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Lectures on Dimension Subgroups

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Key words: dimension subgroups, dimension subgroup problem, dimension subgroup conjecture, solution, counter example.

1. A brief survey of the dimension subgroup problem;
2. A counter-example to the dimension subgroup conjecture;
3. Lie dimension subgroups and the restricted Lie dimension subgroups;
4. A solution of the dimension subgroup problem.

1. A brief survey of the dimension subgroup problem

Let $\Delta(G)$ denote the *augmentation ideal* of the integral group ring ZG. For each $n \geq 1$, the n-th *dimension subgroup* $G \cap (1 + \Delta(G)^n) (= D_n(G))$ is easily seen to contain the n-th lower central subgroup $\gamma_n(G)$ of G. The *dimension subgroup problem* refers to finding the structure of the n-th *dimension quotient* $D_n(G)/\gamma_n(G)$, and the *dimension subgroup conjecture* refers to the statement that the dimension quotients $D_n(G)/\gamma_n(G)$ are trivial for all groups G and all $n \geq 2$, or equivalently, $D_n(G) = \gamma_n(G)$ for all groups G and all integers $n \geq 2$. The equality $D_2(G) = \gamma_2(G)$ follows from the fact that if $g \notin \gamma_2(G)$ then $g - 1 \notin \Delta(G)^2$. When G is the free group F of rank $m \geq 2$ then $D_n(F) = \gamma_n(F)$ for all n, is the well-known Magnus-Grün-Witt theorem proved in the thirties. This is also known as the *fundamental theorem of free group rings.* The origin of the dimension subgroup conjecture can be traced back to Grün (1936) who attributes it to Magnus.

If the *dimension subgroup conjecture* is false then it is already false for some finite $p-$group G (Higman & Reese, see Passi (1968)) who also proved that $D_3(G) = \gamma_3(G)$ for all finite $p-$groups G. While $D_4(G) = \gamma_4(G)$ for all finite $p-$groups G, p odd (Passi 1968), there exists a finite 2-group G such that $D_4(G) \neq \gamma_4(G)$(Rips 1972), so the dimension subgroup conjecture is false for $n = 4$. The structure of the dimension quotient $D_4(G)/\gamma_4(G)$ was resolved by Tahara (1977). Other general results for finite $p-$groups are:

(i) $D_n(G) = \gamma_n(G)$, $n \leq p$ (Moran 1970);

(ii) $D_n(G) = \gamma_n(G)$, $n \leq p + 1$ (Sjorgren 1979);

(iii) $\exists\ c(n)$ (p divides $c(n)$ implies $p \leq n - 2$) such that the exponent of $D_n(G)/\gamma_n(G)$ divides $c(n)$ (Sjorgren 1979, Hartley 1982, Cliff-Hartley 1985, Gupta 1985).

If G is a finitly generated *metabelian* group then there exists $n_0 = n_0(G/G')$

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such that $D_n(G) = \gamma_n(G)$ for all $n \geq n_0$ (Gupta, Hales and Passi 1984).

If G is a metabelian p-group, p odd, then $D_n(G) = \gamma_n(G)$ for all $n \leq p+2$ (Gupta and Tahara 1985).

$D_n(G)/\gamma_n(G)$ has exponent dividing a large power of 2, and it follows that for metabelian p-groups G, p odd, $D_n(G) = \gamma_n(G)$ for all n (N. Gupta 1989).

For each $n \geq 4$, there exists a metabelian 2-groups G_n such that $D_n(G) \neq g_n(G)$ (N. Gupta 1989; cf. Rips 1972, $n = 4$).

Some isolated results of interest.

1. (Gupta-Srivastava 1990) For each $m \geq 1$, there exists a group G such that $D_{4m}(G)$ is not contained in $\gamma_{3m+1}(G)$;

2. (Gupta-Kuzmin 1992) $D_n(G)/\gamma_n(G)$ is abelian for all n and all G;

3. (Gupta - Kuz'min 1995) $D_n(G)/\gamma_n(G)$ is not, in general, central in $G/\gamma_n(G)$. In fact, for any integer s there exists a group G and an integer n such that $D_n(G)/\gamma_n(G)$ is not contained in the s-th upper central subgroup of $G/\gamma_n(G)$.

