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An Algorithm to Classify 3-Manifolds?

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Abstract. Combinatorial topology makes unlimited use of refinements. These refinements translate into an unlimited amount of data to describe objects like 3-manifolds. As a result, procedures to treat the homeomorphism problem, by combinatorial means, become unfeasible.

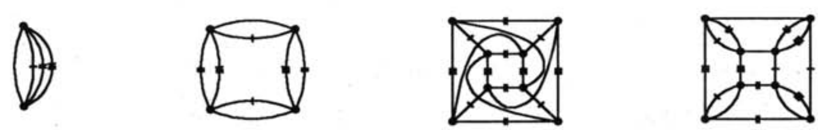
In this article we describe a finite algorithm which attains a combinatorial classification of 3-gems (tridimensional manifolds encoded by graphs), see Chapter 13 of [KL94]. Up to the level studied (3-gems of 30 vertices), the combinatorial classification coincides with the topological one. A general question arises as at what level the combinatorial classes no longer coincides with the classes of homeomorphisms. The hope that this coincidence can always hold enable us to enunciate a conjecture that would computationally classify closed 3-manifolds.

The central point in our approach is that the type of refinements in 3-gems that we permit (the U-move) are only applied in conjunction with some horizontal moves (the TS-moves that do not increase the size of the objects) and by others (ρ-moves) which decrease its size in such a way that the final 3-gem has no more vertices than the original one.

1 Introduction: $(3+1)$-Graphs

An $(n+1)$-graph is an edge-colored finite graph, regular of degree $n+1$, such that the edges incident to each vertex receive distinct colors and where the total number of colors painting the edges is also $n+1$. The colors are named $0, 1, \ldots, n$ and are depicted by the corresponding number of marks in the figures.

The $(n+1)$-graphs are our basic data structure. However simple, they permit the construction and convenient manipulation of PL n-manifolds. We restrict our focus to dimension $n = 3$.



(i) (ii) (iii) (iv)

Fig. 1: Examples of $(3+1)$-graphs

A subclass of the $(3+1)$-graphs encoding closed compact 3-manifolds named *3-gems* (defined in the next section) is our central object of study. All the above examples of $(3+1)$-graphs are 3-gems and they induce, respectively, S^3, S^3, RP^3 and $S^1 \times S^2$.

2 Information about 3-Gems

For us each class of homeomorphism of 3-manifold is an equivalence class of edge-colored graphs named *3-gems*. This is possible due to a Theorem of Ferri and Gagliardi (see below).

An *n-residue* ($0 \leq n \leq 3$) in a $(3+1)$-graph G is a connected component of a subgraph of G induced by all the edges of n chosen colors. Thus the 2-residues are *bi*colored poly*gons* in G, also called *bigons*. A 3-residue which does not use color k is also called a $\hat{k}$-residue. A *3-gem* is a $(3+1)$-graph G in which $v+t=b$, where v is the number of vertices, b is the number of 2-residues and t is the number of 3-residues, all relative to G. It follows easily from the Triangulation Theorem for 3-manifolds of Moise [Moi52], that every closed compact 3-manifold can be induced by a 3-gem.

Construction: From the colors of the edges of a 3-gem we can recover the higher dimensional cells of a ball complex, as follows: attach a 2-disk to each bigon and attach a 3-ball to each 3-residue. This is possible because the disks attached to any 3-residue form a 2-sphere. (This is a consequence of the equality $v+t=b$.) The topological space so generated is a 3-manifold and every 3-manifold appears in this way [LM85].

Here is an example of this construction for the 3-gem of Fig. 1(iii), yielding the real projective space RP^3. Only three of the twelve facial identifications are indicated.

