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Units of Group Rings and Their Group Identities [1]

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Abstract: This paper surveys recent results concerning group rings KG whose group of units satisfies a group identity. For torsion groups we present a solution of a conjecture of Hartley relating the group identities of the group of units of KG to the polynomial identities satisfied by KG.

Key words: group ring, unit group, group identity, polynomial identity, torsion group.

1 Introduction

Let K be a field, G a group and KG the group algebra of G over K. It is in general a difficult problem to describe $U(KG)$, the group of units of KG. Here we want to survey on the progress recently made on the description of KG in case $U(KG)$ satisfies a group identity.

We recall some definitions. Let $X = \{x_1, x_2, \ldots\}$ be a countable set, F the free group on X and $K\{X\}$ the free algebra on X. Recall that a group U is said to satisfy a group identity if there exists a nontrivial word $w = w(x_1, \ldots x_n) \in F$ such that $w(u_1, \ldots, u_n) = 1$ for all $u_1, \ldots, u_n \in U$. Also, a non-zero polynomial $f(x_1, \ldots, x_n) \in K\{X\}$ is a polynomial identity for a K-algebra R if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$.

We shall address ourselves to the following general questions: 1) are the group identities satisfied by $U(KG)$ and the polynomial identities satisfied by KG somehow related ? 2) Can we characterize those group rings whose group of units satisfies a group identity ?

Now, the group algebras satisfying a polynomial identity have been completely described. Recall that a group is called p-abelian if its derived group is a finite p-group. The following theorem holds

Theorem 1 *1) (Isaacs-Passman, [17, Corollary 3.8]) If char$K = 0$ then KG satisfies a polynomial identity if and only if G contains an abelian subgroup of finite index.*

2) (Passman, [17, Corollary 3.10]) If char$K = p > 0$ then KG satisfies a polynomial identity if and only if G contains a p-abelian subgroup of finite index.

The answer to question 1) above is easily seen to be negative in general. In fact, in one direction, if KG satisfies a polynomial identity then one cannot expect in general $U(KG)$ to satisfy a group identity. To see this, take G to be any finite non abelian group and K a field not algebraic over a finite field; if char$K = 0$ or char$K = p > 0$ and $p \nmid o(G)$, then Wedderburn's theorem implies that for

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some noncommutative finite dimensional simple algebra A the group of units of A, $U(A)$, is a homomorphic image of $U(KG)$. Since $U(A)$ contains a free group of rank > 1, (see [10]) it follows that $U(KG)$ cannot satisfy any group identity.

In the other direction, if $U(KG)$ satisfies a group identity, can one expect KG to satisfy a polynomial identity ? Also in this case the answer is easily seen to be negative. In fact, if G is an ordered group, it can be shown (see [21, Proposition 1.6]) that KG has only trivial units i. e., $U(KG) = \pm G$. Now, since every torsion free nilpotent group can be ordered, it follows that $U(KG) = \pm G$ and, so, $U(KG)$ satisfies a group identity. But it is easy to exhibit a torsion free nilpotent group not satisfying the previous Theorem 1; hence KG satisfies no polynomial identities in this case.

We should remark that the above questions can be formulated for group rings RG where R is any ring; some results have been obtained in this setting; anyway in what follows we shall restrict ourselves to the case when $R = K$ is a field or $R = \mathbb{Z}$ is the ring of integers.

2 Some special group identities

In the past years characterizations have been obtained of group rings whose group of units satisfies a specific group identity; here we give a taste of these results by stating some theorems that characterize group rings whose group of units is either nilpotent or solvable. Most of these results are well known and can be found for instance in [21]. Other theorems of the same flavor include for instance results of Cliff and Sehgal [4] and Coelho [5] where a group identity of the form (x_1^n, x_2) was considered.

The case when $U(KG)$ is nilpotent was solved for finite groups by Bateman and Coleman in [2] and in the general case by Khripta in [13] and Fischer, Parmenter and Sehgal in [7]. Here we shall state their theorem only in the case when K is an infinite field. Let us denote with $T(G)$ the set of torsion elements of the group G.