2. Couter-examples to the dimension subgroup conjecture

As mentioned earlier, for each $n \geq 4$, there exists a metabelian 2-group G_n such that $D_n(G) \neq \gamma_n(G)$ (N. Gupta 1990). Here we present without tecnical details a construction based on an idea from Hurley - Sehgal (1991) which has some additional consequences.

Construction of $G = G_{p,q}$ $(p, q \geq 1$ fixed).

Define

$$G_{p,q} = < r,\ a,\ b,\ c,\ x;\ \gamma_{2p+q+1}(G) >$$

subject to the following additional relations **(i)** - **(iv)**:

(i) Commutators with entry patterns:

$(1)\ \{\ r, r\ \};\ (2)\ \{\ a, b, c\ \};\ (3)\{\ a, a, a\ \};\ (4)\ \{\ b, b, b\ \};\ (5)\ \{\ c, c, c\ \};$
$(6)\ \{\ a, b, r\ \};\ (7)\ \{\ a, c, r\ \};\ (8)\ \{\ b, c, r\ \};\ (9)\ \{\ a, a, b\ \};\ (10)\ \{\ a, a, c\ \};$
$(11)\ \{\ a, b, b\ \};\ (12)\ \{\ b, b, c\ \};\ (13)\{\ a, c, c\ \};\ (14)\ \{\ b, c, c\ \};\ (15)\ \{\ x, \ldots, x\ \}$

(x occurs $2p + q - 2$ times).

Define. $[\ a, (m)x, a\ ] = [\ a, x, \ldots, x, a\ ]$ (x repeats m times)

(ii) Commutators of the form:

(16) $[\ a,(m)x,a,\dots\ ]$; (17) $[\ b,(m)x,b,\dots\ ]$; (18) $[\ c,(m)x,c,\dots\ ]$.

(iii) The following commutators:

(19) $[\ a,(p)x\ ]$; (20) $[\ b,(p)x\ ]$; (21) $[\ c,(p)x\ ]$; (22) $[\ r,(q)x\ ]$.

Note that the relation (1) to (22) imply that the only surviving central commutators are:

$[\ r,(q-1)x,a,(p-1)x,a,(p-1)x\ ]\ (=\ [\ r_q,a_p,a_p\ ]\ )$
$[\ r,(q-1)x,b,(p-1)x,b,(p-1)x\ ]\ (=\ [\ r_q,b_p,b_p\ ]\ )$
$[\ r,(q-1)x,c,(p-1)x,c,(p-1)x\ ]\ (=\ [\ r_q,c_p,c_p\ ]\ )$
where

$$a_k=[\ a,(k-1)x\ ],\ b_k=[\ b,(k-1)x\ ],$$
$$c_k=[\ c,(k-1)x\ ],\ r_k=[\ r,(k-1)x\ ].$$

(iv) Final relations for $G_{p,q}$:

(23) $a_p{}^{64}=[r_q,b_p]^4\ [\ r_q,c_p\ ]^2$; (24) $b_p{}^{16}=[\ r_q,a_p]^{-4}[r_q,c_p]$
(25) $c_p{}^4=[\ r_q,a_p\ ]^{-2}\ [\ rq,bp\ ]^{-1}$; (26) $[\ r_q,a_p,a_p\ ]=[\ r_q,b_p,b_p\ ]^2$;
(27) $[\ r_q,b_p,b_p\ ]=[\ r_q,c_p,c_p\ ]^4$.

The construction is complete.

{ Note: (26), (27) imply that $G_{p,q}$ has cyclic centre }.

Summary of important relations of $G_{p,q}$:

(i) $a_p{}^{64}=[r_q,b_p]^4[\ r_q,c_p\ ]^2; b_p{}^{16}=[\ r_q,a_p\ ]^{-4}[\ r_q,c_p\ ], c_p{}^4=[\ r_q,a_p\ ]^{-2}[\ r_q,b_p\ ]^{-1}$;

(ii) $[\ a_p,b_p\ ]^{16}=[\ r_q,a_p,a_p\ ]^4,\ \ [\ b_p,c_p\ ]^4=[\ r_q,a_p,a_p\ ]^4,\ \ [\ a_q,c_p\ ]^4=[\ r_q,a_p,a_p\ ]^2$;

