Resenhas IME-USP 1995, Vol. 2, No. 2, 239–252.

An Overview of Domino and Lozenge Tilings

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Abstract. We consider tilings of quadriculated regions by dominoes and of triangulated regions by lozenges. We present an overview of results concerning tileability, enumeration and the structure of the space of tilings.

1. Introduction

Let R be a finite juxtaposition of unit squares in the plane. We consider tilings of R by *dominoes*, 2×1 rectangles. Alternatively, if R is obtained from unit equilateral triangles, we try to tile R by *lozenges* (or *calissons*), unions of two adjacent triangles. The following questions are natural:

- Can R be tiled?
- How many tilings exist?

A next set of questions is suggested by the concept of *flip*. We perform a flip by lifting two dominoes forming a square or three lozenges forming a hexagon and placing them back after a rotation of 90° or 60°.

- Do all tilings admit flips?
- Are any two tilings joined by a sequence of flips?
- What is the distance (the minimal number of flips) between two tilings?
- Which flips decrease the distance to a given tiling?
- Are there closed geodesics of flips in the space of tilings?

Furthermore, as we shall see, there is a natural bipartition on the space of domino tilings so that flips connect tilings in opposite classes.

- Is the number of tilings in each class the same?

For simply connected regions, these questions are easier. At the other end of the spectrum, they still make sense for quadriculated or triangulated surfaces.

Section 1 deals with the problem of tileability; the main references are [3] and [11]. In section 2, we discuss some techniques for counting tilings, from the beginning of the century with [9] to recent contributions by mathematical physicists ([7] and [8]). Finally, section 3 deals with flips and related issues; the results were originally presented in [10].

2. Existence of tilings

Given a finite set of types of tiles, can we use them to tile the plane? Rather surprisingly, this is an undecidable question [1]. The corresponding problem for

Both authors receive support from MCT and CNPq, Brazil.

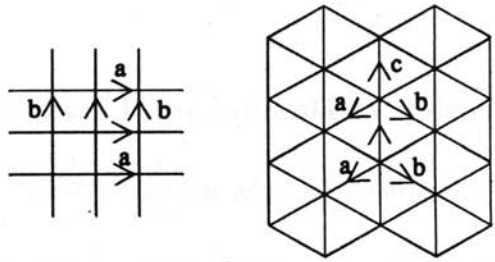

FIGURE 1

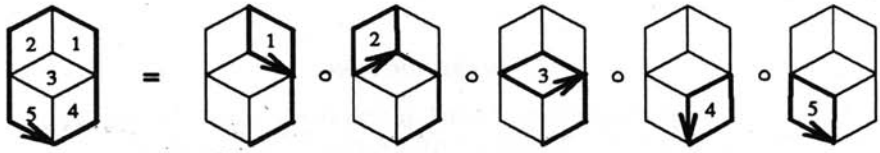

FIGURE 2

finite regions is NP-complete [6]. Colouring arguments are among the simplest to provide necessary conditions for tileability. Both the triangulated and the quadriculated plane are bicolourable and a region must have the same number of triangles (or squares) of each colour to be tiled by lozenges (or dominoes). Conway and Lagarias [3] discovered a method to obtain strong necessary conditions for tileability. Following Thurston [11], we present their method for simply connected quadriculated or triangulated regions, the tiles being dominoes or lozenges.

We start with lozenges. In the triangulated plane, we may label the oriented edges a, b and c as in figure 1.a and paths along edges yield words in the letters a, b and c. Notice that a path is closed iff the total numbers of a's, b's and c's are equal, i.e., iff its word equals the identity in $G_1 = \mathbb{Z}^2$, given in terms of generators and relations as $G_1 = \langle a, b, c | abc = acb = e \rangle$. Consider the group $G = \mathbb{Z}^3 = \langle a, b, c | [a, b] = [a, c] = [b, c] = e \rangle$ (as usual, $[g, h] = g^{-1}h^{-1}gh$): the relations are the boundaries of the three possible orientations of lozenges (i.e., position up to translation). The boundary of a simply connected region R yields a word w and thus an element $g \in G$: a tiling of G gives a way to write w as a juxtaposition of conjugates of boundaries of tiles (see figure 2, or [3] for an actual proof) and thus $g = e$.

