{[k]}(|q|) = \frac{\varepsilon^2}{4k}|q|^2.$$

In the general case, the idea of the construction of Σ is similar. The difficulty is that the set of minimum points of the Lagrangian is not longer invariant.

Let us now conclude this review of the results on the Inversion of the Lagrange–Dirichlet Theorem with the result, [46], which is a generalization of the Theorem 5.6. We consider the Lagrangian (1.1) where again the potential function $U(q)$ has a k-jet starting with a semidefinite negative quadratic form and we assume that the null space of $U_{[2]}(q)$ has even dimension n_1. By using a suitable coordinates system, we write the Lagrangian function directly as follows:

$$L = \frac{1}{2} < (\dot{x},\dot{y}) S(x,y)(\dot{x},\dot{y}) > + < b(x,y), \dot{x} > + \frac{1}{2} < \dot{y}, A^c(x,y)x > +$$
$$\frac{1}{2} < \dot{y}, A(y)y > + \frac{1}{2} < x, l(x,y)x > + V_{[k]}(y) + W(y),$$

where $x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n$, and:

$b \in C^h(\mathbf{R}^n, \mathbf{R}^{n_1})$,

$A^c \in C^h \mathbf{R}^n, L(\mathbf{R}^{n_1}, \mathbf{R}^{n_2}))$

$A \in C^h(\mathbf{R}^n, L(\mathbf{R}^{n_2}, \mathbf{R}^{n_2}))$,

$l \in C^h(\mathbf{R}^n, L(\mathbf{R}^{n_1}, \mathbf{R}^{n_1}))$, $\quad l_{\alpha,\beta}(0,0) = -\omega_\alpha^2 \delta_{\alpha,\beta},\ \omega_\alpha \neq 0, \alpha = 1,..,n_1$

$W(y) = o(|(x,y)|^k)$

Moreover we assume:

(H_i) $A(0)$ is an antisymmetric, nonsingular matrix.

(H_{ii}) For some $\lambda \in R\setminus\{0\}$, there exists a nonzero vector $\tilde{y}$, solving the equation:

$$\frac{\partial V_{[k]}(y)}{\partial y} = \lambda A(0) y.$$

(H_{iii}) Let us denote by $H(\tilde{y})$ the Hessian matrix of $V_{[k]}(y)$ at $y = \tilde{y}$. We consider the two spaces: $T_{\tilde{y}} := \{v \in \mathbf{R}^{n_2},\ \langle v, A(0)\tilde{y}\rangle = 0\}$ and $T_{\tilde{y}}^* := \{v \in T_{\tilde{y}}, \langle v, \tilde{y}\rangle = 0\}$. Then, if min $\{n,k\} \geq 4$,, the quadratic form

$$\langle v, H(\tilde{y})v\rangle$$

is positive (or negative) definite with respect to all v's, $v \in T_{\tilde{y}}^*$.

Then, we have the theorem:

Theorem. *Under the previous assumptions the Lagrangian system admits a solution which is asymptotic to the solution $q = 0$; in particular, this equilibrium is unstable.*

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