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Units in Integral Group Rings - A Survey [1]

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Abstract: This is a short survey on units in integral group rings. It covers partially work done after 1992.

Key words: group rings, torsion units, conjectures.

1 Introduction

Let G be a group and denote by $\mathcal{U}_1\mathbf{Z}G$ the group of units of augmentation one of the integral group ring $\mathbf{Z}G$. Given an element $u \in \mathcal{U}_1\mathbf{Z}G$ we set $T^{(k)}(u) = \sum\limits_{0(g)=k} u(g)$ and $\tilde{u}(g) = \sum\limits_{h\sim g} u(h)$.

We recall some important conjectures of A. A. Bovdi and H. J. Zassenhaus.

BC1: Let G be a group and $u \in \mathcal{U}_1\mathbf{Z}G$ an element of order p^n, p a prime. Then $T^{(p^j)}(u) = 0$, if $j \neq n$ and $T^{(p^n)}(u) = 1$.

BC: Let G be a group and $u \in \mathcal{U}_1\mathbf{Z}G$ an element of order n. Then $T^{(k)}(u) = 0$, if $k \neq n$, and $T^{(n)}(u) = 1$.

ZC1: Let G be a finite group and $u \in \mathcal{U}_1\mathbf{Z}G$ a torsion unit. Then u is conjugate in $\mathbb{Q}G$ to an element of G.

ZC3: Let G be a finite group and $U < \mathcal{U}_1\mathbf{Z}G$ a finite subgroup. Then U is rationally conjugate to a subgroup of G.

BC1 is the original statement of Bovdi's conjecture. These conjectures and others are the inspiration for much of the work that is currently being done in the theory of group rings. The text in **[11]** and **[21]** are good references for the interested reader. This survey covers partially work done after 1992 and originated in a talk given at a meeting held in August 1995.

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2 The Conjectures of Bovdi

In this section we give an account of what is known of BC and BC1. The interested reader can see **[2]**, **[5]**, **[7]** and **[8]** for detailed proofs. We begin with some results which are useful to produce an induction argument.

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Lemma 2.1 *Let G be a finite group and $H \lhd G$ a subgroup of G. Let $\psi : \mathbf{Z}G \to \mathbf{Z}(G/H)$ be the natural projection and let $u \in \mathcal{U}_1\mathbf{Z}G$ be such that $(o(u), |H|) = 1$. If $\beta = \psi(u)$ then $T^{(k)}(u) = T^{(k)}(\beta)$ for every positive integer k such that $(k, |H|) = 1$ and $T^{(k)}(u) = 0$ if $(k, |H|) \neq 1$.*

Proof: Suppose that $(k, |H|) = 1$. Set:

$$\begin{aligned} S &= \{g \in G : o(\psi(g)) = k\} \\ S_1 &= \{g \in S : o(g) > k\} \end{aligned}$$

Note that if $g \in G$ is such that $(o(g), |H|) = 1$ then $o(g) = o(\psi(g))$. Also if $(o(g), |H|) \neq 1$ then $\widetilde{u}(g) = 0$ by [**21**, Lemma 38.11]. Hence, $\tilde{u}(g) = 0$ for all $g \in S_1$. Since S_1 is a normal subset of G we have that $\sum_{g \in S_1} u(g) = 0$. Using these facts we have:

$$\begin{aligned} T^{(k)}(\beta) &= \sum_{o(\psi(g))=k} u(g) = \sum_{g \in S} u(g) = \sum_{o(g)=k} u(g) + \sum_{g \in S_1} u(g) \\ &= \sum_{o(g)=k} u(g) = T^{(k)}(u) \end{aligned}$$

The second part follows by [**21**, Lemma 38.11] and the fact that $G(k)$, the set of elements of order k, is a normal subset of G. □

Lemma 2.2 *Let p be a prime rational integer and G a finite group. Suppose that G contains a unique subgroup H of order p. Let $u \in \mathcal{U}_1\mathbf{Z}G$ be such that $o(u) = p^n$. Then, with the notation of Lemma 2.1, we have that $T^{(p^{j+1})}(u) = T^{(p^j)}(\beta)$ for $j \geq 1$ and $T^{(p^1)}(u) \in \{0, 1\}$.*
In particular if **BC1** *holds for G/H then* **BC1** *holds for G.*

Proof: Let $g \in G$ be an element of order p^{j+1}. If $j = 0$ then this follows from a well-known theorem of D.S. Bermans. So, suppose that $j > 0$. Then $g^{p^j} \in H$, by the uniqueness of H. Hence $o(\psi(g)) = p^j$. Also if $o(\psi(g)) = p^j$ then $p^j \in H \setminus \{1\}$. Hence $o(g) = p^{j+1}$. Using these facts we have that

$$T^{(p^j)}(\beta) = \sum_{o(\psi(g))=p^j} u(g) = \sum_{o(g)=p^{j+1}} u(g) = T^{(p^{j+1})}(u) .$$

