Resenhas IME-USP 1996, Vol. 2, No. 3, 309 - 312.

On free groups in the unit group of integral group rings [1]

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Abstract: We present a short history of the following problem: Classify the finite groups G, so that the group of units of the integral group ring ZG contains a subgroup of finite index isomorphic to a direct product of nonabelian free groups.

Key words: Unit groups, Group rings.

Let G be a finite group and $U(\mathbf{Z}G)$ the group of units of the integral group ring $\mathbf{Z}G$. There is great interest in describing $U(\mathbf{Z}G)$ up to finite index in various ways (see [9]). One point of view is the one in [9] where the following question is considered: When does some class of units (e.g. trivial, cyclic and/or bicyclic) generate a subgroup of finite index of $U(\mathbf{Z}G)$? Another point of view could be to consider the following question: When is there a subgroup H of finite index of $U(\mathbf{Z}G)$ which belongs to a "nice" class of groups (free, abelian,...)? This short note is concerned with this second point of view, where the "nice" class of groups is the one formed by the direct products of free groups; i.e., we shall consider the following problem: When does $U(\mathbf{Z}G)$ contain a subgroup of finite index which is isomorphic to a direct product of (nonabelian) free groups.

Of course a previous question would be the following: When does $U(\mathbf{Z}G)$ contain a nonabelian free group. The classical results by G. Higman [5] and B. Hartley and P.F. Pickel [4] give an answer to this question.

The cyclic group with n elements is denoted by C_n. The dihedral (resp. quaternion) group of order $2m$ is denoted D_{2m} (resp. Q_{2m}).

Theorem 0.1 (Higman [5]) *For a finite group G, we have that* $\mathrm{U}(\mathbf{Z}G) = \pm G$ *if and only if either G is abelian of exponent* $2, 3, 4$ *or* 6 *or then* $G \simeq Q_8 \times C_2^{(k)}$ *for some* $k \geq 0$.

Theorem 0.2 (Hartley-Pickel [4]) *Assume that G is nonabelian. Then* $\mathrm{U}(\mathbf{Z}G)$ *contains a noncommutative free group if and only if G is not a Hamiltonian 2-group.*

This two theorems show that either $U(\mathbf{Z}G)$ is finite or it contains a nonabelian free group. Next we recall some results which show several examples of groups G such that $U(\mathbf{Z}G)$ contains a nonabelian free subgroup of finite index.

Theorem 0.3 (Jespers-Leal [1]) *The bycyclic units of* $\mathrm{U}(\mathbf{Z}D_8)$ *generate a free subgroup W of rank* 3 *which is a normal complement of* D_8.

[1] Research partially supported by DGICYT (PB93-0515-C02-02) and Comunidad Autónoma de Murcia (PIB 95/61).

Theorem 0.4 (Jespers-Leal [1]) *Let* $P = \langle a, b \mid a^4 = b^4 = 1, [a, b] = a^2 \rangle$. *Then* P *has a normal complement which is a free group of rank* 9.

Theorem 0.5 (Jespers-Parmenter [3]) *The byclic units of* $\mathrm{U}(\mathbf{Z}S_3)$ *generate a free subgroup* W *of rank* 3 *which is a normal complement of* S_3.

Theorem 0.6 (Parmenter [7]) *Let* $T = \langle a, b \mid a^6 = 1, b^2 = a^3, ba = a^5 b \rangle$. *Then* T *has a normal complement which is a free group of rank* 5.

The first general result is due to E. Jespers, G. Leal and A. del Río [2] who classified the nilpotent groups G such that $U(\mathbf{Z}G)$ contains a subgroup of finite index which is isomorphic to a direct product of (nonabelian) free groups by means of the Wedderburn decomposition of $\mathbf{Q}G$.

We need some notation.

For every $n \geq 1$, W_n denotes the group given by the following presentation:

$$W_n = \langle x_1, x_2, \ldots, x_n \mid x_i^4 = [x_i, [x_j, x_k]] = [x_i, x_j^2] = 1, 1 \leq i, j, k \leq n \rangle$$

For every $1 \leq i, j \leq n$, we denote $t_{ij} = [x_i, x_j] \in W_n$.

