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On Finite Irreducible Subgroups of GL(p,D) [1]

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Abstract: We outline the main steps in the classification of finite irreducible subgroups of $GL(p, D)$, where p is a prime and D is a division ring of characteristic zero.

Key words: Finite simple group, division ring, centre-by-finite group, irreducible representation.

1. Introduction, History, and Questions

What can one say about the structure of a finite group, given that it has a faithful irreducible representation of a given degree? In fact, there is more than one way of interpreting this question. The most comprehensive is via the classification of all finite subgroups of $GL(n, K)$, for suitable fields K (usually the complex numbers) for successive values of n. The most up-to-date work seems to be that of Feit [4], where all finite groups having a finite irreducible quasi-primitive complex representation of degree $8 \leq n \leq 10$ are classified. Since the earlier work of Blichfeldt, Brauer, Lindsey, and Wales (see Feit's article for references) had already covered the cases $n \leq 7$, the classification is now complete up to $n = 10$. The corresponding information, however, is much more limited for fields that are not algebraically closed.

A second approach involves the study of groups having a faithful irreducible complex representation whose degree is small compared to some prime divisor of the order of the group. The best result to date is that of Zhang Jiping [14], who classified such groups G if the degree of the representation is $\leq p-1$, where p is a prime dividing the order of the group (earlier work had been done by Brauer, Tuan, Ferguson, and Feit). The proof uses the classification of finite simple groups, as well as the well-known fact that the Sylow p-subgroups of G are abelian (for the particular prime p), and this fact is quite crucial in the proof. The result can be paraphrased by saying that G has a normal subgroup H such that $C_G(H) \subseteq H$, and H is a "known" group.

The above results make essential use of the fact that the underlying field is commutative and algebraically closed, although it also becomes apparent that the groups in the families do not resemble each other as one moves from dimension n to $n+1$. Now when dealing with small fields (such as the rationals), one has to worry about the *Schur index* of the representation as well as the matrix degree. Indeed, a simple component $D^{n \times n}$ of the rational group algebra of G corresponds to a family of (conjugate) complex representations of degree nm, where m is

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the corresponding Schur index. It turns out that, in this generality, the decisive factor influencing the structure of the group is the complexity of the degree n. In addition, one can use essentially the same techniques to study centre-by-finite groups having representations of particular degrees. This, of course, is the same as studying the projective representations of the central quotient group. Finite subgroups of $GL(1, D) = D^*$ were classified by Amitsur [1], so one may turn to finite subgroups of $GL(p, D)$, where D is a division ring of characteristic zero, and p is a prime.

Question 1. For what finite groups G is there a decomposition of the rational group algebra $\mathbf{Q}G = \ldots + D^{p\times p} + \ldots$? That is, what finite groups have at least one *rational reduced degree* equal to a prime?

Much is known about the structure of the groups if all the reduced degrees are powers of the same prime (see, for example, the work of Huppert and Gow in [8] and [9]), but little is known about the question which is posed here, except when $p = 2$, where the classification was done by Banieqbal [2], based on earlier work of Hartley and Shahabi [6] (which in turn depended on Gorenstein and Harada's classification of groups of sectional 2-rank at most 4).

We can make Question 1 more specific, as follows:

Question 2. If $k = Z(D^{p\times p})$ is the centre of $D^{p\times p}$, and G is a (centre-by-finite subgroup of $(D^{p\times p})$ such that $k[G] = D^{p\times p}$, then what can be said about the structure of G and/or D?

The project discussed in this paper aims, as a first step, at the classification of the finite groups G. In this case, the division rings D are well-known to belong to the *Schur subgroup* of the Brauer group of k, which was first determined by Yamada [13] for local and global fields k. If k is a local or a global field, then Lorenz and Opolka [10] show that every class of the Brauer group of k contains an algebra of the form $k[G] = D^{n\times n}$, where G is centre-by-finite. Note that the n here is not necessarily a prime. Not much is known about the structure of such G, even when $n = 1$, although one at least knows the structure of $G/\zeta_1(G)$ (where $\zeta_1(G)$ denotes the centre of G) (see [11], where the finite subgroups of $PGL(1, D)$ are classified).

For the rest of the paper, G is a finite subgroup of a central simple k-algebra $D^{p\times p} = k[G]$, (i.e., G contains a k-basis for the algebra). If N is any normal subgroup of G, then G acts transitively (by conjugation) on the simple components of $k[N]$ ([12], 1.1.6).