The second statement is a consequence of the first part and Lemma 2.1. □

Theorem 2.3 *Let G be a finite solvable group and $u \in \mathcal{U}_1\mathbf{Z}G$ an element of order p^n. Suppose that G contains an abelian Sylow p-subgroup. Then $T^{(p^j)}(u) \in \{0, 1\}$.*

Lemma 2.4 *Let G be a noetherian group containing a normal torsion-free subgroup H. If $u \in \mathcal{U}_1\mathbf{Z}G$ is a torsion element then, with the notation of Lemma 2.1, we have that $T^{(k)}(u) = T^{(k)}(\beta)$. In particular* **BC1** *holds for G if it holds for G/H.*

Proof: Let $g \in G$ be an element of finite order. We set $\overline{g} = \psi(g)$ *and* $\overline{G} = \psi(G)$. Then, since H is torsion free, $o(\overline{g}) = o(g)$. Hence we have that $\psi^{-1}(\overline{G}(k)) = G(k) \cup \{g \in G : o(g) = \infty, o(\overline{g}) = k\}$. Now $S = \{g \in G : o(g) = \infty, o(\overline{g}) = k\}$ is a normal subset of G and hence it is a disjoint union of conjugacy classes. So, by [**21**, 47.5], $\sum_{g \in S} u(g) = 0$ and thus $T^{(k)}(\beta) = T^{(k)}(u)$. □

Corollary 2.5 *Let $\mathcal{F}$ be a family of finite groups satisfying BC (or BC1). Let G be a polycyclic-by-finite group and suppose that G has a normal torsion free subgroup H such that G/H is in $\mathcal{F}$. Then BC (or BC1) holds for G.*

If $\mathcal{F}$ is the family of nilpotent groups then BC holds for G. This is because ZC3 is true for finite nilpotent groups [**23**]. Note that G itself needs not be nilpotent. In fact, this is a large family of groups including that of nilpotent groups (see also [**1**]). The Lemma also gives us an inductive argument.

Theorem 2.6 *BC (or BC1) holds for polycyclic-by-finite groups if and only if it holds for finite groups.*

Proof: By [**17**, Theorem 10.2.5] G has a normal torsion free subgroup of finite index. Hence the result follows. □

The Theorem shows that, at least for polycyclic-by-finite groups, if one wants to find new families of groups which satisfy BC (or BC1) one must look for new families of finite groups which satisfy these conjectures. See also [9] for a weaker version of ZC1.

The following result is easy to prove.

Lemma 2.7 *Let H be an abelian Sylow p-subgroup of a finite solvable group G. Then one of the following holds:*
i) $H \lhd G$
ii) $O_{p'}(G) \neq 1$.

Theorem 2.8 *Let G be a finite nilpotent-by-nilpotent group. Then BC1 holds for G. In particular BC1 holds for finite supersoluble and finite metabelian groups.*

Sketch of the Proof: This follows by induction on $|G|$, Lemma 2.1 and [**21**, 41.12]. □

As a consequence we have the following result.

Theorem 2.9 *Let G be a polycyclic-by-finite group. If G is nilpotent-by-nilpotent then BC1 holds for G. In particular BC1 holds for supersoluble groups.*

Using the above results we can prove BC1 for other families of groups.

Theorem 2.10 *BC1 holds for the following groups:*

1. *Frobenius Groups.*
2. *Solvable groups whose Sylow subgroups are abelian or generalized quaternion groups.*
3. *Groups whose order is not divisible by the fourth power of any prime.*

For BC we have:

Theorem 2.11 *BC holds for metabelian groups.*

We finish this section mentioning a result which settles another conjecture of Bovdi.

Theorem 2.12 *Let $n = \exp(G/\mathcal{Z}(G))$ be finite, where $\mathcal{Z}(G)$ denotes the center of G. If $u \in \mathcal{U}_1\mathbf{Z}G$ is a torsion unit and m is the smallest positive integer such that $u^m \in G$, then m divides n.*

3 The Conjectures of Zassenhaus

These conjectures have been established for various kinds of groups, although they remain open in general. The most far reaching result is due to A. Weiss which shows that ZC3 is true for finite nilpotent groups. ZC1 was proved for metacyclic groups $G = \langle a \rangle \rtimes \langle x \rangle$, with a, x of coprime order, by C. Polcino Milies, J. Ritter and S. K. Sehgal. A. Valenti has shown ZC3 for groups of the form $G = \langle a \rangle \rtimes X$ with X abelian and the orders of a and X are relatively prime. It is interesting to know that K. W. Roggenkamp and L. Scott found a counterexample (in fact a whole family) to ZC3 and later L. Klinger found another counterexample. ZC1 has also been established for some isolated groups. For example N. Fernandes proved it for S_4, I. S. Luthar and P. Trama for S_5 and I. S. Luthar and I. B. S. Passi for A_5 (see references).