(iii) $b_p{}^{128}c_p{}^{64}=d(a)^{64},\ \ d(a)\in sgp\{\ [\ r,a\ ],\ [\ r,b\ ],\ [\ r,c\ ]\ \}$;

$a_p{}^{128}c_p{}^{-32}=d(b)^{16},\ \ d(b)\in sgp\{\ [\ r,a\ ],\ [\ r,b\ ],\ [\ r,c\ ]\ \}$;

$a_p{}^{64}b_p{}^{32}=d(c)^4,\ \ d(c)\in sgp\{\ [\ r,a\ ],\ [\ r,b\ ],\ [\ r,c\ ]\ \}$;

[e.g. $a_p{}^{-128}c_p{}^{32}=[\ r_q,b_p\ ]^{-8}\ [\ r_q,c_p\ ]^4\ [\ r_q,a_p\ ]^{-16}\ [\ r_q,b_p\ ]^{-8}$

$$=[\ r_q,c_p\ ]^4\ [\ r_q,c_p,c_p\ ]^{-6}\ [\ r_q,b_p\ ]^{-16}\ [\ r_q,a_p\ ]^{-16}$$

$$=[\ r_q,c_p,c_p\ ]^{-6}\ [\ r_q,b_p\ ]^{-16}\ [\ r_q,a_p\ ]^{-16}$$

$$=(\ [\ r_q,a_p,a_p\ ]^{-6}\ [\ r_q,b_p\ ]^{-1}\ [\ r_q,a_p\ ]^{-1}\ )^{16}\ ].$$

(iv) $g = [\, a_p, b_p \,]^{128} \, [\, a_p, c_p \,]^{64} \, [\, b_p, c_p \,]^{32} = [\, r_q, a_p, a_p \,]^{32} \neq 1$;

(v) $g^2 = 1$.

Consider the group $G \; (= \; G_{p,p})$ and let

$$g = [\, a_p, b_p \,]^{128} [\, a_p, c_p \,]^{64} \, [\, b_p, c_p \,]^{32}.$$

Then

(i) $g \notin \gamma_{3p+1}(G)$, and the expansion of $g-1$ gives

(ii) $g-1 \in (\, \gamma_{2p}(G)-1 \,)^2 + (\, \gamma_{2p}(G)-1 \,)(\, \gamma_p(G)-1 \,)^2 + (\, \gamma_p(G)-1 \,)^2 (\, \gamma_{2p}(G)-1 \,) \leq ZG(\, G-1 \,)^{4p}$.

We thus have proved that for each $m \geq 1$, there exists a group G such that $D_{4m}(G)$ is not contained in $\gamma_{3m+1}(G)$.

3. Lie dimension and restricted Lie dimension subgroups

Consider the group $G + G_{p,q}$ as constructed in the previous section. Then the group element

$$g = [\, a_p, b_p \,]^{128} \, [\, a_p, c_p \,]^{64} \, [\, b_p, c_p \,]^{32}$$

has the property that $g \notin \gamma_{2p+q+1}(G)$, and the expansion of $g-1$ gives

$$g - 1 \in (\gamma_{p+q}(G) - 1)^2 + (\gamma_{p+q}(G) - 1)(\gamma_p(G) - 1)^2 +$$

$$+ (\gamma_p(G) - 1)^2 (\gamma_{p+q}(G) - 1) + (\gamma_p(G) - 1)^4$$

$$\leq {\Lambda_{p+q}}^2 + \Lambda_{p+q}{\Lambda_p}^2 + {\Lambda_p}^2 \Lambda_{p+q} + {\Lambda_p}^4, \qquad (*)$$

where $\Lambda_i = ZG(\gamma_i(G) - 1)$.

Define Lie powers $\Delta^{(n)}$ as follows: $\Delta^{(1)} = \Delta$,

$$\Delta^{(n+1)} = ZG(\Delta^{(n)}, \Delta) = Ideal_{ZG}\{\, (u, v); u \in \Delta^{(n)}, v \in \Delta \,\}$$

where $(u, v) = uv - vu$.

Let $D_{(n)}(G) = G \cap (1 + \Delta^{(n)}(G))$ denotes the n-th *Lie dimension subgroup*.