2. Clifford Theory and minimality

Suppose G has a normal subgroup N such that $S = k[N]$ is *not* a simple ring

Then S is a direct sum of p copies of D, permuted transitively by G, and so G is a subgroup of a permutational wreath product $H \; wr \; G_0$, where H is a subgroup of D^*, and G_0 is a primitive permutation group of degree p (so G_0 acts on the p copies of H through its action on the index set $\{1, \ldots, p\}$). The structure of the Amitsur groups H is well-known (Amitsur [1], or see [12], 2.1.4 and 2.1.5). As for G_0, we know that either G_0 is soluble or it is 2-transitive. In the first case, Galois already knew that $\mathbf{F}_p \leq G_0 \leq \mathbf{F}_p \mathbf{F}_p^*$ (e.g. [7], II.3.6), while the O'Nan-Scott Theorem together with the classification of finite simple groups implies that G_0 is one of the following (cf. [3], Theorem 5.3):

- $Alt(p)$, $Sym(p)$, $PSL(d,q)$ or $PGL(d,q)$ with $p = \frac{q^d-1}{q-1}$.
- $PSL(2,11)$, $PGL(2,11)$, or M_{11} when $p = 11$.
- M_{23} when $p = 23$.

It is not known whether there exist infinitely many pairs (q,d) for which $\frac{q^d-1}{q-1}$ is a prime.

It may be useful to mention the idea of minimality at this point, as it is useful in setting up inductive arguments. Suppose that G has a proper normal subgroup N such that $k[N] = k[G] = R$, say. Then it can be shown that there exists a subgroup $d_R(N)$ of R such that $N \leq G \leq d_R(N)$, N is normal in $d_R(N)$, and if $\mu(k)$ denotes the group of roots of unity in k^*, then $d_R(N)/\mu(k)N$ is a subgroup of the outer automorphism group $Out(N)$. Therefore, once one knows the structure of N and $k[N] = D^{p\times p}$, the structure of G may be effectively determined too. We say that G has a **minimal embedding** into $k[G] = D^{p\times p}$ if $k[N] \neq k[G]$ for every proper normal subgroup N of G.

We may now suppose, for the rest of the paper, that for every proper normal subgroup N of G, the ring $k[N]$ is simple and distinct from $k[G]$. Of course, this means that $k[N]$ has the form Δ or $\Delta^{p\times p}$ for some division ring Δ.

3. Components of G

Recall that a *component* of a finite group is a subnormal subgroup $X \neq 1$ such that $X = X'$ and $X/\zeta_1(X)$ is a simple group (so non-abelian).

In our situation, it can be shown that every component of G is actually normal in G, and so either $X = SL(2,5)$ generates a division ring, or $k[X] = \Delta^{p\times p}$ for some Δ. It is then trivial to see that G can have at most one component of each type (otherwise the normal subgroup $E(G)$ generated by all the components spans a matrix ring of degree divisible by p^2).

The next easiest case is when the Fitting subgroup of G is strictly greater than the centre of $E(G)$. This is because (if q is a prime dividing the order of $F = Fitt(G)$)

the Sylow q-subgroup Q of F can only generate simple rings of dimension a power of q. Therefore, Q is cyclic if $q \neq 2$ or p, Q is cyclic or quaternion if $q = 2$, and Q is extra-special of order p^3 if $q = p$ (supposing that p is an odd prime). These are of course immediate consequences of P. Hall's classification of p-groups all of whose characteristic abelian subgroups are cyclic ([7], III.13.10). It is then relativly easy (though very tedious) to classify G itself. This, in particular, gives the structure of all soluble G.

Finally, the case where $Fitt(G) = \zeta_1(E(G))$ has to be handled. This requires detailed information about the automorphism groups of covering groups of finite simple groups and their representations. The following examples may provide an indication of the sort of argument that is required.

4. EXAMPLES

Example 1. Let $p = 3$, and suppose G has a component $X = 2.J_2$, the covering group of the second Janko group. What can we say about G? It is known that the rational group ring of X has a simple component $\mathbf{Q}[X] = \Delta^{3\times 3}$, where Δ is the quaternion algebra $(\frac{-1,-1}{Q(\sqrt{5})})$ (see, e.g., Feit [5]). It follows from the structure of Δ that every cyclotomic field $\mathbf{Q}(\epsilon_n)$, where ϵ_n is a primitive n-th root of unity with $n \geq 3$, splits Δ. This immediately implies that $Fitt(G) = \zeta_1(X)$ has order 2, so $C_G(X) = \zeta_1(X)$. Therefore, $X \leq G \leq Aut(X)$. Since $Out(X)$ has order 2, it only remains to check whether $G = Aut(X)$ is a possibility. It is not, so $G = X$.

Example 2. Suppose $p = 23$, and G has a component $X = M_{24}$ (recall that the Schur multiplier of M_{24} is trivial). There is a ring $\mathbf{Q}[X] = \mathbf{Q}^{23\times 23}$. Since $Out(X) = 1$, it is immediate that $G = M_{24} \times H$, where H is an Amitsur group.

Example 3. Suppose G is a **Quasi-Amitsur** group, i.e., G is not an Amitsur group, but every proper characteristic subgroup of G can be embedded into a division ring. Such groups may, in a sense, be regarded as generalizations of p-groups all of whose characteristic abelian subgroups are cyclic. These groups have very special properties, and can be classified. Many (though not all) have faithful representations of prime degree over division rings of characteristic zero.

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