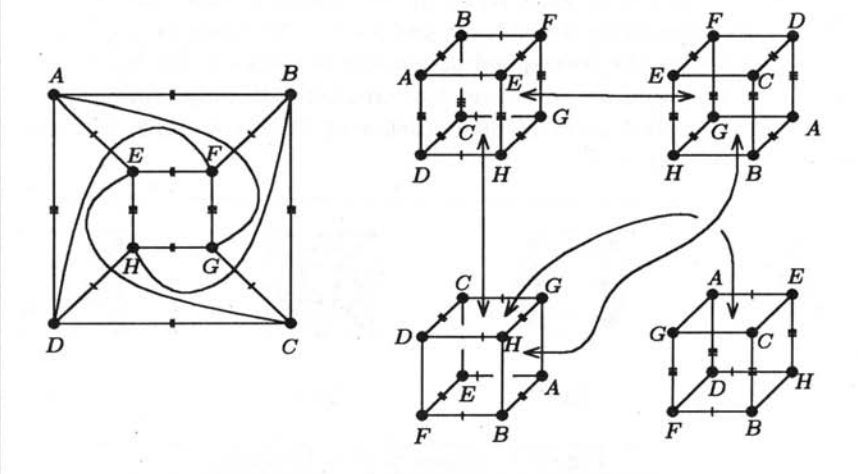


Fig. 2: Construction of RP^3 from a 3-gem

The equivalence class (which is the combinatorial counterpart of homeomorphism among 3-manifolds) is generated by two simple moves due to Ferri and Gagliardi [FG82]:

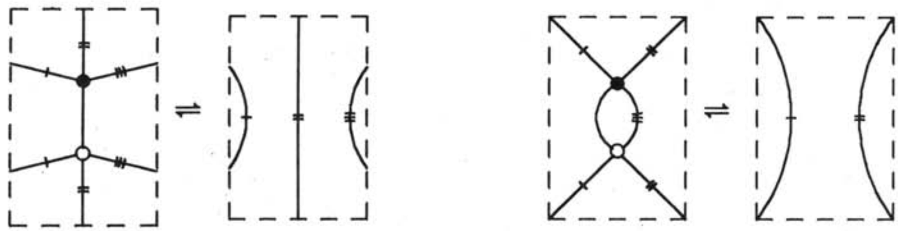


Fig. 3: 1-Dipole and 2-Dipole Moves

We suppose, in the first picture, that the two vertices are in distinct $\hat{0}$-residues and in the third that they are in distinct 12-gons. Under these hypotheses, the configurations are called *1-dipole involving color 0* and a *2-dipole involving colors 0 and 3*, respectively. There are, of course, four types of 1-dipoles and six types of 2-dipoles. The *involved colors* are the colors of the i edges linking the two vertices which form an i-dipole.

A *crystallization* is a 3-gem without 1-dipoles.

Theorem 1 (Ferri and Gagliardi, [FG82]) *Two crystallizations G and H induce the same 3-manifold if and only if H is obtained from G by a finite number of moves where each move is one of the following three:*

- *a 2-dipole cancellation;*
- *a 2-dipole creation;*
- *a 1-dipole creation followed by a 1-dipole cancellation.*

After the Ferri-Gagliardi Theorem, the problem of deciding whether two given 3-manifolds are homeomorphic becomes the one of deciding whether two 3-gems inducing them are linked by a finite number of cancellations and creations of 1- and 2-dipoles. Of course this is, still, an exceedingly difficult problem, which is not satisfactorily solved even if one of the manifolds is the 3-sphere. The reason for the difficulty is that there are no bounds for the number of consecutive 2-dipole creations (which increase the number of vertices of the 3-gems).

Our central point is to provide evidence that these moves can be replaced by others (more complicated) but which do not increase the number of vertices. The evidence is given up to 3-gems of 30 vertices.

All the orientable 3-manifolds induced by 3-gems up to 30 vertices have been generated and classified. (This is joint work with C. Durand and S. Sidki). Up to 28 vertices the proof of this fact appears in [Lin95]). Except for computing time and memory requirement, the methods developed could go on.

A general question arises: at which level do the techniques employed break down? If they never break, this would imply a general algorithm to classify closed compact 3-manifolds.

The classification was achieved by: (a) the computational possibility of generating all the relevant 3-gems (the rigid ones). See Section 5.1 of [Lin95] and (b) the computational possibility of identifying the *attractor* for each of the 3-manifolds involved. See Section 5.4 of [Lin95].