Using $\Lambda_i \Lambda_j \leq \Delta^{(i+j-1)}$ (cf. Passi - Sehgal 1975) gives

$$\begin{aligned} {\Lambda_{p+q}}^2 + \Lambda_{p+q}{\Lambda_p}^2 + {\Lambda_p}^2 \Lambda_{p+q} + {\Lambda_p}^4 &\leq \Delta^{(2p+2q-1)} + \Delta^{(3p+q-2)} + \Delta^{(4p-3)} \\ &\leq \Delta^{(2p+q+1)} \text{ if } 2 \leq q \leq 2p-4. \end{aligned}$$

As a corollary we deduce the following result,

Theorem (Hurley - Sehgal 1991). For each $n \geq 9$ there exists a group $G\ (= G_n)$ such that $D_{(n)}(G) \neq \gamma_n(G)$.
[proof. Choose $n = 2p+q+1$ with $2 \leq q \leq 2p-4$. Then $p \geq 3$ and $q \geq 2$ so $n \geq 9$]

Define *restricted* Lie powers $\Delta^{[n]}$ as follows: $\Delta^{[1]} = \Delta$,

$$\Delta^{[n]} = Ideal_{ZG}\{\ (g_1, \ldots, g_n); g_i \in G\ \},\quad n \geq 2.$$

Define the *restricted* Lie dimension subgroups by

$$D_{[n]}(G) = G \cap (1 + \Delta^{[n]}(G)).$$

Using $\Lambda_i\Lambda_j \leq \Delta^{[i+j-2]}$ (cf Gupta-Levin 1983) gives

$$\begin{aligned}{\Lambda_{p+q}}^2 + \Lambda_{p+q}{\Lambda_p}^2 + {\Lambda_p}^2\Lambda_{p+q} + {\Lambda_p}^4 &\leq \Delta^{(2p+2q-2)} + \Delta^{(3p+q-4)} + \Delta^{(4p-6)} \\ &\leq \Delta^{(2p+q+1)} \ \text{ if } 3 \leq q \leq 2p-7.\end{aligned}$$

As a corollary we deduce the following result,

Theorem (Hurley - Sehgal 1991). For each $n \geq 14$ there exists a group $G\ (= G_n)$ such that $D_{[n]}(G) \neq \gamma_n(G)$.
proof. [Choose $n = 2p+q+1, 3 \leq q \leq 2p-7$. Then $p \geq 5$ and $q \geq 3$ so $n \geq 14$].

An improvement (Gupta - Srivastava 1991): Given $n \geq 9$, there exists a group G satisfying $D_{(n)}(G) \neq D_{[n]}(G)$.

The study of Lie dimension subgroups was stimulated by Sandling (1972) who proved, among others things, that

$$D_{(n)}(G) = \gamma_n(G) \ \text{ for } n \leq 6$$

whereas Hurley - Sehgal (1991) proved that there exists, for each $n \geq 9$, a group G such that $D_{(n)}(G) \neq \gamma_n(G)$. This leaves the problem open for $n = 7$ and 8. However, we have

Theorem (Gupta - Tahara 1993). $D_{(n)}(G) = \gamma_n(G)$ for $n \leq 8$ and $D_{[n]}(G) = \gamma_n(G)$ for $n \leq 8$.

This completely resolves the Lie dimension subgroup conjecture and the restricted Lie dimension subgroup conjecture.

4. A solution of the dimension subgroup problem

It has now become possible to describe structure of the n-th dimension quotient $D_n(G)/\gamma_n(G)$ of an arbitrary finitely generated group G. From the structure one deduces that the dimension quotients have exponent dividing 2. Thus, whereas it known that there exists 2-groups G with non-trivial dimension quotients for $n \geq 4$, for p-groups G with p odd, $D_n(G)$ coincides with $\gamma_n(G)$ for all n, so the dimension subgroup conjecture holds for p-groups, p odd.