These group-theoretical considerations produce Conway-Lagarias's condition for tileability of simply connected regions. We state their result for triangulated (or quadriculated) regions:

Theorem 1. *Let R be a triangulated (or quadriculated) simply connected bounded region and Σ a finite set of (connected and simply connected) tile shapes. Let G be the group with generators corresponding to edges and relations given by boundaries*

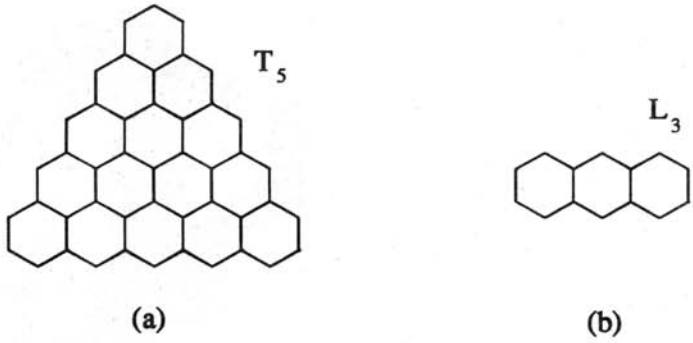

FIGURE 3

of elements of Σ. Then a necessary condition for the tileability of R by translates of elements of Σ is that the word induced by the boundary of R is trivial in G.

For lozenges, this condition reduces to the simple colour condition. More interesting applications of the group theoretic conditions yield the following results [3]. Let T_n be the triangular region of size n in the hexagonal lattice (fig 3.a) and L_3 be the *tribone*, the juxtaposition of three hexagons in a line (fig 3.b).

Theorem 2. *The triangular region T_n can be tiled by copies of T_2 iff $n \equiv 0, 2, 9$ or* 11 (mod 12). *Also, T_n cannot be tiled by congruent copies of L_3.*

David and Tomei noticed that a lozenge tiling (as in figure 4.a) may literally be seen as a picture of a bunch of cubes stored in a big box (as in figure 4.b). Align the sides of the box with the coordinate axes so that vertices of triangles receive integer x, y, z coordinates. In figure 4, for example, the box has size $5 \times 5 \times 5$ and there are 66 cubes in it. The figure is an orthogonal projection of the three dimensional configuration on the plane $x + y + z = 0$. If we think of gravity as pointing "downwards" to $(-1, -1, -1)$, the cubes form a gravitationally stable *pile* iff they represent a tiling. By projecting on the coordinate planes, we immediately obtain for hexagonal regions the following invariance result [4]:

Theorem 3. *Let R be an arbitrary triangulated planar region. Then the numbers n_{xy}, n_{xz}, n_{yz} of lozenges in each orientation do not depend on the tiling.*

The visual argument works for simply connected regions; for general regions we might use *height sections* (to be introduced later) but we present an elementary proof which appears to be folklore. Consider vectors going from the origin of the plane to the centre of each white triangle and from the centre of each black triangle to the origin and add them up obtaining a vector v. Let w_{xy} be the vector joining the centres of triangles in a xy-lozenge, from black to white; define w_{xz} and w_{yz} similarly. Given a tiling, we may group pairs of vectors associated to the same lozenge to obtain:

$$v = n_{xy} w_{xy} + n_{xz} w_{xz} + n_{yz} w_{yz}.$$

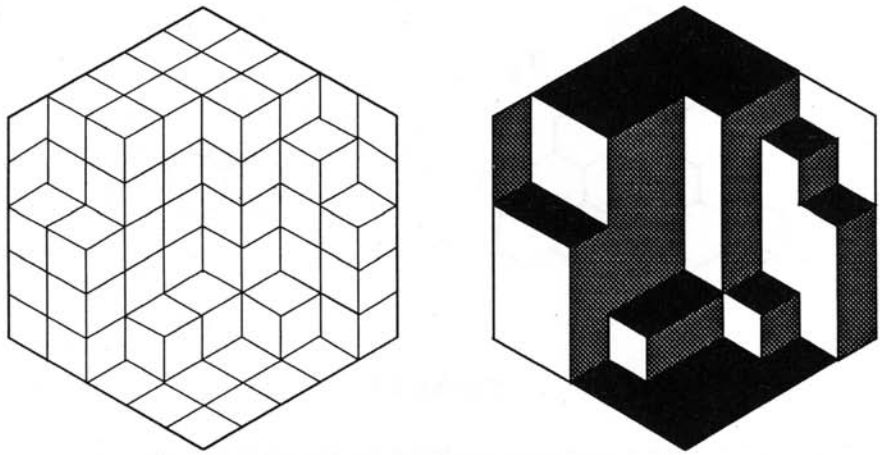

FIGURE 4

Since $n_{xy} + n_{xz} + n_{yz}$ does not depend on the tiling and $w_{xy} - w_{xz}$ and $w_{xy} - w_{yz}$ are linearly independent, the theorem follows.

Thurston obtained the same three dimensional interpretation as a spinoff of the group-theoretical construction. Consider the Cayley diagram Γ for G and its natural projection onto Γ_1, the Cayley diagram of its quotient G_1 (Γ_1 is, of course, the triangulated plane): a tiling of a simply connected region lives in Γ_1 but may be lifted to Γ (since each tile can). Imbed the Cayley graph Γ of $G = \mathbb{Z}^3$ in $\mathbb{R}^3$ (in the obvious way) and Γ_1 in the plane $x + y + z = 0$: the projection of Γ onto Γ_1 is the restriction of the orthogonal projection onto this plane. This lifting assigns to each vertex of a triangle the same coordinates as the more visual procedure represented in figure 4.

The extra information obtained by lifting is $x + y + z$, the height with respect to the plane $x + y + z = 0$: we call the function which assigns to each vertex the sum of its coordinates a *height function*, as shown in figure 5.a. Of course, there are other expressions which convey the 3D information equally well but, for height functions of lozenge tilings, we always take $x + y + z$. Height functions are therefore well defined for a tiling by lozenges of a simply connected region up to an additive constant. For a given tiling, a simple local rule obtains the height function: when following an edge at the border of a tile, add (resp. subtract) 1 to the height function at the source to get the height function at the target if the edge has a black triangle to the right (resp. left). Also, height functions of a simply connected region R are easy to characterize: height functions have the mod 3 values shown in figure 5.b, change by at most 2 along any edge in R and change by 1 along an edge in the boundary of R. An unexpected consequence of this characterization is the following [10]:

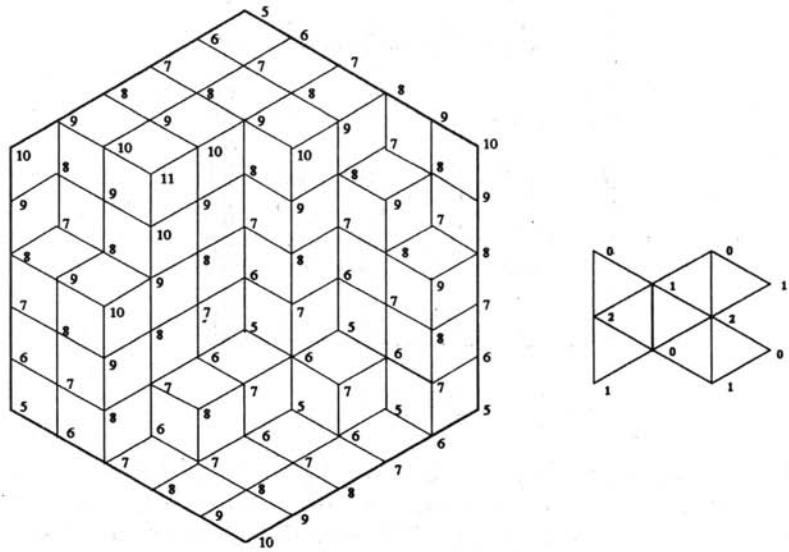

FIGURE 5

Theorem 4. *For a simply connected region, let T be the space of its lozenge tilings. The set T is a lattice: given two tilings, the maximum (or minimum) of their height functions is the height function of some tiling.*

In terms of the pile-of-cubes interpretation, the minimum tiling corresponds to the empty box. Thurston [11] uses height functions to prove:

Theorem 5. *There is a polynomial time algorithm to decide whether a simply connected region R is tileable by lozenges.*

In a nutshell, start by computing the height function along the boundary of R and move inwards assigning to each point the smallest value satisfying the characterization. The algorithm either fills the region, obtaining the minimal tiling, or runs into a contradiction, indicating non-tileability of R.