We define *the attractor of* M^3 as the set of *all* 3-gems inducing M^3 and having the minimum number of vertices. For many interesting 3-manifolds the attractor is formed by a single 3-gem. In this case the 3-gem is called the *superattractor for the 3-manifold.*

Clearly, the attractor of any 3-manifold exists and is unique, being formed by a finite number of 3-gems. Given a 3-gem G inducing M^3 we submit it to a *combinatorial simplification dynamics.* This combinatorial simplification dynamics has been enough, in practice, to find all the gems in the attractors of all the manifolds (induced by 3-gems up to 30 vertices).

3 Simplifying Dynamics

We give a brief presentation of all the operations that we employ to achieve the simplifying dynamics. These operations are discussed with details in Section 4.1 of [Lin95]

3.1 1-Dipole Cancellation, ρ-Pair Switching and ρ-move

Beyond looking for 1-dipoles in order to cancel them, our simplifying dynamics look for *ρ-pairs.* A *ρ-pair* in a $(3+1)$-graph is a pair of equaly colored edges that are together in 2 or 3 bigons. The *switching of a ρ-pair* is the passage from a gem G to a $(3+1)$-graph G' obtained by replacing $\{a_1, a_2\}$ by new edges $\{a'_1, a'_2\}$ having the same ends and preserving the $(\circ, \bullet)$-bipartition (which is defined even if G is not bipartite — see Subsection 2.3.2 of [Lin95]):

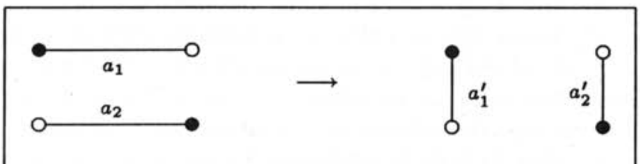


Fig. 4: Switching a ρ-pair

A ρ-pair which is in i bigons, $(i = 2, 3)$, is called a ρ_i-pair. The result of switching a ρ-pair $\{a, b\}$ in a gem G is another 3-gem, denoted $G^{swt}_{a,b}$. The switching of a ρ-pair causes the appereance of 1-dipoles and so, smaller 3-gems inducing the same manifolds (up to connected sums with $S^1 \times S^2$, in the case of ρ_3-pairs — see Proposition 20 of [Lin95]). Thus we may suppose 3-gems with dipoles or with ρ-pairs as irrelevant and concentrate in 3-gems without them. These are named *rigid 3-gems.* A *ρ-move* is either the the cancellation of a 1-dipole or else the switching of a ρ-pair (which creates 1-dipoles) followed by the cancellation of a 1-dipole.

3.2 TS-Configurations and TS-Moves

If a 3-gem is rigid our simplifying dynamics starts looking for the availability of TS-*moves*. These moves are based on the configurations below:

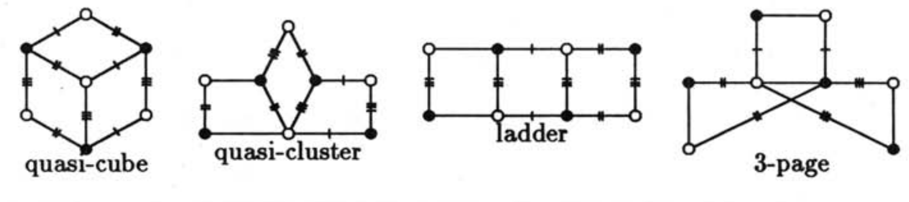


Fig. 5: The TS-configurations

The first three TS-moves are available when it occurs the first configuration of 3 squares named a *quasi-cube*.

