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A Measure Theoretic Erdős-Rado Theorem

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Infinite combinatorics has become an essential tool to handle a significant number of problems in Analysis, Topology and Set Theory. The central result in these applications is a theorem of P. Erdős and R. Rado [ER1, ER2], although versions due to N. A. Shanin on Δ-systems [Sh] and S. Mazur [Ma] should also be mentioned. The reader can find a wealth of information on the use of infinite combinatorics in Topology and Analysis in [C]. For Set Theory, we suggest the exposition found in [K].

If A is a set, $|A|$ denotes the cardinal of A. The real unit interval is denoted by $[0,1]$; we write λ for Lebesgue measure in $[0,1]$ and λ^* for the outer measure associated to λ.

A basic concept is described by

Definition 1 *A family $\{S_i : i \in I\}$ of sets is* **a Δ-system** *or a* **quasi-disjoint family** *if there is a fixed set J such that $S_i \cap S_{i'} = J$ for all distinct i, $i' \in I$. In particular, a family of pairwise disjoint sets is quasi-disjoint.*

The flavor of the Erdős-Rado theorem can be sampled through the following result, one of its most often used versions (Theorem II.1.5, p. 49, [K]).

Theorem A *If $\mathcal{A}$ is any uncountable family of finite sets, there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ which forms a Δ-system.* □

If we assume the Continuum Hypothesis (CH), that is, that the cardinal of the reals, 2^ω, is the first uncountable cardinal ω_1, we may restate Theorem A as

Theorem B (CH) *Let $\{S_\xi : \xi \in A\}$ be a family of finite sets with $|A| = 2^\omega$. Then there is $B \subseteq A$ such that $|B| = 2^\omega$ and $\{S_\xi : \xi \in B\}$ is a quasi-disjoint family.* □

These results are a consequence of the following theorem, which appears as Theorem 1.4 (p. 5) in [C] and Theorem II.1.6 (p. 49) in [K]. Recall that if κ and α are cardinals, α is called **strongly κ-inaccessible**, written $\kappa \ll \alpha$, if $\kappa < \alpha$ and $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \kappa$.

Theorem C (Erdős-Rado) *Let κ, α be infinite cardinals such that $\kappa \ll \alpha$, with α regular. Let $\{S_\beta : \beta \in \alpha\}$ be a family of sets such that $|S_\beta| < \kappa$ for all $\beta \in \alpha$. Then there exist $A \subseteq \alpha$ with $|A| = \alpha$ and a set J such that*

$$S_\beta \cap S_{\beta'} = J$$

for all distinct β, $\beta' \in A$. □

In [C], the reader will find extensions of Theorem C to singular cardinals (Theorem 1.9, p. 13). One of the independent proofs, due to S. Argyros, was motivated by problems in the isomorphic embedding theory of Banach spaces (see [A] and the references therein).

When dealing with measure theoretic questions in infinite dimensional Banach spaces, it became apparent to us that there was a need, in certain situations, of a refinement of these results, in the sense that it was important to control the measure, and not just the cardinality, of the subset that yielded a quasi-disjoint family. We were thus led to establish the following measure theoretic analogue of Theorem B, where λ^* stands for the Lebesgue outer measure.

Theorem 2 (CH) *Let A be a subset of $[0,1]$ with $\lambda^* A > 0$ and $\{S_t\colon t \in A\}$ a family of finite sets. Then there is $B \subseteq A$ with $\lambda^* B > 0$ such that $\{S_t\colon t \in B\}$ is a quasi-disjoint family of sets.*

Before the proof of Theorem 2, we need some preparatory steps.

Remark 3 If $C \subseteq [0,1]$ is such that $\lambda^* C = \delta > 0$, then

$$\begin{aligned}\lambda^* &= \inf\Big\{\sum \lambda I_n\colon C \subseteq \bigcup I_n \text{ and every } I_n \text{ is an interval}\Big\}\\ &= \inf\Big\{\sum \lambda I_n\colon C \subseteq \bigcup I_n \text{ and every } I_n \text{ is an interval}\\ &\qquad\qquad\qquad\qquad \text{with rational endpoints}\Big\}.\end{aligned}$$

The set $\{J\colon J \subseteq [0,1] \text{ and } J \text{ is an interval with rational endpoints}\}$ is countable. So, if we assume CH, the collection

$$\begin{aligned}\mathcal{I}_\delta &= \Big\{(I_n)_{n\geq 1}\colon \sum \lambda I_n < \delta \text{ and every } I_n \text{ is an interval}\\ &\qquad\qquad\qquad\qquad \text{with rational endpoints}\Big\}\end{aligned}$$

has cardinal $\omega_1 = 2^\omega$; we **fix** a bijection $h\colon [1,\omega_1) \to \mathcal{I}_\delta$ and write $h(\alpha) = (I_n^\alpha)_{n\geq 1}$.