Theorem 2 *If $charK = p > 0$ and G has a central element of order p then $U(KG)$ is nilpotent if and only if G is nilpotent and G' is a finite p-group. In all other cases $U(KG)$ is nilpotent if and only if $T(G)$ is a central subgroup.*

The corresponding characterization for integral group rings is due to Polcino Milies [19] for finite groups and to Sehgal and Zassenhaus in [23] for the general case. This theorem can be formulated in the following way

Theorem 3 *$U(\mathbb{Z}G)$ is nilpotent if and only if G is nilpotent, every subgroup of $T(G)$ is normal in G and one of the following holds:*

1. *$T(G)$ is central in G;*

2. *$T(G)$ is an abelian 2-group and for all $x \in G$, $t \in T(G)$ there exists $\epsilon = \epsilon(x) \in \{1, -1\}$ such that $x^{-1}tx = t^{\epsilon}$;*

3. *$T(G)$ is a Hamiltonian 2-group.*

The now turn to the case when $U(KG)$ is solvable. The result proved in this case is for finite groups and is due to Bateman [1] (see also [3] and [16]). We state the theorem for fields of characteristic $\neq 2, 3$.

Theorem 4 *Let G be a finite group. If $charK = 0$, $U(KG)$ is solvable if and only if G is abelian. If $charK = p > 0, p \neq 2, 3$, $U(KG)$ is solvable if and only if G' is a p-group.*

The corresponding result for integral group rings holds for general groups and is due to Sehgal [21].

Theorem 5 *If $U(\mathbb{Z}G)$ is solvable then*

1. *$T(G)$ is a subgroup of G which is abelian or a Hamiltonian 2-group;*

2. *every subgroup of $T(G)$ is normal in G.*

The converse holds provided G is a solvable group and $G/T(G)$ is nilpotent.

3 Torsion Groups

The examples given in the introduction seem to leave little room for positive results concerning question 1) in general; nevertheless for torsion groups Hartley made the following

Conjecture. If G is a torsion group and $U(KG)$ satisfies a group identity then KG satisfies a polynomial identity.

The first positive results were obtained in the early 80s by D. S. Warhurst, a student of Hartley, in her Ph.D. thesis [24]; she studied some special cases when G is a p-group and K has characteristic p. Another approach in this setting was suggested by Menal [15].

Goncalves and Mandel in [11] characterized group algebras of torsion groups over an infinite field for which $U(KG)$ satisfies a semigroup identity, proving this way Hartley's conjecture in the case of semigroup identities. Recall that if w_1, w_2 are distinct words of a free semigroup, then $w_1 = w_2$ is a semigroup identity for a group U if it holds for every substitution of the variables by elements of U. Clearly semigroup identities are special instances of group identities and a result of Malcev [14] implies a sufficient condition for a group to satisfy a semigroup identity. As a consequence, nilpotent-by-finite groups satisfy a semigroup identity.

Another result in [11] related to this conjecture is the following

Theorem 6 *Let K be an infinite field, G a torsion-generated group and Z its center. If $U(KG)$ satisfies a semigroup identity, then*

1. *if $charK = 0$, G is abelian;*

2. *if $charK = p > 0$, G/Z is a p-group of bounded exponent.*

In the setting of semigroup identities and integral group rings we have to record the following theorem of Dokuchaev and Goncalves [6]. We say that a group G satisfies a semigroup identity-by-torsion if G is an extension of a group satisfying a semigroup identity by a torsion group.

Theorem 7 *If $\mathbb{Z}G$ satisfies a semigroup identity-by-torsion and $T(G)$ is a subgroup of G, then every subgroup of $T(G)$ is normal in G and $T(G)$ is either abelian or a Hamiltonian 2-group.*

Hartley's conjecture has been recently proved over an infinite field for semiprime group algebras by Giambruno, Jespers and Valenti in [8] and in the general case by Giambruno, Sehgal and Valenti in [9].

For a group G, let $\phi(G)$ denote the FC-subgroup of G.

Theorem 8 *Let K be an infinite field. If G is a torsion group and $U(KG)$ satisfies a group identity, then KG satisfies a polynomial identity.*

Proof. Let N be the sum of all the nilpotent ideals of KG. The proof is essentially divided into three cases: 1) $N = 0$, i. e., KG is semiprime; 2) $N \neq 0$ is nilpotent; 3) N is a nil non-nilpotent ideal. Now, case 1) will be treated in the next section. By taking the quotient of G with the p-part of its FC-subgroup, case 2) can be deduced from case 1). We give the proof of case 3).