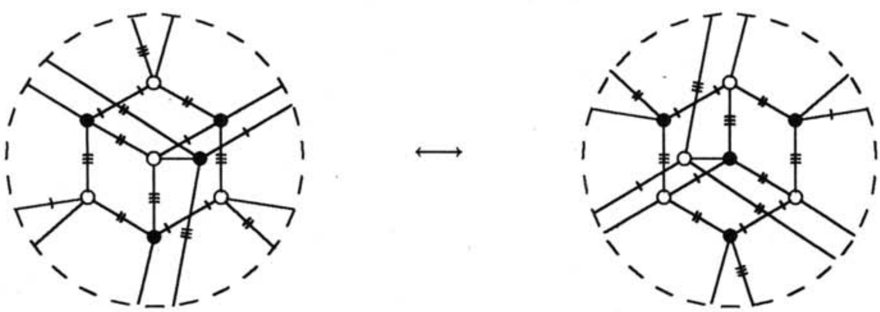


Fig. 6: Definition of TS_1-move up to edge-color permutation

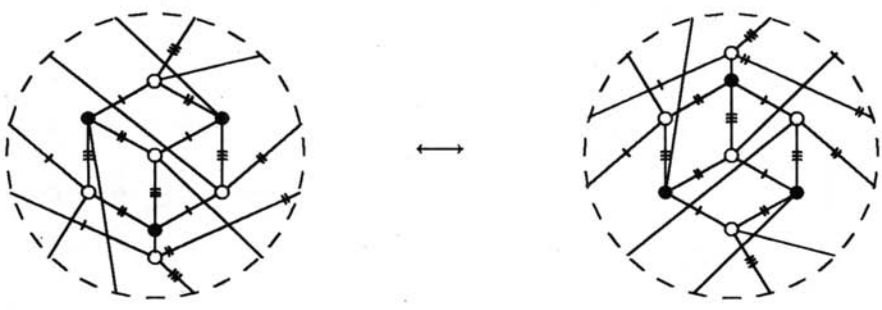


Fig. 7: Definition of TS_2-move up to edge-color permutation

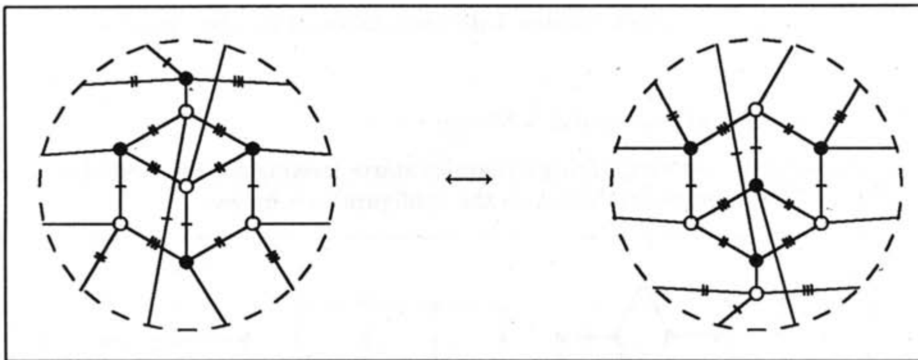


Fig. 8: Definition of TS_3-move up to edge-color permutation

The fourth TS-move is available whenever three squares forming a *quasi-cluster* is found:

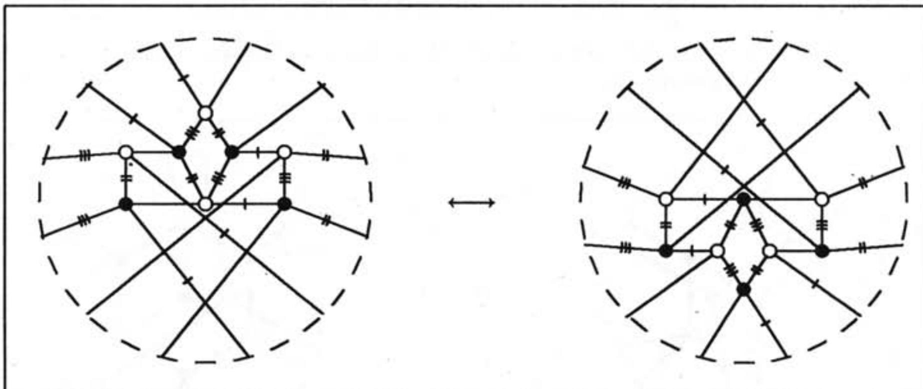


Fig. 9: Definition of TS_4-move up to edge-color permutation

The fifth TS-move is available whenever three squares forming a *ladder* is found:

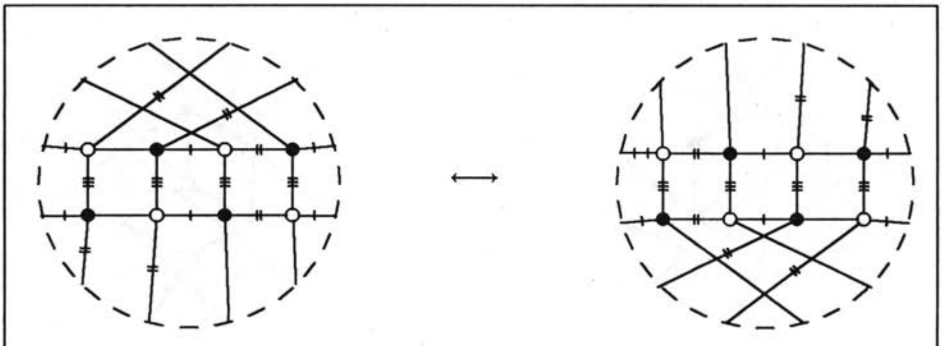


Fig. 10: Definition of TS_5-move up to edge-color permutation

The sixth TS-move is available whenever three squares forming a *3-page* is found:

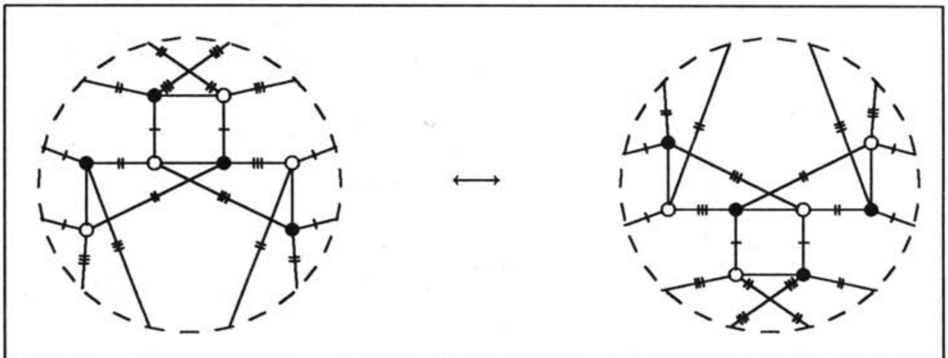


Fig. 11: Definition of TS_6-move up to edge-color permutation

Starting with a (rigid) 3-gem G we form a finite graph denoted Γ_G^{TS}. The vertices of this graph are in 1-1 correspondence with the 3-gems that are obtained from G by a finite number of TS-moves. An edge in this graph corresponds to a single TS-move. All the vertices of Γ_G^{TS} are 3-gems with the same number of vertices and inducing the same 3-manifold: the one induced by G. If there is a vertex H in Γ_G^{TS} which is not a rigid 3-gem, a smaller 3-gem inducing the same manifold is easily produced from H. See Section 2.3.2 and 3.2.5 of [Lin95]. We start all over again with this reduced 3-gem. This is the basis for the TS_ρ-algorithm (Subsection 4.1.5 of [Lin95]), which leaves very few topological uncertainties, which were resolved with the TS_ρ^U-algorithm of Subsection 4.1.8 of [Lin95].

3.3 The U-Move

The last element in our simplifying dynamics is a move, named U-move, which increases the number of vertices (!). Whenever two bigons of complementary colors meet in a single vertex v, a U-move can be applied. The vertex v is called a *monopole*. In Figure 12 we present how a U-move looks in the $\hat{0}$-residue. Note that v is the single meeting of two color-complementary bigons of 6 edges each. The 0-colored edges are presented in a dashed form.

