Resenhas IME-USP 1996, Vol. 2, No. 3, 333 - 346.

Matrix Algebras with Transpose or Symplectic Involution and their *-Polynomial Identities [1]

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Abstract: We look at the theory of $*$-polynomial identities of the algebra of $n \times n$ matrices over a field. The representation theory of the hyperoctahedral group and of the general linear group are applied for a quantitative study of the theory in characteristic zero. We examine the problem of determining $*$-polynomial identities of minimal degree for symplectic and transpose involution and new $*$-polynomial identities of degree $2n-1$ are constructed.

Key words: matrices, involution, polynomial identities.

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1 Generalities

Let F be a field of characteristic different from 2, $X = \{x_1, x_2, \ldots\}$ a countable set of unknowns and $F\{X, *\} = F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ the free algebra with involution $*$ over F. If R is an F-algebra with involution $*$, we shall consider only involutions such that $(\alpha a)^* = \alpha a^*$ for all $\alpha \in F, a \in R$. Recall that a polynomial $0 \neq f(x_1, x_1^*, \ldots, x_m, x_m^*)$ in $F\{X, *\}$ is a $*$-polynomial identity ($*$-PI) for R if $f(r_1, r_1^*, \ldots, r_m, r_m^*) = 0$ for all $r_1, \ldots, r_m \in R$.

If one wants to study the $*$-PI's of an algebra R as a whole, then the right concept is that of $*$-T-ideal i. e., an ideal of the free algebra $F\{X, *\}$ invariant under all endomorphisms of $F\{X, *\}$ that commute with the involution $*$.

The connection between $*$-T-ideals and $*$-PI's is the following: if R is an F-algebra with involution,

$$T(R, *) = \{f(x_1, x_1^*, \ldots, x_m, x_m^*) \in F\{X, *\} \mid f(x_1, x_1^*, \ldots, x_m, x_m^*)$$

[1] Research partially supported by MURST of Italy and FAPESP of Brazil.
AMS Classification: 16R50, 15A24.

is a $*-$PI for $R\}$

is a $*$-T-ideal of $F\{X,*\}$. Moreover, if J is a $*$-T-ideal, $T(F\{X,*\}/J) = J$, so every $*$-T-ideal of the free algebra is of this type.

Let now $R = M_n(F), n \geq 2$, be the algebra of $n \times n$ matrices over F. In $M_n(F)$ one can define several involutions; two of them play a very important role in the study of the $*$-PI's of $M_n(F)$: the transpose involution, denoted $* = t$, and the canonical symplectic involution, denoted $* = s$.

Recall that s is defined only in case $n = 2m$ is even and it is given by the rule: if $A \in M_n(F)$, let $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ where $B, C, D, E \in M_m(F)$ and set

$$A^s = \begin{pmatrix} E^t & -C^t \\ -D^t & B^t \end{pmatrix},$$

where t is the usual transpose.

Let us write $(M_n(F),*)$ for the ring of $n \times n$ matrices with the involution $*$. The importance of the above two involutions is given in the following ([17, Theorem 3.1.62])

Theorem 1.1 *Let F be an infinite field. If $*$ is an involution in $M_n(F)$, then either $T((M_n(F),*)) = T((M_n(F),t))$ or $T((M_n(F),*)) = T((M_n(F),s))$.*

Let us now give few examples of $*$-PI's for $M_n(F)$, for small values of n. They can all be checked by direct computation. Clearly every polynomial identity for $M_n(F)$ is a $*$-polynomial identity for $M_n(F)$. Let $S_m(x_1, \ldots, x_m)$ denote the standard polynomial of degree m.

Examples

1. $x_1 - x_1^* \in T((M_1(F),t)$.
2. $[x_1 - x_1^*, x_2 - x_2^*] \in T((M_2(F),t)$.
3. $[x_1 + x_1^*, x_2] \in T((M_2(F),s)$.
4. $[S_3(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*), x_4] \in T((M_3(F),t)$.
5. $[[x_1 + x_1^*, x_2 - x_2^*]^2, x_3] \in T((M_4(F),s)$.
6. $S_6(x_1 - x_1^*, \ldots, x_6 - x_6^*) \in T((M_4(F),t)$.