Suppose that there exists a nil non-nilpotent ideal I of KG. Let t be an indeterminate and consider $K\{X\}[[t]]$ the power series ring over the free algebra $K\{X\}$. For any n, the elements $1 + x_1t, \ldots, 1 + x_nt$ are units of $K\{X\}[[t]]$ and they generate a free group of rank n. Hence, if $w = w(x_1, \ldots, x_n)$ is the group identity satisfied by $U(KG)$, $w(1+x_1t, \ldots, 1+x_nt) \neq 1$. By writing out explicitly, we then obtain an expression of the form

$$\sum_{i \geq 1} p_i(x_1, \ldots, x_n)t^i \neq 0,$$

where for $i \geq 1, p_i(x_1, \ldots, x_n) \in K\{X\}$.

Since the above expression is non-zero, there exists $m \geq 1$ such that $p_m(x_1, \ldots, x_n) \neq 0$; notice that $p_m(x_1, \ldots, x_n) \in K\{X\}$ is a homogeneous polynomial of degree m.

Let now $\lambda \in K$ and $r_1, \ldots, r_n \in I$; the elements $1 + r_i\lambda$ are invertible in KG with inverse

$$(1 + r_i\lambda)^{-1} = (1 - r_i\lambda + r_i^2\lambda^2 - \cdots).$$

It follows that if we now evaluate $w(x_1,\ldots,x_n)$ on the elements $1+r_1\lambda,\ldots,1+r_n\lambda$ we get

$$\sum_{i=1}^{k} p_i(r_1,\ldots,r_n)\lambda^i = 0,$$

for some $k \geq 1$ and $p_l(r_1,\ldots,r_n) = 0$ for all $l > k$. Now, since K is an infinite field, a Vandermonde determinant argument implies that $p_1(r_1,\ldots,r_n) = \ldots = p_k(r_1,\ldots,r_n) = 0$. Thus $p_m(r_1,\ldots,r_n) = 0$ in any case.

We have proved that $p_m(x_1,\ldots,x_n)$ is a polynomial identity for the ideal I. But then by [20] I satisfies a multilinear polynomial identity $f(x_1,\ldots,x_m)$. Since I is not nilpotent there exist elements $a_1,\ldots,a_m \in I$ such that $a_1\cdots a_m \neq 0$. It follows that $f(a_1x_1,\ldots,a_mx_m)$ is a non-degenerate multilinear generalized polynomial identity for KG (see [17, pag 201]). But then by a theorem of Passman [17, Theorem V.3.15], $[G : \phi(G)] < \infty$ and $\mid \phi(G)' \mid < \infty$. The proof is now completed by showing that $\phi(G)'$ is a p-group where $p = \mathrm{char}K$ and then by applying Theorem 1. □

Actually in [9] it was proved that if G is torsion and KG is semiprime (for instance if $\mathrm{char}K = 0$) then a group identity in $U(KG)$ forces G to be abelian. In [18] Passman has pushed further the results in [9] obtaining a characterization of the group algebras whose unit group satisfies a group identity. The final theorem is the following (compare with Theorem 4)

Theorem 9 *Let KG be the group algebra of a torsion group G over an infinite field K.*

1. *If $\mathrm{char}K = 0$, $U(KG)$ satisfies a group identity if and only if G is abelian.*
2. *If $\mathrm{char}K = p > 0$, $U(KG)$ satisfies a group identity if and only if G has a normal p-abelian subgroup of finite index, and G' is a p-group of bounded period if and only if $U(KG)$ satisfies $(x,y)^{p^k} = 1$ for some $k \geq 0$.*

4 The general case

The following result concerning the group of units of a general ring was proved in [8]

Theorem 10 *Let R be an algebra over an infinite commutative domain and suppose that the group of units $U(R)$ satisfies a group identity of degree d. Then there exists a positive integer $m = m(d)$, such that if $a, b, c \in R$ and $a^2 = bc = 0$ then $bacR$ is a nil right ideal of R of exponent bounded by m.*

An easy consequence of this theorem is the following

Corollary 11 *Let R be a semiprime algebra over an infinite commutative domain A. If $U(R)$ satisfies a group identity then every idempotent of $A^{-1}R$ is central.*

Recall that semiprime group algebras were characterized by Passman (see [17, Theorem 2.12 and Theorem 2.13]): if char$K = 0$, KG is always semiprime; if char$K = p > 0$, KG is semiprime if and only if $\phi(G)$ is a p'-group.