One might think that proving ZC1 or ZC3 for just one specific group is not particularly interesting but, as we shall see shortly, this is indeed very important. In fact some results of the previous section depend on these particular ones.

In this section we concentrate mostly on a p-subgroup version of ZC3. The interested reader should consult [**3**], [**4**] and [**7**] for detailed proofs.

p-ZC3: *If $H < \mathcal{U}_1\mathbf{Z}G$ is of prime power order then H is rationally conjugate to a subgroup of G i.e. there exists a unit $\alpha \in \mathbb{Q}G$ such that $\alpha^{-1}H\alpha \subset G$.*

In particular, if p-ZC3 is true for a group G, then any Sylow subgroup of $\mathcal{U}_1\mathbf{Z}G$ is rationally conjugate to a subgroup of G. Conjugation of those Sylow subgroups of $\mathcal{U}_1\mathbf{Z}G$ which can be embedded into a group basis was investigated in [**10**], [**11**]. We begin with some reduction results.

Let N be a normal subgroup of G, and let $\overline{G} = G/N$, $\Psi : \mathbf{Z}G \to \mathbf{Z}(G/N)$ be the natural map, ; set $\overline{g} = \Psi(g)$ for $g \in G$.

Theorem 3.1 *Let H be a finite subgroup of $\mathcal{U}_1\mathbf{Z}G$ such that $(|H|,|N|) = 1$ and G_0 be a subgroup of G with $(|G_0|,|N|) = 1$. Then H is rationally conjugate to G_0 if and only if $\Psi(H)$ is conjugate to $\Psi(G_0)$ in $\mathbb{Q}\overline{G}$.*

Sketch of the Proof: We only have to prove the converse. Let $\overline{H} = \Psi(H)$ and $\overline{G}_0 = \Psi(G_0)$. Let $\gamma^{-1}\overline{H}\gamma = \overline{G_0}$ for some $\gamma \in \mathbb{Q}\overline{G}$, $\alpha \in H$ and β be as above. We see that $h_\alpha = \gamma^{-1}\beta\gamma$ is, up to conjugacy, the unique element of $\overline{G}$ with $\widetilde{\beta}(h_\alpha) \neq 0$. From [**21**, Lemma 38.11] it follows that $(o(h_\alpha),|N|) = 1$ and the Schur-Zassenhaus Theorem shows that we can choose $g_\alpha \in G$ such that $h_\alpha = \Psi(g_\alpha)$ and $(o(g_\alpha),|N|) = 1$. Then it follows from [**21**, Lemma 38.11] and [**3**, Lemma 2.1] that, up to conjugacy, g_α is the unique element of G with $\tilde{\alpha}(g_\alpha) \neq 0$. Since $(|G_0|,|N|) = 1$, the restriction of Ψ to G_0 gives an isomorphism between G_0 and $\overline{G}_0$. Denote by Ψ_1 the inverse of this isomorphism and define a homomorphism $\phi : H \longrightarrow G_0$ by setting $\phi(\alpha) = \Psi_1(\gamma^{-1}\beta\gamma)$. Since $(o(\phi(\alpha)),|N|) = 1$, [**3**, Lemma 2.1] implies that $\widetilde{\alpha}(\phi(\alpha)) = \widetilde{\beta}(\Psi\phi(\alpha)) = \widetilde{\beta}(h_\alpha) \neq 0$ and $\phi(\alpha)$ is conjugate to g_α. It follows by [**21**, Lemma 41.4] that H is rationally conjugate to G_0. □

Remark: We have proved that if $H < \mathcal{U}_1\mathbf{Z}G$ and $(|H|,|N|) = 1$ then Ψ is injective on H.