The $*$-PI's of $(M_2(F),*)$, $*=$ s or t, of minimal degree are well known; moreover, in characteristic zero, Levcenko in [10] exhibited a basis for the $*$-T-ideals $T(M_2(F),s)$ and $T(M_2(F),t)$. The result is the following

Theorem 1.2 *Let $charF = 0$.*

1. The $*$-polynomials

$$[(x-x^*)(y-y^*),z],$$

$$[x-x^*,y-y^*],$$

$$[x_1+x_1^*,x_2+x_2^*][x_3+x_3^*,x_4+x_4^*]+[x_2+x_2^*,x_3+x_3^*][x_1+x_1^*,x_4+x_4^*],$$

$$+[x_3+x_3^*,x_1+x_1^*][x_2+x_2^*,x_4+x_4^*],$$

$$[x-x^*,y+y^*,z-z^*,t+t^*]-4(x-x^*)(z-z^*)[t+t^*,y+y^*]$$

are a set of generators for the $*$-T-ideal $T(M_2(F),t)$.

2. The $*$-polynomial

$$[x+x^*,y]$$

generates the $*$-T-ideal $T(M_2(F),s)$.

2 S_n and GL-cocharacters

Methods of representation theory of the hyperoctahedral group and of the general linear group have been introduced in [4] and [5] to study $T(R,*)$ in general. We will now sketch these methods.

Let H_n be the hyperoctahedral group of degree n. If $C_2=\{1,*\}$ is the multiplicative group of order 2 and S_n is the symmetric group of degree n, then H_n is the wreath product C_2^n wr S_n and we write

$$H_n=\{(a_1,\ldots,a_n;\sigma)\mid a_i\in C_2,\sigma\in S_n\}$$

with multiplication defined by

$$(a_1,\ldots,a_n;\sigma)(b_1,\ldots,b_n;\tau)=(a_1b_{\sigma^{-1}(1)},\ldots,a_nb_{\sigma^{-1}(n)};\sigma\tau).$$

We say that a $*$-polynomial $f(x_1,x_1^*,\ldots,x_n,x_n^*)$ is multilinear if in every monomial of f, x_i or x_i^*, $i=1,\ldots,n$, appears exactly once. Then

$$V_n(*)=\mathrm{Span}_F\{x_{\sigma(1)}^{a_1}\cdots x_{\sigma(n)}^{a_n}\mid(a_1,\ldots,a_n;\sigma)\in H_n\}$$

is the space of multilinear $*$-polynomials in $x_1,x_1^*,\ldots,x_n,x_n^*$.

This space is strictly related to the group algebra of H_n; in fact the map

$$V_n(*)\to FH_n$$

given by

$$\sum_{(a;\sigma)\in H_n}\alpha_{(a;\sigma)}x_{\sigma(1)}^{a_{\sigma^{-1}(1)}}\cdots x_{\sigma(n)}^{a_{\sigma^{-1}(n)}}\to\sum_{(a;\sigma)\in H_n}\alpha_{(a;\sigma)}(a_1,\ldots,a_n;\sigma)$$

is an F- linear isomorphism of $V_n(*)$ onto FH_n. This map clearly induces a structure of S_n-module on $V_n(*)$. Let T be a $*$-T-ideal of $F\{X,*\}$. Then, under the above identification, $T_n = T \cap V_n(*)$ becomes a left ideal of FH_n.

Suppose char $F = 0$. Then every $*$-T-ideal is determined by its multilinear polynomials, hence to study T it is enough to study $\{T_n\}_{n\geq 1}$.

Actually it is more convenient to study the sequence of left H_n-modules $\{V_n(*)/T_n\}_{n\geq 1}$. Let us denote by $\chi_n(T,*)$ the H_n-character of $V_n(*)/T_n$ and let us call $\{\chi_n(T)\}_{n\geq 1}$ the sequence of H_n-cocharacters of T. Since every character $\chi_n(T)$ is a sum of irreducible H_n-characters, the problem of determining $\chi_n(T)$ is reduced to that of computing the multiplicities of each irreducible H_n-character in such decomposition.