The following theorem [8] can now be easily deduced (compare with Theorem 3 and Theorem 5).

Theorem 12 *If $U(\mathbb{Z}G)$ satisfies a group identity then*

1. *$T(G)$ is a subgroup of G which is abelian or a Hamiltonian 2-group;*

2. *every subgroup of $T(G)$ is normal in G.*

The converse holds provided $G/T(G)$ is a nilpotent group.

Proof. Let $g \in T(G)$ be of order n; by the previous corollary the element $(1+g+g^2+\cdots+g^{n-1})/n$ is a central idempotent of the rational group algebra $\mathbb{Q}G$. It follows that the cyclic group generated by g is normal in G. This proves 2) and $T(G)$ is either an abelian or a Hamiltonian group. In case $T(G)$ is a Hamiltonian group, then by applying [12, Theorem 1] to a finite non-abelian subgroup of $T(G)$, it follows that $T(G)$ must be a 2-group.

To prove the converse, since $G/T(G)$ is nilpotent, as in [21, Theorem VI 4.8] it follows that $U(\mathbb{Z}G) = U(\mathbb{Z}T(G))G$. Since the units of the integral group ring of a Hamiltonian 2-group are trivial, the result follows. □

The corresponding result for semiprime group algebras KG was proved in [8] and [9]; for simplicity we restrict ourselves to the case of characteristic different from 2.

Theorem 13 *Let K be an infinite field, char$K \neq 2$. If KG is semiprime and $U(KG)$ satisfies a group identity, then $T(G)$ is an abelian group and*

1. *If char$K = 0$, every subgroup of $T(G)$ is normal in G.*

2. *If char$K = p > 0$, every p'-subgroup of $T(G)$ is normal in G.*

Proof. Let $g \in T(G)$ be of order n and suppose, in case char$K = p > 0$ that $p \nmid n$. As in the previous proof we deduce that $\langle g \rangle$, the cyclic subgroup generated by g, is normal in G. This proves 1) and 2).

We are left with proving that $T(G)$ is abelian. If char$K = p$, the p'-elements form a subgroup Q of G. To simplify the terminology, in case char$K = 0$, we write $T(G) = Q$. Since every subgroup of Q is normal, Q is abelian or Hamiltonian. If Q is Hamiltonian, then Q and, so, G would contain H, the quaternion group of order 8. But KH has a summand which is either a quaternion division algebra or the algebra of 2×2 matrices over K. By [10] then $U(KG)$ would contain a free group of rank > 1, which is impossible. Thus Q is abelian. In characteristic zero this says that $T(G)$ is abelian and we are done. Thus we may assume that char$K = p > 0$.

Let now $g, h \in T(G)$ be two p-elements and let

$$\widehat{h} = 1 + h + h^2 + \cdots + h^{o(h)-1}.$$

Since $1-g$ is nilpotent and $\widehat{h}(1-h) = 0$, by [9, Lemma 2.1] $\widehat{h}(1-g)(1-h) = 0$. Hence $\widehat{h}gh = \widehat{h}g$ and it follows that for some $i > 0, g = h^i gh$ and, so, $ghg^{-1} = h^{-i}$. This says that

$$P = \{g \in G \mid o(g) = p^k, \text{ for some } k\}$$

is a subgroup of G and every subgroup of P is normal in P. Thus P is abelian or Hamiltonian. Since P is a p-group and $p \neq 2$, P must be abelian.

Take now $g \in Q$ and $h \in P$. Since $\langle g \rangle \triangleleft G$, it follows that $H = \langle g, h \rangle$ is a finite subgroup of G. By [9, Lemma 2.3] H is p-abelian, so the commutator (g, h) is a p-element in $\langle g \rangle$. Hence $(g, h) = 1$ and $gh = hg$. It follows that $T(G) = PQ$ is abelian. □

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