An equivalent criterion, computationally harder but easier to state, is the following: consider positively oriented simple closed curves which, outside the boundary of the region, have a black triangle to their left—the numbers of enclosed black and white triangles are equal.

The group theoretical construction easily applies to domino tilings. Label oriented edges in the quadriculated plane as in figure 1.b, obtaining the Cayley diagram Γ_1 for $G_1 = \langle a, b | [a, b] = e \rangle = \mathbb{Z}^2$. Taking into account the shape of the tiles, define $G = \langle a, b | [a, b^2] = [a^2, b] = e \rangle$: it turns out that G is a non-abelian group and we describe a construction of its Cayley diagram Γ in $\mathbb{R}^3$, obtaining a height function for domino tilings.

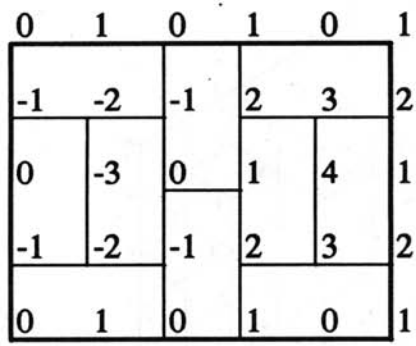

FIGURE 6

Introducing a third generator $c = [a, b]$, we have $G = \langle a, b, c | ac = c^{-1}a, bc = c^{-1}b, ab = bac \rangle$ and, by standard arguments, any element of G can be written as $c^w b^y a^x$ for unique integers x, y, w. We may thus take $\mathbb{Z}^3$ to be the set of vertices of a Cayley graph for G: connect (x, y, w) by an a-edge to $(x+1, y, w)$ (because $c^w b^y a^x \cdot a = c^w b^y a^{x+1}$), by a b-edge to $(x, y+1, w)$ if x is even or to $(x, y+1, w+(-1)^y)$ if x is odd and by a c-edge to $(x, y, w+(-1)^{x+y})$. A somewhat nicer picture is obtained by taking

$$z = 4w + \begin{cases} 0 & \text{if } x \text{ and } y \text{ are both even,} \\ 1 & \text{if } x \text{ is odd and } y \text{ is even,} \\ -2 & \text{if } x \text{ and } y \text{ are both odd,} \\ -1 & \text{if } x \text{ is even and } y \text{ is odd.} \end{cases}$$

The vertices of Γ are then the points $(x, y, z) \in \mathbb{Z}^3$ for which

$$z \underset{(\text{mod } 4)}{\equiv} \begin{cases} 0 & \text{if } x \text{ and } y \text{ are both even,} \\ 1 & \text{if } x \text{ is odd and } y \text{ is even,} \\ 2 & \text{if } x \text{ and } y \text{ are both odd,} \\ 3 & \text{if } x \text{ is even and } y \text{ is odd,} \end{cases}$$

and, for the original presentation of G, there are edges joining two vertices iff the euclidean distance is $\sqrt{2}$. Notice that this graph projects onto Γ_1 by omitting the z coordinate.

By construction, any domino tiling lifts to this Cayley graph. In figure 4 we show a tiling and its related height function. The simple procedure to obtain height functions is: colour squares of the region in a checkerboard pattern, assign 0 to the origin and, for an edge not covered by a domino, assign to the target vertex the value at the source plus (resp. minus) 1 if the square to the left is black (resp. white).

Again, for simply connected regions, it is easy to give a local characterization of height functions, from which the counterparts of theorems 4 and 5 follow [10] and [11].