In characteristic zero it is known that there exists a one-to-one correspondence between non-equivalent irreducible representations of H_n and pairs of partition (λ, μ) where λ is a partition of k, μ is a partition of $n-k$ and $k = 0, \ldots, n$. We write briefly $\mid \lambda \mid + \mid \mu \mid = n$. So, let us denote by $\chi_{\lambda,\mu}$ the irreducible H_n-character associated to the pair (λ, μ).

If T is a $*$-T-ideal of $*$-PI's of the algebra R, then we write $\chi_n(T) = \chi_n(R,*)$ and we have

$$\chi_n(R,*) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

where $m_{\lambda,\mu}$ is the multiplicity of $\chi_{\lambda,\mu}$ in the given decomposition.

If $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ is a partition of n, we call $r = h(\lambda)$ the height of λ ($h(\lambda)$ is the height of the corresponding Young diagram). We have

Theorem 2.1 *([4, Theorem 6.2]) Let* $r = \frac{k(k+1)}{2}$ *and* $u = \frac{k(k-1)}{2}$. *Then*

$$\chi_n(M_k(F), t) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leq r \\ h(\mu)\leq u}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

and

$$\chi_n(M_k(F), s) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leq u \\ h(\mu)\leq r}} m_{\lambda,\mu}\chi_{\lambda,\mu}.$$

We now turn to a description of the finite-dimensional polynomial representations of $GL(n) \times GL(n)$. This is intimately connected to partition of $n-k$ and $k = 0, \ldots, n$. We write briefly $\mid l1, 2, \ldots$ define $s_i = x_i + x_i^*$ and $k_i = x_i - x_i^*$; then, since $charF \neq 2$, $F\{X,*\} = F\{s_1, k_1, s_2, k_2, \ldots\}$ has a natural multigrading obtained by counting the degrees in the symmetric variables s_i and in the skew variables k_i. For a fixed n, let U and V be n-dimensional vector spaces over F with bases $\{s_1, \cdots, s_n\}$ and $\{k_1, \ldots, k_n\}$ respectively. Let

$$W = (U \oplus V) \otimes \cdots \otimes (U \oplus V) = (U \oplus V)^{\otimes n}.$$

W can be identified with the space of homogeneous $*$-polynomials of degree n in the variables x_i and x_i^*. The group $GL(U) \times GL(V) \equiv GL(n) \times GL(n)$ acts naturally on the space $U \oplus V$ and we extend this action diagonally to an action on W.

The representation theory of $GL(U) \times GL(V)$ acting on W is well known: there exists a one-to-one correspondence between irreducible non-equivalent polynomial representations of $GL(U) \times GL(V)$ and pairs of partitions (λ, μ) where λ is a partition of k, μ is a partition of $n-k$ and $k = 0, \ldots, n$. So, let us denote by $\psi_{\lambda,\mu}$ the irreducible character of $GL(U) \times GL(V)$ associated to the pair (λ, μ). Also, if M is a $GL(U) \times GL(V)$-module, let us write $\psi(M)$ for the character of M.

If T is a $*$-T-ideal, then $T \cap W$ is the space of homogeneous $*$-polynomials of degree n in $T \cap F\{x_1, x_1^*, \ldots x_n, x_n^*\}$ and $T \cap W$ is a $GL(U) \times GL(V)$-module. The $GL(U) \times GL(V)$ structure of $\frac{W}{T \cap W}$ and the H_n structure of $V_n(*)/T_n$ are related by the following result ([5, Theorem 3])

Theorem 2.2 *Let T be a $*$-T-ideal of $F\{X,*\}$ and $\psi_n(T,*)$ the $GL(U) \times GL(V)$-character of $\frac{W}{T \cap W}$. If*

$$\psi_n(T,*) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \psi_{\lambda,\mu}$$

and

$$\chi_n(T,*) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu} \chi_{\lambda,\mu}$$

then $m_{\lambda,\mu} = m'_{\lambda,\mu}$.

A quantitative study of $T((M_2(F), t))$ and $T((M_2(F), s))$ in characteristic zero was done by Drensky and Giambruno in [2]. They obtained among other things the exact values of the multiplicities in the cocharacter sequence of the $*$-PI's for the 2×2 matrices with symplectic or transpose involution.