Theorem 0.7 (Jespers-Leal-del Río [2]) *For any nonabelian nilpotent finite group* G *the following conditions are equivalent:*

1. $U(\mathbf{Z}G)$ *contains a subgroup of finite index which is isomorphic to a finite direct product of noncyclic free groups.*
2. *For every primitive central idempotent* e *of* $\mathbf{Q}G$, *either* $\mathbf{Q}Ge$ *is a commutative field or, it is isomorphic to either the ring of rationl quaternions* $\mathbf{H}(\mathbf{Q})$ *or to* $M_2(\mathbf{Q})$;
3. $G \simeq H \times C_2^k$ *for some* $k \geq 0$ *and* H *is isomorphic to exactly one of the following groups*
   (a) W_2.
   (b) $W_2/\langle x_1^2 \rangle$.
   (c) $W_2/\langle x_1^2 t_{12} \rangle$.
   (d) $W_2/\langle x_1^2, x_2^2 \rangle$.
   (e) $W_2/\langle x_1^2 t_{12}, x_2^2 t_{12} \rangle$.
   (f) $G^1_{2^{2n}} = W_n/\langle t_{ij} \mid 2 \leq i < j \leq n \rangle \times \langle x_i^2 \mid 2 \leq i \leq n \rangle$ $(n \geq 3)$.
   (g) $G^2_{2^{2n}} = W_n/\langle t_{ij} \mid 2 \leq i < j \leq n \rangle \times \langle x_i^2 t_{1i} \mid 2 \leq i \leq n \rangle$ $(n \geq 3)$.
   (h) $G^1_{2^{2n-1}} = W_n/\langle t_{ij} \mid 2 \leq i < j \leq n \rangle \times \langle x_i^2 \mid 1 \leq i \leq n \rangle$ $(n \geq 3)$.
   (i) $G^2_{2^{2n-1}} = W_n/\langle t_{ij} \mid 2 \leq i < j \leq n \rangle \times \langle x_i^2 t_{1i} \mid 2 \leq i \leq n \rangle \times \langle x_1^2 \rangle$ $(n \geq 3)$.
   (j) $G^3_{2^{2n-1}} = W_n/\langle t_{ij} \mid 2 \leq i < j \leq n \rangle \times \langle x_i^2 t_{1i} \mid 2 \leq i \leq n \rangle \times \langle x_1^2 t_{12} \rangle$ $(n \geq 3)$.

Note that $W_2/\langle x_1^2t_{12}, x_2^2t_{12}\rangle \simeq Q_8$, so that $U(\mathbf{Z}G)$ is finite. Moreover $W_2/\langle x_1^2, x_2^2\rangle \simeq D_8$ and $W_2/\langle x_1^2t_{12}\rangle \simeq P$ are the groups considered in Theorems 0.3 and 0.4. On the other hand this classification does not include S_3 and T (the group in Theorem 0.6), which are not nilpotent.

The work in [2] has been recently completed by Leal and del Río showing a similar result for non-nilpotent groups.

Following [8] we denote by $\left(\frac{a,b}{\mathbf{Q}}\right)$ the quaternion algebra defined by $a, b \in \mathbf{Q}$.

For an abelian group A and a cyclic group $\langle x\rangle$ of even order $A *_\sigma \langle x\rangle$ denotes the semidirect product defined by the action $\sigma : \langle x\rangle \to \mathrm{Aut}(A)$ given by $\sigma(x)(g) = g^{-1}$.

Theorem 0.8 (Leal-del Río [6]) *The following conditions are equivalent for a finite nonnilpotent group G.*

1. *$\mathcal{U}(\mathbf{Z}G)$ contains a subgroup of finite index which is isomorphic to a direct product of nonabelian free groups.*

2. *For every primitive central idempotent e of $\mathbf{Q}G$ the simple ring $\mathbf{Q}Ge$ is either a commutative field or isomorphic to $M_2(\mathbf{Q})$ or $\left(\frac{-1,-3}{\mathbf{Q}}\right)$.*

3. *G is isomorphic to $(C_3^m *_\sigma C_k) \times C_2^n$ for some $n \geq 0$, $m \geq 1$ $k = 2, 4$.*

Note that the groups considered in Theorems 0.5 and 0.6 are included here. Indeed, $C_3 *_\sigma C_2 \simeq S_3$ and $C_3 *_\sigma C_4 \simeq T$.

Theorems 0.7 and 0.8 suggest the following two problems:

Problem 1: Characterize all finite groups G such that $\mathcal{U}(\mathbf{Z}G)$ contains a subgroup of finite index which is isomorphic to a direct product of (abelian or nonabelian) free groups.

Problem 2: Characterize all finite groups G such that $\mathcal{U}(\mathbf{Z}G)$ contains a subgroup of finite index which can be constructed from (abelian or nonabelian) free subgroups and "basic" operations, such as semidirect (rather than direct) products..

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