Using a free representation ($1 \to R \to F \to G \to 1$) of the group G, the dimension quotients $D_n(G)/\gamma_n(G)$ translate to the quotients

$$F \cap (1 + \mathbf{r} + \mathbf{f}^{\,n})/R\,\gamma_n(F),$$

where $\mathbf{f} = ZF(F-1)$ $(= \Delta(F))$ and $\mathbf{r} = ZF(R-1)$ $(= ZF\Delta(R))$ are the fundamental ideals of ZF. A filteration through the derived series of F/R then reduces the problem to solving, for each $k \geq 0$, $2^k < n$, the *k-th partial dimension congruence:*

$$(*) \qquad w_k - 1 \equiv 0 \text{ modulo } \mathbf{f}^{(k)}\,\mathbf{r}^{(k)} + \mathbf{r}^{(k)}\,\mathbf{f}^{(k)} + \mathbf{r}^{(k+1)} + \mathbf{f}^{(k,n)},$$

in the free group ring $ZF^{(k)}$, where $F^{(k)} = < x_{k,1}, \ldots, x_{k,m(k)} >$ is a certain finitely generated free group contained in the k-th derived group $\delta_k(F)$, admitting the free presentation of $\delta_k\,(F/\gamma_n(F))$, $\mathbf{f}^{(k)} = ZF^{(k)}(F^{(k)}-1)$, $\mathbf{r}^{(k)} = ZF^{(k)}(R^{(k)}-1)(R^{(k)} = R \cap F^{(k)})$ and

$$\mathbf{f}^{(k,n)} = ideal_{ZF^{(k)}}\{(x_{k,i(1)}-1)\ldots(x_{k,i(q)}-1),$$

$q \geq 2$, $\mathbf{wt}\ x_{k,i(1)} + \ldots + \mathbf{wt}\ x_{k,i(q)} \geq n$.

The solution of the congruence $(*)$ comprises of a *specific* element $g_k \in [F^{(k)}, F^{(k)}]$ together with elements $r_k \in R \cap [F^{(k)}, F^{(k)}]$ and $h_k \in F^{(k,n)} < [F^{(k)}, F^{(k)}] \cap \gamma_n(F)$ such that with $w_{k+1} = {r_k}^{-1}{h_k}^{-1}{g_k}^{-1}w_k$, the problem shifts to solving the next partial dimension congruence:

$$(**) \qquad w_{k+1} - 1 \equiv 0 \text{ modulo } \mathbf{f}^{(k+1)}\,\mathbf{r}^{(k+1)} + \mathbf{r}^{(k+1)}\,\mathbf{f}^{(k+1)} + \mathbf{r}^{(k+2)} + \mathbf{f}^{(k+1,n)}.$$

Since, for sufficiently large t (e.g. $2^{t+1} \geq n$), $g_{t+1} = r_{t+1} = h_{t+1} = 1$, it follows that $w = g_0 h_0 r_0 \ldots g_t h_t r_t \equiv g_0 \ldots g_t$ (mod $R\,\gamma_n(F)$) is the required solution of the dimension congruence: $w - 1 \equiv 0$ modulo $\mathbf{r} + \mathbf{f}^n$. We prove, in addition, that the g_i's commute and $g_i^2 = 1$ for all i.

Reduction to the partial dimension congruences

Let $n \geq 3$ be an arbitrary but fixed positive integer. For each $k \geq 0$ with $2^k < n$, consider the free group $F^{(k)} = < x_{k,1}, \ldots, x_{k,m(k)} >$, and define certain ideals of the free group rings $ZF^{(k)}$ as follows:

$$\mathbf{f}^{(0)} = \mathbf{f} = ZF(F-1), \quad \mathbf{r}^{(0)} = \mathbf{r} = ZF(R-1),$$

$$\mathbf{f}^{(0,n)} = \mathbf{f}^{\,n} = Z\text{-span}\{\ (x_{i(1)^{\pm 1}} - 1)\ldots(x_{i(t)^{\pm 1}} - 1)\ \},\ t \geq n,$$

are ideals of $ZF = ZF^{(0)}$, where without ambiguity $(x_{i(1)^{\pm 1}} - 1) = (x_{i(1)} - 1)$ or $(x_{i(1)^{-1}} - 1)$; and for $k \geq 1$, define the corresponding ideals in $ZF^{(k)}$ as

$$\mathbf{f}^{(k)} = ZF^{(k)}(F^{(k)} - 1),\quad \mathbf{r}^{(k)} = ZF^{(k)}(R^{(k)} - 1),$$

$$\mathbf{f}^{(k,n)} = Z\text{-span}\{\ (x_{k,i(1)^{\pm 1}} - 1)\ldots(x_{k,i(1)^{\pm 1}} - 1)\ \},\ t \geq 2,$$

$$x_{k,i(j)} \in \{x_{k,1}, \ldots, x_{k,m(k)}\},\ \sum_{1\leq j\leq t} \mathbf{wt}\ x_{k,i(j)} \geq n \ \text{ and}$$

$$(x_{k,i(j)^{\pm 1}} - 1) = (x_{k,i(j)} - 1) \text{ or } (x_{k,i(j)^{-1}} - 1).$$