3 $*$-Polynomial identities of minimal degree

In this section we want to discuss the following

Problem 3.1 *Find $*$-polynomial identities of minimal degree satisfied by $M_n(F)$.*

The standard polynomial

$$S_m(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} (sgn\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)}$$

plays a very important role in PI-theory.

Let $T(M_n(F))$ denote the T-ideal of ordinary identities (without involution) satisfied by $M_n(F)$. The Amitsur-Levitzki theorem shows that

$$T(M_1(F)) \supset T(M_2(F)) \supset T(M_3(F)) \supset \cdots$$

is a properly descending chain of T-ideals whose intersection is zero and also gives the least degree of a polynomial satisfied by $M_n(F)$.

Theorem 3.2 *(Amitsur-Levitzki)([17, Theorem1.4.1]) $S_{2n}(x_1, \ldots, x_{2n})$ is a polynomial identity for $M_n(F)$ of minimal degree and any other multilinear polynomial identity of $M_n(F)$ of degree $2n$ is a scalar multiple of S_{2n}.*

The original proof of Theorem 3.2 was a clever induction using matrix units but subsequently there have been several different proofs given by Kostant [11], based on the cohomology of $M_n(C)$ as a Lie algebra, by Swan [19], using graph theory, by Razmyslov [14] and Rosset [15] based on the Cayley-Hamilton Theorem.

The $*$-PI's of $M_n(F)$ of minimal degree do not seem to be well understood. In fact, the smallest possible degree of a $*$-PI for arbitrary n is not known. An easy argument shows that such degree must be at least n. The best lower bound is found in the following ([6, Theorem 1])

Theorem 3.3 *If f is a $*$-PI for $(M_n(F), *)$ and $n > 2$, then $deg(f) \geq n + 1$.*

It follows that, if f is a $*$-PI for $(M_n(F), *)$ of minimal degree and $n > 2$ then

$$2n \geq deg f > n.$$

By recalling that $F\{X, *\} = F\{s_1, k_1, s_2, k_2, \ldots\}$ then equivalently one can consider only polynomials in symmetric and skew variables.

About positive results, in [11] Kostant proved that if n is even then $S_{2n-2}(k_1, \ldots, k_{2n-2})$ is a $*$-PI for $(M_n(F), t)$. Rowen in [16] extended this result to arbitrary n and later in [18] he also proved that $S_{2n-2}(s_1, \ldots, s_{2n-2})$ is a $*$-PI for $(M_n(F), s)$.

No $*$-PI's of degree lower than $2n - 2$ are known for arbitrary n. Moreover, $2n-2$ is not minimal for all n. In fact, for instance, if $* = s$, then $[x_1+x_1^*, x_2+x_2^*]$ is a $*$-PI of minimal degree for $(M_2(F), s)$, and since $[s_1, s_2]^2$, $[s_1, k_2]^2$, when evaluated in $(M_4(F), s)$, take values in the center (see [3, pag 203]), it follows that $(M_4(F), s)$ has $*$-PI's of degree 5. Moreover if $* = t$, Racine constructed $*$-PI's of degree 5 for $M_4(F)$.

Definition 3.4 *For $m, r \geq 0$ define $w_i = \begin{cases} s_i & \text{if } i = 1, \ldots, m \\ k_i & \text{if } i = m+1, \ldots, m+r \end{cases}$ and let*

$$W_{m,r} = Span_F\{w_{\sigma(1)} \cdots w_{\sigma(m+r)} \mid \sigma \in S_{m+r}\}$$

be the space of multilinear polynomials in m symmetric variables and r skew variables.

The problem of determining the $*$-PI's of minimal degree can be reformulated in the following two problems

Problem 3.5 *For $m = 0, 1, \ldots, n$, find the least r such that*

$$W_{m,r} \cap T((M_n(F), *)) \neq 0$$

and exhibit a basis of this space.

Problem 3.6 *For $r = 0, 1, \ldots, n$, find the least m such that*

$$W_{m,r} \cap T((M_n(F), *)) \neq 0$$

and exhibit a basis of this space.