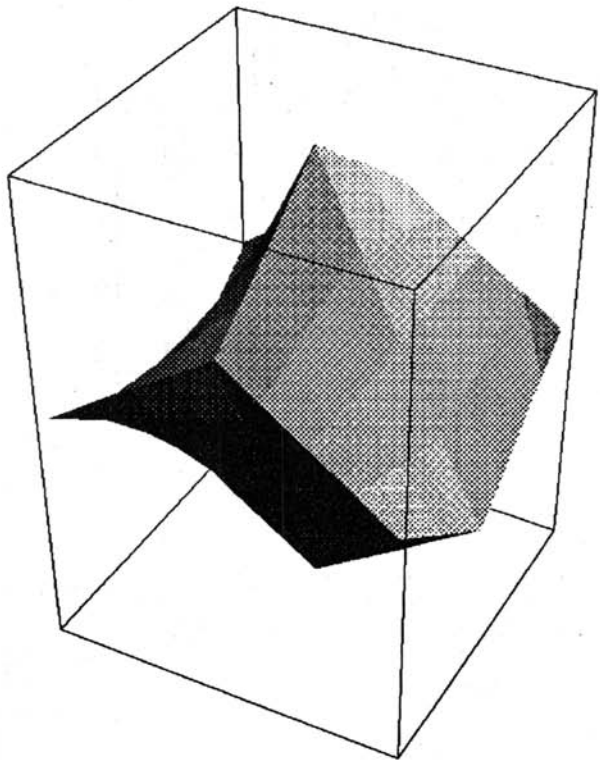

FIGURE 7

The pile-of-cubes interpretation for lozenge tilings has its analogue for domino tilings. The vertices of the three dimensional solid involved belong to Γ as above; the solid itself can be taken to be

$$\{(x, y, z) \mid |x| \leq 1, |y| \leq 1, |x| + 3|y| - 2|xy| \leq z \leq 4 - 3|x| - |y| + 2|xy|\},$$

shown in figure 7. Notice that, when seen from the top (resp. bottom), the solid covers two vertical (resp. horizontal) dominoes. We call this solid a block and there is, indeed, a *pile-of-blocks interpretation*; height functions, however, nicely encapsulate the relevant properties of such arrangements and we may therefore avoid visualization altogether.

The procedures in [3] and [11] do not apply to regions which are not simply connected. To study domino tilings, we introduced *height sections* [10], a generalization of height functions which applies to any quadriculated surface (e.g., a torus or Klein bottle). Instead of a formal definition, we give a typical example; vertices in the interior of the surface must be surrounded by 4 squares. A similar

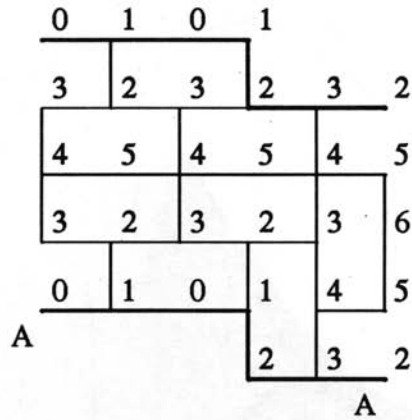

FIGURE 8

construction applies for lozenge tilings of triangulated surfaces (interior vertices in a triangulated surface are surrounded by 6 triangles).

Consider a cylinder with boundary, as in figure 8, tiled by dominoes. The two vertical sides are identified as indicated by the labels: two dominoes trespass the cut. Begin by assigning the value 0 to the vertex on the left labelled A and follow the local instructions for constructing height functions to obtain the value of the height section at all vertices. The point A in the cylinder is represented by two vertices in the diagram which received different numbers: 0 at the left and 2 at the right. The same ambiguity happens for the other points along the cut. How is this to be interpreted? We might take values mod 2 but this would amount to throwing away the bundle with the section. Instead, we construct a fibre bundle with fibre $\mathbb{Z}$ and the cylinder as base space (a 2-manifold with boundary). Over any simply connected open subset of the basis, the leaves can be labelled by the integers: do this for the complement of the cut—the values of the height section are to be understood in terms of these local coordinates. The different values for the points in the cut indicate how to glue leaves along the cut: in our example, the n-leaf at the left is glued to $n+2$-leaf at the right. Each number in the figure labels one element of the fibre over the corresponding vertex and the different numbers over distinct representations of the same vertex label, by construction, the same element. These elements form a section of the restriction of this *height bundle* to the vertices of the basis: the *height section*. Summing up, the procedure simultaneously constructs the height section and the height bundle.