[We remind that, by definition, $\mathbf{wt}\ x_{k,i(j)} = s$ implies $x_{k,i(j)^{\pm 1}} \in \gamma_s(F)\backslash\gamma_{s+1}(F)$]

Also define higher ideals,

$$\mathbf{a}^{(0)} = \mathbf{a} = ZF(F' - 1),\quad \mathbf{a}^{(k)} = ZF^{(k)}([F^{(k)}, F^{(k)}] - 1),$$
$$\text{and}\quad \mathbf{r}^{(k)+} = ZF^{(k)}([R^{(k)}, F^{(k)}] - 1).$$

Consider now the following series of subgroups of $F^{(k)}$:

$$D(n, \mathbf{r}) = P(n, \mathbf{r}^{(0)}) \geq P(n, \mathbf{r}^{(1)}) \geq \ldots \geq P(n, \mathbf{r}^{(k)}) \geq \ldots$$

where

$$P(n, \mathbf{r}^{(k)}) = F^{(k)} \cap (1 + \mathbf{r}^{(k)} + \mathbf{f}^{(k,n)}).$$

For each $k \geq 0$, consider the subgroup $G(n, \mathbf{r}^{(k)})$ of $F^{(k)}$ defined by

$$G(n, \mathbf{r}^{(k)}) = F^{(k)} \cap (1 + \mathbf{r}^{(k)}\mathbf{f}^{(k)} + \mathbf{f}^{(k)}\mathbf{r}^{(k)} + \mathbf{r}^{(k+1)} + \mathbf{f}^{(k,n)}).$$

Then, clearly

$$P(n, \mathbf{r}^{(k+1)}) \leq G(n, \mathbf{r}^{(k)}) \leq P(n, \mathbf{r}^{(k)}).$$

Since $u \in \mathbf{r}^{(k)}$ implies that, modulo $\mathbf{r}^{(k)}\mathbf{f}^{(k)} + \mathbf{f}^{(k)}\mathbf{r}^{(k)} + \mathbf{r}^{(k+1)}$,

$$u \equiv \sum_i n_i(r_{k,i} - 1) \equiv (\prod_i r_{k,i^{n_i}} - 1) = r(k) - 1,\ r(k) \in R^{(k)},$$

it follows that

$$P(n, \mathbf{r}^{(k)}) = R^k G(n, \mathbf{r}^{(k)}).$$

Define

$$F^{(k,n)} = sgp\{\ [x_{k,i(1)^{+}_{-}1}, \ldots, x_{k,i(t)^{+}_{-}1}]\ \},\ t \geq 2,$$

$$x_{k,i(j)} \in \{x_{k,1}, \ldots, x_{k,m(k)}\},\ i(1) > i(2) \leq \ldots \leq i(t),\ \sum_{1 \leq j \leq t} \mathbf{wt}\ x_{k,i(j)} \geq n,$$

to be the subgroup of $F^{(k)}$ contained in the commutator subgroup $[F^{(k)}, F^{(k)}]$, so that

$$F^{(k,n)} \leq \gamma_n(F) \cap [F^{(k)}, F^{(k)}].$$

Finally, let us assume that we can identify, for each $k \geq 0$, the quotient

$$G(n, \mathbf{r}^{(k)})/F^{(k,n)}P(n, \mathbf{r}^{(k+1)}).$$

In other words, assume that for any $k \geq 0$, $f \in G(n, \mathbf{r}^{(k)})$ implies that there exists $g_{k+1} \in [F^{(k)}, F^{(k)}]$, $r_{k+1} \in [R^{(k)}, F^{(k)}]$ and $h_{k+1} \in F^{(k,n)}$ such that $g_{k+1}^{-1} r_{k+1}^{-1} h_{k+1}^{-1} f \equiv 1 \bmod P(n, \mathbf{r}^{(k+1)})$.