4 Transpose Involution

If $* = t$ it is easy to show that $S_{2n}(s_1, \ldots, s_{2n})$ is a $*$-PI for $(M_n(F), t)$ of minimal degree among $*$-PI's in symmetric variables.

It was shown in [12] by Ma and Racine that in general $*$-PI's of minimal degree need not resemble standard polynomials. In fact, they constructed a multilinear polynomial $f_{2n}(s_1, \ldots, s_{2n})$ such that any other homogeneous $*$-PI in symmetric variables of degree $2n$ is a consequence of f_{2n} $(n \neq 3)$ for large characteristic of the field.

Let $W_{\infty,1} = \bigcup_{m \geq 1} W_{m,1}$. In [7] it was constructed a polynomial in $W_{\infty,1} \cap T((M_n(F), t))$ of degree $2n - 1$ of minimal degree among all $*$-PI's in one skew variable and the remaining symmetric variables; moreover it was shown that any other multilinear $*$-PI in $W_{\infty,1} \cap T((M_n(F), t))$ of degree $2n-1$ is a scalar multiple of this polynomial.

Definition 4.1 *Let $r \geq 2$ and write $y = x_r$; if $1 \leq j \leq r$ define*

$$P_r^{(j)}(x_1, \ldots, x_{r-1}; y) = \sum_{\sigma \in S_{r-1}} (sgn\sigma) x_{\sigma(1)} \cdots x_{\sigma(j-1)} y x_{\sigma(j)} \cdots x_{\sigma(r-1)}$$

and, if $[\alpha]$ denotes the integral part of the real number α, define

$$P_r(x_1, \ldots, x_{r-1}; y) = \sum_{i=1}^{[\frac{r+1}{2}]} (-1)^{i+1} P_r^{(2i-1)}(x_1, \ldots, x_{r-1}; y).$$

Notice that if $r \equiv 3 \pmod 4$, then we can also write, for $r = 4m - 1$,

$$\begin{aligned} &P_{4m-1}(x_1, \ldots, x_{4m-2}; y) \\ &\quad = \sum_{i=0}^{m-1} P_{4m-1}^{(4i+1)}(x_1, \ldots, x_{4m-2}; y) - \sum_{i=1}^{m} P_{4m-1}^{(4i-1)}(x_1, \ldots, x_{4m-2}; y). \end{aligned}$$

The result is the following

Theorem 4.2

1. $P_{2n-1}(s_1, \ldots, s_{2n-2}; k_{2n-1})$ *is a* $*$-*PI for* $(M_n(F), t)$.

2. *If* $f(s_1, \ldots, s_{2n-2}, k_{2n-1})$ *is a multilinear* $*$-*PI for* $(M_n(F), t)$ *then* $f = \alpha P_{2n-1}$ *for some* $\alpha \in F$.

3. *If* $f(k_1, s_2, \ldots, s_l)$ *is a* $*$-*PI for* $(M_n(F), t)$ *such that* $deg_{k_1}(f) = 1$ *then* $deg(f) \geq 2n - 1$.

The proof of this theorem is based on a technique of Razmyslov [14] used to give a proof of the Amitsur-Leviztki theorem and in the following remarks

Remark 4.3 *If* A *is a symmetric* $n \times n$ *matrix under the symplectic involution, then* $n = 2m$ *and* A *satisfies a polynomial of degree* m

$$p(x) = x^m - \mu_1 x^{m-1} + \mu_2 x^{m-2} - \ldots + (-1)^m \mu_m$$

such that $p(x)^2$ *is the characteristic polynomial of* A.

In case char $F = 0$, it is possible to derive formulas, analogous to Newton's formulas, which allow to write the coefficients μ_i as polynomials in $tr(A), tr(A^2), \ldots, tr(A^m)$ with rational coefficients; they are given inductively by the formulas

$$\mu_0 = 1, \;\; 2i\mu_i = \sum_{j=1}^{i} (-1)^{j-1} \mu_{i-j} tr(A^j)$$

(see [17, pag 150]).

Remark 4.4 *Let* A *be a commutative ring. If* $U_1, \ldots, U_{2l} \in M_n(A)$ *then* $trS_{2l}(U_1, \ldots, U_{2l}) = 0$.