Height sections allow us to give a tileability algorithm similar to Thurston's. It would be interesting to detail such an efficient algorithm and generalize the alternate criterion (based on closed curves) to arbitrary quadriculated surfaces. For planar regions, the space of tilings is still a lattice.

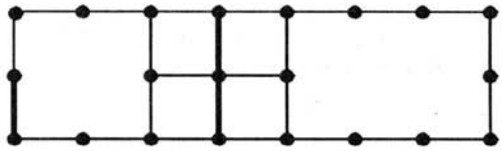

FIGURE 9

3. COUNTING TILINGS

For a quadriculated or triangulated planar region R, consider a graph Λ whose vertices are the k unit faces (squares or triangles) of R, connected by an edge iff the faces have a common side. Since faces in R can be coloured black and white, Λ is bipartite; we assume the number of black faces to equal the number of white faces, a necessary condition for the existence of tilings. Let A be the $k \times k$ adjacency matrix of Λ: by labelling black vertices first, we may write

$$A = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}.$$

In a language familiar to physicists, tilings of R correspond to *dimer coverings* of Λ, i.e., partitions of the set of vertices into pairs of adjacent vertices.

Each dimer covering corresponds to a monomial in the expansion of $\det B$. Suppose that $\tilde{B}$ is obtained from B by changing the signs of some entries and that all monomials in the expansion of $\det \tilde{B}$ have the same sign. Then the number N of tilings of R is $|\det \tilde{B}|$. Amazingly, for simply connected triangulated regions, we can take $\tilde{B} = B$. For quadriculated regions, Kasteleyn [7] showed that such a $\tilde{B}$ can be obtained from B by changing the signs of the 1's corresponding to edges on alternate vertical lines. Lieb and Loss [8] generalized this idea to prove a formula for the number of dimer partitions of an arbitrary bipartite planar graph. The following theorem, a special case of their results, unifies the previous claims.

Theorem 6. *Let Λ be a planar bipartite graph with adjacency matrix*

$$A = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}.$$

Choose a set of edges in Λ such that on each minimal loop of size $4n$ (resp. $4n+2$) there is an odd (resp. even) number of chosen edges. Change the sign of the 1's in B corresponding to chosen edges obtaining $\tilde{B}$. Then the number of dimer partitions of Λ is $|\det \tilde{B}|$.

The required choice of edges is always possible. An example is given in figure 9, where there are 6 minimal loops of sizes 8, 4, 4, 4, 4 and 10 and the chosen edges are thick.

The sign of the monomial in $\det B$ associated to a tiling is not natural, depending on the labelling of vertices. Given two tilings, however, equality of their signs is well defined and thus tilings naturally split in two classes. This splitting is very different in the quadriculated and triangulated cases:

Theorem 7. *Let R be a simply connected region, N the number of tilings of R and B as above. If R is quadriculated,*

$$|\det B| \leq 1:$$

the two classes are almost balanced. If R is triangulated,

$$|\det B| = N:$$

one of the classes is empty.

The quadriculated case is due to Deift and Tomei [5]. For general planar regions this difference assumes any value. The triangulated case is a corollary of theorem 6 but we will see another proof in the next section.

We present closed formulae for N in special cases. Kasteleyn [7] obtained a closed form for the number of tilings of a rectangle:

Theorem 8. *If R is a $n \times m$ rectangle,*

$$\prod_{k=1}^{m/2} \prod_{\ell=1}^{n} 2 \left(\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{\ell\pi}{n+1} \right)^{1/2},$$

assuming m even.

Kasteleyn's proof boils down to computing the determinant of $\tilde{B}$, as above. He essentially finds the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & \tilde{B} \\ \tilde{B}^t & 0 \end{pmatrix}^2;$$

the eigenvalues are the terms in the product with doubles multiplicities and the eigenvectors have the form

$$\sin \frac{k\pi i}{m+1} \sin \frac{\ell\pi j}{n+1}.$$

The pile-of-cubes interpretation of lozenge tilings reduces their counting for center-symmetric hexagonal regions to that of *plane partitions*. To describe a tiling, count the number of cubes above each square in the xy plane (here, the "up" direction is $(0,0,1)$ and not $(1,1,1)$). The example in figure 3 would be encoded as below.