The strength of this move is that it induces many TS-configurations which in turn may provide various simplifications. In conjunction with the ρ-moves and the TS-moves, the U-moves achieve the complete topological classification of 3-gems up to 30 vertices. In particular, we have concretely obtained all the attractors for the orientable 3-manifolds induced by 3-gems up to 30 vertices. We provide in Section 5.1 and in the Appendix of Section 8.1 of [Lin95] a complete catalogue of all rigid bipartite 3-gems up to 28 vertices. From this catalogue and the classification performed in Section 5.4 of [Lin95] one can explicitly display all the attractors up to this level.

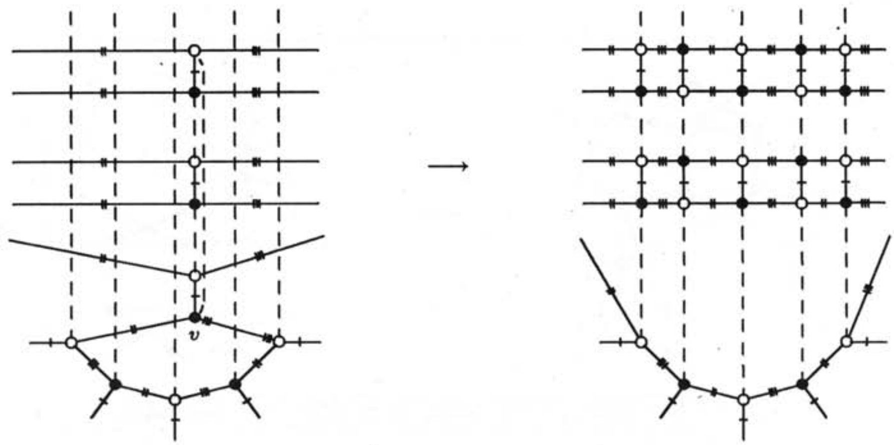


Fig. 12: An example of a U-move at v, single meeting of two hexagonal bigons

4 A Theorem and two Conjectures

A *u^0-move* on a 3-gem is either a ρ-move or a TS-move, whereas a *u^0_*-move* is the identity or a finite sequence of u^0-moves. A *u^1-move* is a move of type Uu^0_* *which may decrease but does not increase the number of vertices.* A *u^1_*-move* on a 3-gem is a finite sequence of u^1- and u^0-moves. In general, let a *u^n-move* be a move of the type Uu^{n-1}_*, *which may decrease but does not increase the number of vertices.* Let finally a *u^n_*-move* be a finite sequence of u^m's moves with $m \leq n$.

A 3-gem is *u^n-essential* if it cannot be simplified by u^n-moves. A *u^n-class* of 3-gems is a maximal set of 3-gems such that given any ordered pair of in the set, the second 3-gem can be obtained from the first by a single u^n_*-move. Note that 3-gems in a u^n-class have the same number of vertices and induce the same 3-manifold.

A u^n-class is *essential* if each of its members is u^n-essential.

The U-moves form a nice computational counterpart of the TS and ρ-moves. As we showed in Chapter 5 of [Lin95], the topological uncertainties remaining with the u^0-classes were resolved when we put a single U-move into scene.

Theorem 2 *For the bipartite 3-gems up to 30 vertices the attractors of the induced 3-manifolds coincide with the u^1-essential classes.*

This Theorem has been proved up to 28 vertices in [Lin95] and, as we have said, it was recently extended to 30 vertices in a joint work with Cassiano Durand and Said Sidki.

Conjecture 1 (Weak Conjecture) *There exists a computable function*

$$f : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$$

(which might be constant — maybe always 1) so that two 3-gems having p and q vertices induce the same 3-manifold if and only if they are linked by a single u_*^n-move, where $n = f(p,q)$.

We remark that the number of relevant u_*^n-moves is finite. Therefore, having a positive answer for the Conjecture, given the computability of f, would imply a finite algorithm to decide homeomorphism among 3-manifolds.

Our data so far, suggests more, that, maybe,

Conjecture 2 (Strong Conjecture) *The attractor for a 3-manifold is formed by the 3-gems in the unique u^1-essential class of 3-gems inducing this manifold.*

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