Corollary 3.2 *Let N be a normal subgroup of G and H be a finite subgroup in $1 + \Delta(G,N)$. If p is a prime which divides $|H|$ then p divides $|N|$. In particular, if N is a Hall subgroup of G then $|H|$ divides $|N|$.*

Proof: We already know that $|H|$ is a divisor of $|G|$. Suppose that there exists a rational prime p that divides $|H|$ and does not divide $|N|$. Let $\alpha \in H$ be a unit of order p. By the Remark α is not mapped to 1 in $\mathbf{Z}(G/N)$, a contradiction. □

Theorem 3.3 *Let $G = N \rtimes X$, where the orders of N and X are relatively prime. Then any finite subgroup H of $\mathcal{U}_1\mathbf{Z}G$ such that $(|H|,|N|) = 1$ is rationally conjugate to a subgroup of $\mathcal{U}_1\mathbf{Z}X$.*

Sketch of the Proof: For $\alpha \in H$ we write $\alpha = vw$ with $v \in \mathcal{U}(1+\Delta(G,N))$ and $w \in \mathcal{U}_1\mathbf{Z}X$. By [**3**, Lemma 2.5] the isomorphism $\alpha = vw \in H \mapsto w$ satisfies the hypothesis of [**3**, Lemma 2.6]. Hence H is conjugate to H_0 in $\mathbb{Q}G$, where H_0 is the image of H in $\mathcal{U}_1\mathbf{Z}X$. □

Proposition 3.4 *Let P be an abelian Sylow p-subgroup of a solvable group G. If H is a finite p-subgroup of $\mathcal{U}_1\mathbf{Z}G$ then H is rationally conjugate to a subgroup of G.*

Proof: By [**21**, Theorem 41.12] we may assume that P is not normal in G. It follows from Lemma 2.7 that $N = O_{p'}(G) \neq 1$. Since the factor group G/N satisfies our hypothesis we can use Theorem 3.1 and induction to conclude that H is rationally conjugate to a subgroup of G. □

Theorem 3.5 *ZC3 holds for S_4 and for the Binary Octahedral Group.*

Theorem 3.6 *Let G be a nilpotent-by-nilpotent group. Then p-ZC3 holds for G. In particular p-ZC3 holds for finite metabelian and supersoluble groups.*

Proof: Let U be a p-subgroup of $\mathcal{U}_1\mathbf{Z}G$ and H be a normal nilpotent subgroup of G so that G/H is nilpotent. If H is not a p-group, then G possesses a normal p'-subgroup N. It follows from Theorem 3.1 and induction on the order of G that U is conjugate in $\mathbb{Q}G$ to a subgroup of G. If H is a p-group, then the Sylow p-subgroup of G is normal and [**21**, Lemma 41.12] implies that U is rationally conjugate to a subgroup of G. □

Theorem 3.7 *Let G be a solvable group such that Sylow subgroups of G are either abelian or generalized quaternion. Then G satisfies p-ZC3.*

Sketch of the Proof: Let U be a finite p-subgroup of $\mathcal{U}_1\mathbf{Z}G$. In view of Proposition 3.4 we may assume that $p = 2$ and the Sylow 2-subgroups of G are generalized quaternion. If the Fitting subgroup F of G is not a 2-group, then G contains a non-trivial normal subgroup N of odd order. Since the factor group G/N satisfies the assumption of the theorem we use Theorem 3.1 and induction on $|G|$.

Let F be a 2-group. One can argue that G can be supposed to be isomorphic to the Binary Octahedral Group. Hence Theorem 3.5 gives us the final conclusion. □

We can now prove the following result.

Theorem 3.8 *p-ZC3 holds for the following groups:*

1. *Soluble Frobenius groups.*

2. Groups whose order is not divisible by the fourth power of any prime.

Sketch of the Proof: Let G be a finite soluble Frobenius group. By [**20**, 10.5.6] $G = N \rtimes X$ where N is nilpotent, $(|N|, |X|) = 1$ and the Sylow p-subgroups of X are either abelian or generalized quaternion. The result follows now easily.

Suppose now that $|G|$ is not divisible by the fourth power of any prime. If $F = Fit(G)$ is not of prime power order we apply induction and Theorem 3.1. If it is of prime power order then one can show that G must be isomorphic to S_4 and hence Theorem 3.5 gives us the final conclusion. □

We want to mention that the case in which G is a non-soluble Frobenius group is considered in [**4**]. In that paper ZC3 is also proved for S_5, $SL(2,5)$ and A_5. Finally we point out that p-ZC3 implies a positive solution of [**21**, Problem 32]. Note that Corollary 3.2 already gives a partial solution.

Proposition 3.9 *Let N be a normal subgroup of a group G which satisfies p-ZC3. If U is a finite subgroup of $\mathcal{U}(1 + \Delta(G, N))$ then $|U|$ divides $|N|$.*

Proof: Let U_p be a Sylow p-subgroup of U. By p-ZC3, U_p is rationally conjugate to a subgroup N_p of G. Going down modulo N we see that $N_p \subset N$. Hence $|U_p|$ divides $|N|$, and consequently $|U|$ divides $|N|$. □

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