$$\begin{matrix} 5 & 5 & 5 & 5 & 3 \\ 5 & 5 & 5 & 5 & 3 \\ 5 & 5 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{matrix}$$

Following Bender and Knuth [2], a plane partition of n (the total number of cubes in the pile) is an array of non-negative integers

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

for which $\sum_{ij} n_{ij} = n$, $n_{ij} \geq n_{(i+1)j}$ and $n_{ij} \geq n_{i(j+1)}$ (these conditions correspond to the pile being gravitationally stable). If $n_{ij} = 0$ for all $i > r$ (resp. $j > c$), the partition is called *r-rowed* (resp. *c-columned*) (this is the size of the region in the x (resp. y) direction). If $n_{11} \leq m$, we say the parts *do not exceed* m (the size in the z direction). Let π_n be the number of plane r-rowed, c-columned partitions of n with parts not exceeding m. MacMahon [9] obtained

$$\sum_{n=0}^{\infty} \pi_n x^n = \prod_{i=1}^{r} \prod_{j=1}^{m} \frac{1 - x^{c+i+j-1}}{1 - x^{i+j-1}},$$

yielding the following formula:

Theorem 9. *The number of lozenge tilings of a center-symmetric hexagonal region of sides r, c and m is*

$$\prod_{i=1}^{r} \prod_{j=1}^{m} \frac{c+i+j-1}{i+j-1}.$$

4. Flips and the space of tilings

The space T of all tilings of a given region R admits a natural graph structure: two tilings are adjacent iff they differ by a *flip*, the action of lifting two dominoes forming a square or three lozenges forming a hexagon and placing them back after a rotation of 90° or 60°. Two tilings are joined by a flip iff their height functions are different in one vertex only; this difference must be 3 (resp. 4) in the triangulated (resp. quadriculated) case. Equivalently, in the pile-of-cubes (or pile-of-blocks) interpretation, a flip corresponds to adding or removing a cube (or block). This suggests a basic property of T when R is simply connected:

Theorem 10. *If R is a simply connected region, T is connected.*

The inductive step is to prove that if a tiling t_1 is larger than a tiling t_2, we may perform a flip on t_1 to obtain t_3 with $t_1 > t_3 \geq t_2$. Let θ_i be the height function for t_i; remember that all height functions coincide along the boundary. To find out where to flip, consider the set of vertices where $\theta_1 - \theta_2$ is maximal and, within this set, choose the vertex with maximum θ_1. Since all height functions coincide in the boundary, this point is a local maximum of θ_1. By the characterization of height functions, a flip is possible at that point (we found a removable cube or block).

The triangulated version of theorem 7 follows from theorem 10: for triangulated surfaces, flips preserve parity of the monomials in the determinant expansion of B.

As usual, define distance in the graph T as the minimum number of edges (flips) in a path joining two vertices. The following distance formula follows from the same inductive step.

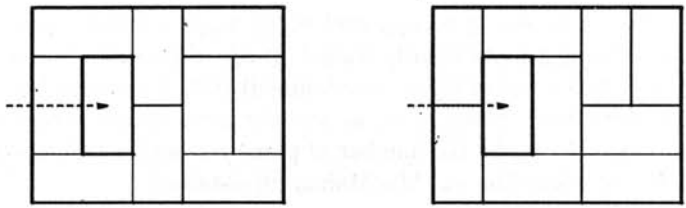

FIGURE 10

Theorem 11. *Let R be a simply connected region. The distance between tilings t_1 and t_2, with height functions θ_1 and θ_2, is*

$$d(t_1, t_2) = \frac{1}{X} \sum_v |\theta_1(v) - \theta_2(v)|,$$

where $X = 3$ or 4 depending whether R is triangulated or quadriculated.

As usual, we call such minimal paths *geodesics*. Some geodesics are pretty much the same: we may freely change the order of two successive flips (f_1 and f_2) if they act on disjoint hexagons or squares. We call two such geodesics equivalent and glue a 2-cell (along the loop $f_1 f_2 f_1^{-1} f_2^{-1}$) to the graph T to make them homotopic. Similarly, glue k-cells to identify all permutations of k independent flips, obtaining from T the CW-complex $\mathcal{T}$.