We can then solve the dimension subgroup problem as follows:

Let $w = w_0 \in D(n, \mathbf{r}) = P(n, \mathbf{r}^{(0)})$. Then as above there exists $r(0) \in R^{(0)}$ such that $r(0)^{-1} w_0 \in G(n, \mathbf{r}^{(0)})$. Similarly, there exists $g_1 \in [F^{(0)}, F^{(0)}]$, $r_1 \in [R^{(0)}, F^{(0)}]$ and $h_1 \in F^{(0,n)}$ such that $w_1 = g_1^{-1} h_1^{-1} r_1^{-1} r(0)^{-1} w_0 \in P(n, \mathbf{r}^{(1)})$. Repeating the argument, there exists $g_2 \in [F^{(1)}, F^{(1)}]$, $r_2 \in [R^{(1)}, F^{(1)}]$ and $h_2 \in F^{(1,n)}$ such that

$$g_2^{-1} h_2^{-1} r_2^{-1} r(1)^{-1} w_1 = g_2^{-1} h_2^{-1} r_2^{-1} r(1)^{-1} g_1^{-1} h_1^{-1} r_1^{-1} r(0)^{-1} w_0 \in P(n, \mathbf{r}^{(2)}).$$

By iteration, for any $t \geq 1$, we have

$$w \equiv r(0) r_1 h_1 g_1 \ldots r(t-1) r_t h_t g_t \ \text{ modulo } P(n, \mathbf{r}^{(t+1)})$$

which gives, for sufficiently value of t, the congruence

$$w \equiv g_1 \ldots g_t \text{ modulo } R\ \gamma_n(F),$$

which gives a complete solution to the dimension subgroup problem. It suffices therefore to identify the partial dimension quotients.

In particular, the validity of the following theorem for all $k \geq 0$ yields a complete solution of the dimension subgroup problem.

<u>Theorem</u> (a) Let $w_k - 1 \equiv 0$ modulo $\mathbf{r}^{(k+1)} + \mathbf{f}^{(k,n)}$, then there exists $h_{k+1} \in F^{(k,n)}$ such that

$$h_{k+1}^{-1} w_k - 1 \equiv 0 \text{ modulo } \mathbf{r}^{(k+1)} + \mathbf{f}^{(k+1,n)};$$

(b) Let $w_k - 1 \equiv 0$ modulo $\mathbf{r}^{(k)}\mathbf{f}^{(k)}\mathbf{f}^{(k)} + \mathbf{f}^{(k)}\mathbf{r}^{(k)}\mathbf{f}^{(k)} + \mathbf{f}^{(k)}\mathbf{f}^{(k)}\mathbf{r}^{(k)} + \mathbf{r}^{(k+1)} + \mathbf{f}^{(k,n)}$, then there exist $r_{k+1} \in [R^{(k)}, F^{(k)}, F^{(k)}]$ and $h_{k+1} \in F^{(k,n)}$ such that

$$h_{k+1}^{-1} r_{k+1}^{-1} w_k - 1 \equiv 0 \text{ modulo } \mathbf{r}^{(k+1)} + \mathbf{f}^{(k+1,n)};$$

(c) Let $w_k - 1 \equiv 0$ modulo $\mathbf{r}^{(k)}\mathbf{f}^{(k)} + \mathbf{f}^{(k)}\mathbf{r}^{(k)} + \mathbf{r}^{(k+1)} + \mathbf{f}^{(k,n)}$, then there exist $g_{k+1} \in [F^{(k)}, F^{(k)}]$, $r_{k+1} \in [R^{(k)}, F^{(k)}]$, $h_{k+1} \in F^{(k,n)}$, such that

$$g_{k+1}^{-1} h_{k+1}^{-1} r_{k+1}^{-1} w_k - 1 \equiv 0 \text{ modulo } \mathbf{r}^{(k+1)} + \mathbf{f}^{(k+1,n)}.$$

[A proof of this theorem will be published elsewhere.]

References

[Not all references are cited in the text. These are included for the interested reader who wants to explore the subject further. The list is by no means claimed to be complete. See Gupta (1987)b for some additional references and historical remarks].

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