This can be proved by noticing that if w_1, w_2 are two monomials in the U_i's which are cyclic permutations of one another, then on one side $tr(w_1) = tr(w_2)$ and on the other side w_1 and w_2 have opposite signs in $S_{2l}(U_1, \ldots, U_{2l})$.

Let us denote by $M_n^+(F)$ and $M_n^-(F)$ the spaces of symmetric and skew elements of $(M_n(F), *)$ respectively.

Proof of Theorem 4.2 We shall sketch the proof of the first and second part of the theorem.

Since P_{2n-1} has coefficients ± 1, it is enough to prove the theorem in case char $F = 0$. Suppose first that $n = 2m$ is even. In this case if $C \in M_n^-(F)$ is invertible, then the map $\phi : A \to C^{-1} A^t C$ defines an involution of symplectic type on $M_n(F)$. If $B \in M_n(F)$, $B = -B^t$ then BC under this involution is a symmetric element. Thus, by Remark 4.3, BC satisfies a polynomial of degree m; let

$$(BC)^m - \mu_1 (BC)^{m-1} + \cdots + (-1)^m \mu_m = 0$$

where the μ_is are polynomial expressions with rational coefficients in $tr(BC)$, $tr(BC)^2, \ldots, tr(BC)^m$.

Now, by a Zariski density argument it follows that we may take B, C arbitrary elements in $M_n(F)^-$ under the transpose involution; also, if we multilinearize the above relation we get

$$\sum_{\sigma,\tau\in S_m} B_{\sigma(1)}C_{\tau(1)}B_{\sigma(2)}C_{\tau(2)}\cdots B_{\sigma(m)}C_{\tau(m)} + Q = 0 \tag{1}$$

where $B_1, C_1, \ldots, B_m, C_m \in M_n^-(F)$ and Q is a linear combination of products of the form $B_{\sigma(i_1)}C_{\tau(i_1)}\cdots B_{\sigma(i_l)}C_{\tau(i_l)}$, with $1 \le i_1 < i_2 < \ldots < i_l \le m$, $l < m$ with coefficients polynomials in traces of the form

$$tr(B_{\sigma(j_1)}C_{\tau(j_1)}\cdots B_{\sigma(j_r)}C_{\tau(j_r)}).$$

Take $A_1, \ldots A_{2n-2} \in M_n^+(F)$ and $B \in M_n^-(F)$; then $[A_i, A_j] = A_iA_j - A_jA_i \in M_n^-(F)$ and we make in (1) the substitutions

$$\begin{array}{ll} B_1 = B, & C_1 = [A_1, A_2], \\ B_2 = [A_3, A_4], & C_2 = [A_5, A_6], \\ \vdots & \vdots \\ B_m = [A_{2n-5}, A_{2n-4}], & C_m = [A_{2n-3}, A_{2n-2}]. \end{array}$$

We obtain from (1) a new equation and we write the left hand side as $f(A_1, \ldots, A_{2n-2}, B)$, a polynomial with traces.

Now in (1) make the substitutions

$$\begin{array}{ll} B_1 = [A_1, A_2], & C_1 = B, \\ B_2 = [A_3, A_4], & C_2 = [A_5, A_6], \\ \vdots & \vdots \\ B_m = [A_{2n-5}, A_{2n-4}], & C_m = [A_{2n-3}, A_{2n-2}]; \end{array}$$

we obtain a new equation and let $g(A_1, \ldots, A_{2n-2}, B)$ be the resulting left hand side.

Let $\overline{f}$ and $\overline{g}$ be the skewsymmetrizations of f and g respectively with respect to the variables $A_1, \ldots, A_{2n-2}$; it follows that

$$\begin{aligned} 0 &= \overline{f}(A_1, \ldots, A_{2n-2}, B) - \overline{g}(A_1, \ldots, A_{2n-2}, B) \\ &= 2^{m-1}P_{2n-1}(A_1, \ldots, A_{2n-1}; B) \\ &\qquad + \text{a linear combination of terms with } tr. \end{aligned}$$

Thus, in order to complete the proof of the theorem in case n is even, it is enough to prove that all the coefficients of the remaining terms (which involve traces) are zero.