Proposition 4 (CH) *Let A be a subset of $[0,1]$ with $\lambda^* A > 0$ and $\{S_t\colon t \in A\}$ be a family of singletons. Then there is $B \subseteq A$ such that $\lambda^* B > 0$ and $\{S_t\colon t \in B\}$ is a quasi-disjoint family of sets. Thus, one of the following two possibilities occurs:*

(*i*) *There is an element a such that $S_t = \{a\}$, for all $t \in B$; or else*
(*ii*) *The sets S_t ($t \in B$) are pairwise disjoint.*

Proof. The proof involves a diagonal argument and transfinite induction. Let $\delta = \lambda^* A$. Suppose $S_t = \{a_t\}$ ($t \in A$).

Let $\mathcal{A} = \{a_t\colon t \in A\}$; for $x \in \mathcal{A}$, set $A_x = \{t \in A\colon S_t = \{x\}\}$. If, for some $x \in \mathcal{A}$, we have $\lambda^* A_x > 0$, then it is sufficient to take $B = A_x$ and $S_t = \{x\}$ for all $t \in B$. Thus, we may assume that for each $x \in \mathcal{A}$ we have $\lambda^* A_x = 0$.

Let $(I_n^\alpha)_{n\geq 1}$, $\alpha < \omega_1$, be the enumeration of $\mathcal{I}_\delta$ constructed in Remark 3. We claim that, by transfinite induction on ω_1, we may choose a sequence $\{t_\alpha : \alpha < \aleph_1\}$ such that if ν, β are distinct countable ordinals then $t_\nu \neq t_\beta$ and $S_{t_\nu} \cap S_{t_\beta} = \emptyset$.

For the first step in the induction, since $\lambda^* A = \delta > 0$ and $\sum \lambda I_n^1 < \delta$, there is $t_1 \in A$ such that $t_1 \notin \bigcup I_n^1$. Note that $t_1 \in A_{x_1}$, where $S_{t_1} = \{x_1\}$.

Having constructed t_η $(\eta < \alpha)$, and recalling that $\bigcup_{\eta<\alpha} A_{x_\eta}$ has measure zero (it is a countable union of sets of measure zero), we may choose

$$t_\alpha \in A - \Big(\bigcup_{\eta<\alpha} A_{x_\eta} \cup \bigcup I_n^\alpha\Big).$$

It is straightforward that the sequence $B = \{t_\alpha : \alpha < \omega_1\}$ has the claimed properties. It remains to verify that $\lambda^* B > 0$.

If $\lambda^* B = 0$ then there is a covering (I_n) of B by intervals with rational endpoints such that $\sum \lambda I_n < \delta/2$. By Remark 3, we can find $\alpha < \omega_1$ such that $(I_n) = (I_n^\alpha)$. Thus, $B \subseteq \bigcup I_n^\alpha$. But this is impossible since $t_\alpha \in B$ was chosen outside $\bigcup I_n^\alpha$.

□

Remark 5 The proof of Proposition 3 actually shows that if there is no subset of C of A of positive outer measure such that $S_t = S_{t'}$ for all t, t' in C, then there is $B \subseteq A$ such that $\lambda^* B = \lambda^* A$ and $S_t \cap S_{t'} = 0$ for all distinct t, t' in B.

Proof of Theorem 2

We begin by generalizing Proposition 4.

Fact *Let A be a subset of $[0,1]$ with $\lambda^* A > 0$ and $\{S_t : t \in B\}$ a family of finite sets, each of them with $k \geq 0$ elements. Then there is $B \subseteq A$ such that $\lambda^* B > 0$ and $\{S_t : t \in B\}$ is a quasi-disjoint family.*

Proof. We proceed by induction on $k \geq 0$. The case $k = 0$ is trivial and the case in which $k = 1$ is taken care of by Proposition 4. Therefore suppose the statement holds for each $j \leq k$; we are going to verify it for $k+1$. Let $\lambda^* A = \delta > 0$ and define $\mathcal{A} = \bigcup_{t\in A} S_t$.

If $u \subseteq \mathcal{A}$, set $A_u = \{t \in A : u \subseteq S_t\}$. Note that if we write $S_t = \{t_i : 1 \leq i \leq k+1\}$ then $t \in \bigcup_{i=1}^{k+1} A_{\{t_i\}}$. We shall use the enumeration of $\mathcal{I}_\delta$ described in Remark 3.