Theorem 12. *If R is simply connected, $\mathcal{T}$ is contractible.*

As a consequence, all geodesics between two given point are equivalent and all closed curves null-homotopic.

We now consider non-simply connected planar regions R. The graph T is now usually disconnected: to see this, we consider the following invariant under flips. Take a *cut*, i.e., a simple oriented curve in R consisting of edges with extrema in different boundary components of R. If R is triangulated (resp. quadriculated), define the *flow* of a tiling across a cut as the number of lozenges (resp. dominoes) crossing the cut, counted with a sign determined by the colour of the triangle (resp. square) to the left of the cut. For instance, the two tilings in figure 10 are not in the same connected component of T because the flows across the indicated cut are 1 and 0. The reader can easily see that homologous cuts yield equivalent conditions, the flows differing by a fixed constant from one cut to the other. In terms of height sections, it turns out that the height bundle does not depend on the tiling and two tilings have the same flow across all cuts iff their height sections coincide on the whole boundary.

Theorem 13. *Let R be a quadriculated planar region. Two tilings of R belong to the same connected component of T iff their flows coincide across all cuts. For two tilings in the same connected component of T, the distance formula in theorem 11 holds. Furthermore, connected components of $\mathcal{T}$ (constructed as before) are contractible.*

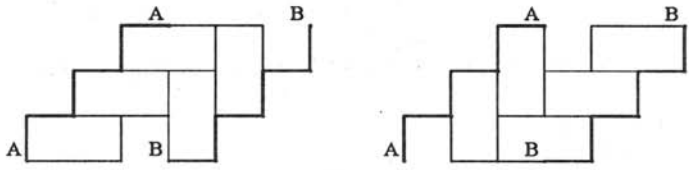

FIGURE 11

It would be interesting to see what happens for triangulated planar regions. A further generalization consists in considering arbitrary triangulated or quadriculated surfaces. The cut and flow invariants still make sense if the surface is orientable and bicoloured (no longer necessary conditions!). The appropriate generalization to the general case involves homology and cohomology with local coefficients, which we shall not discuss. In figure 11, we show that cut and flow conditions are no longer sufficient, even assuming orientability and bicolourability. For obvious reasons, we call these configurations *ladders*; rather surprisingly, their exact position turns out to be the only additional condition for domino tilings to be in the same connected component of T. As a further complication, if the surface is a (bicolourable) torus, certain connected components of $\mathcal{T}$ are no longer contractible, but homotopically equivalent to $\mathbb{S}^1$ and therefore contain "closed geodesics".

Theorem 14. *Let R be an arbitrary quadriculated surface. If the boundary of R is non-empty, connected components of $\mathcal{T}$ are contractible. In general, they are either contractible or homotopically equivalent to circles.*

REFERENCES

1. R. Berger, *The undecidability of the domino problem,* Mem. Amer. Math. Soc. **66** (1966).
2. E. A. Bender and D. E. Knuth, *Enumeration of plane partitions,* J. Comb. Theor. **A13**, 40-54 (1972).
3. J. H. Conway and J. C. Lagarias, *Tilings with polyominoes and combinatorial group theory,* J. Comb. Theor. **A53**, 183-208 (1990).
4. G. David and C. Tomei, *The problem of the calissons,* Amer. Math. Monthly, **96**, 429-431 (1989).
5. P. A. Deift and C. Tomei, *On the determinant of the adjacency matrix for a planar sublattice,* J. Comb. Theor. **B35**, 278-289 (1983).
6. M. Garey and D. S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness,* Freeman, San Francisco, 1979.
7. P. W. Kasteleyn, *The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice,* Phisica **27**, 1209-1225 (1961).
8. E. H. Lieb and M. Loss, *Fluxes, Laplacians and Kasteleyn's theorem,* Duke Math. Jour., **71**, 337-363 (1993).
9. P. A. MacMahon, *Combinatory analysis,* vol. 2, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960.

10. N. C. Saldanha, C. Tomei, M. A. Casarin Jr. and D. Romualdo, *Spaces of domino tilings,* Discr. Comp. Geom., to appear.
11. W. P. Thurston, *Conway's tiling groups,* Amer. Math. Monthly, **97**, 8, 757-773 (1990).

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