Case 1 There is $u \subseteq \mathcal{A}$ with $1 \leq |u| \leq k+1$ such that $\lambda^* A_u > 0$.

Then $\{S_t - u : t \in A_u\}$ is a family of sets with $k + 1 - |u| \leq k$ elements such that $\lambda^* A_u > 0$. By induction, there is $B \subseteq A_u$ with $\lambda^* B > 0$, such that $\{S_t - u : t \in B\}$ is quasi-disjoint. It is clear that $\{S_t : t \in B\}$ has the same property.

Case 2 For all $u \subseteq \mathcal{A}$ with $|u| \leq k+1$, we have $\lambda^* A_u = 0$.

Here the method is as in the proof Proposition 4. By transfinite induction on $\alpha \in \omega_1$, it is possible to construct a sequence $B = \{t_\alpha: \alpha < \omega_1\}$ of elements of A such that

$$t_\alpha \in A - \left(\bigcup I_n^\alpha \cup \bigcup_{\beta<\alpha} (A_1^\beta \cup \cdots \cup A_{k+1}^\beta) \right),$$

where $S_{t_\beta} = \{x_i^\beta: 1 \leq i \leq k+1\}$ and $A_j^\beta = \{t \in A: x_j^\beta \in S_{t_\lambda}\} = A_{\{x_j^\beta\}}$.

The inductive step comes, just as in the proof Proposition 4, from the fact that $\lambda^* A_j^\beta = 0$ for all $\beta < \alpha$ and $j \leq k+1$ and the hypothesis assumed in Case 2. Moreover, the constructed sequence satisfies

$$\beta < \eta \Longrightarrow t_\eta \notin A_1^\beta \cup \cdots \cup A_{k+1}^\beta \Longrightarrow S_{t_\beta} \cap S_{t_\eta} = \emptyset,$$

and $\{S_t: t \in B\}$ is a disjoint family of sets. The diagonal argument used in the proof of Proposition 4 shows that $\lambda^* B > 0$, verifying the Fact.

To finish the proof of Theorem 2, write $A = \bigcup_{k \geq 1} A_k$, where $A_k = \{t \in A: S_t$ has cardinal $k\}$. Since A has strictly positive outer measure, the same must be true of at least one A_k. The desired conclusion follows from an application of the Fact to A_k. □

An analogous argument will establish

Theorem 6 (CH) *Let (M, Σ, μ) be a separable probability space and let μ^* be the outer measure associated to μ. Let $\{S_\xi: \xi \in A\}$ be a family of finite sets such that $\mu^*(A) > 0$. Then there is $B \subseteq A$ such that $\mu^*(B) > 0$ and $\{S_\xi: \xi \in B\}$ is a quasi-disjoint family.* □

The statement of Theorem 6 applies, for instance, to the topological group 2^ω with its canonical Haar measure and, in fact, to any finite regular Borel measure in a separable complete metric space. It can be easily extended to the case of σ-finite regular Borel measures in Polish spaces. However, as it stands, it will not apply to the topological group 2^α with Haar measure for a cardinal α greater than that of the continuum.

It would be interesting to find other classes of measure spaces for which this generalization of the Erdős-Rado theorem holds true. It is clear that, under CH, Theorems A and B follow from Theorem 2.

Remark 7 The role of the Continuum Hypothesis in our discussion is to control the measure of the union of sets of measure zero. Our proof will also go through with other axioms of Set Theory, such as Martin's Axiom, which can guarantee that the union of less than continuum many sets of measure zero has measure zero.

Theorem 2 (or Theorem 6) has many interesting applications in integration theory in Banach spaces, some of which are presented in [AM1] and [AM2]. To cite an example, we set down

Definition 8 *A Banach space X is said to be* **Pm** *if every bounded Pettis integrable map $f\colon [0,1] \to X$ is measurable.*

Information concerning Pettis integrable functions can be found in [P], [DU] and [T]. It can be shown that all separable Banach spaces are Pm and the question arises if there are non-separable spaces with this property. Theorem 2 is instrumental in showing the existence of large families of non-separable Pm spaces, due to the following

Theorem 9 (Thm 4.2 [AM1]) (CH) *The operation of ℓ_1 sum preserves property Pm. In fact, if $\{X_i\colon i \in I\}$ is any family of Banach spaces, the following conditions are equivalent:*

1. *The ℓ_1 sum of $\{X_i\colon i \in I\}$ is Pm.*
2. *For all $i \in I$, the Banach space X_i is Pm.* □

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