Notice that, the traces in which B does not appear in $\overline{f}$ and $\overline{g}$ are of the form $tr(S_{2j}(A_{i_1}, \ldots, A_{i_{2j}}))$ and this is zero by Remark 4.4. Also, one checks that in $\overline{f} - \overline{g}$ the traces involving B vanish too. This proves the theorem in case n is even.

In case n is odd, embed $M_n(F)$ in the upper left corner of $M_{n+1}(F)$ with induced transpose involution and write

$$P_{2(n+1)-1}(s_1, \ldots, s_{2n}; k_{2n+1}) \\ = P_{2n-1}(s_1, \ldots, s_{2n-2}; k_{2n+1})s_{2n-1}s_{2n} + h(s_1, \ldots, s_{2n}, k_{2n+1})$$

where no monomial of $h(s_1, \ldots, s_{2n}, k_{2n+1})$ ends with $s_{2n-1}s_{2n}$. Now take $j_0 \in \{1, \ldots, n\}$. Then $e_{j_0\, n+1} + e_{n+1\, j_0}$, $e_{n+1\, n+1} \in M_n^+(F)$ and let $A_1, \ldots, A_{2n-2} \in M_n^+(F)$, $B \in M_n^-(F)$. Since $n+1$ is even, by the first part of the proof, $P_{2(n+1)-1}(s_1, \ldots, s_{2n}; k_{2n+1})$ is a $*$-PI for $(M_{n+1}(F), t)$; thus

$$\begin{aligned} 0 &= P_{2(n+1)-1}(A_1, \ldots, A_{2n-2}, e_{j_0\, n+1} + e_{n+1\, j_0}, e_{n+1\, n+1}; B)e_{n+1\, n+1} \\ &= P_{2n-1}(A_1, \ldots, A_{2n-2}; B)(e_{j_0\, n+1} + e_{n+1\, j_0})e_{n+1\, n+1} \\ &\qquad + h(A_1, \ldots, A_{2n-2}, e_{j_0\, n+1} + e_{n+1\, j_0}, e_{n+1\, n+1}, B)e_{n+1\, n+1} \\ &= P_{2n-1}(A_1, \ldots, A_{2n-2}; B)e_{j_0\, n+1}. \end{aligned}$$

Since j_0 is arbitrary in $\{1, \ldots, n\}$, we get $P_{2n-1}(A_1, \ldots, A_{2n-2}; B) = 0$ for all $A_1, \ldots, A_{2n-2} \in M_n^+(F)$ and $B \in M_n^-(F)$ and we are done.

To prove the second part of the theorem, one proves by induction on n that if $f(s_1, \ldots, s_{2n-2}, k)$ is a $*$-PI for $M_n(F), t)$ then

$$\begin{aligned} & f(s_1, \ldots, s_{2n-2}, k) \\ &= \alpha \sum_{\substack{i,j=1 \\ i<j}}^{2n-2} (-1)^{i+j}[s_i, s_j]P_{2n-3}(s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{2n-2}; k) \\ &\qquad + k \cdot h(s_1, \ldots, s_{2n-2}) \end{aligned}$$

for some $\alpha \in F$.

Now, an easy calculation shows that in general

$$\begin{aligned} & P_{2n-1}(s_1, \ldots, s_{2n-2}; k) \\ &= \sum_{\substack{i,j=1 \\ i<j}}^{2n-2} (-1)^{i+j}[s_i, s_j]P_{2n-3}(s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{2n-2}; k) \\ &\qquad + k \cdot S_{2n-2}(s_1, \ldots, s_{2n-2}). \end{aligned}$$

Thus, since P_{2n-1} and f are $*$-PI's for $(M_n(F), t)$ also

$$f - \alpha P_{2n-1} = k \cdot (h(s_1, \ldots, s_{2n-2}) - \alpha S_{2n-2}(s_1, \ldots, s_{2n-2}))$$

vanishes in $(M_n(F), t)$.

Let $\overline{K}$ denote the subring generated by the skew elements of $M_n(F)$; then, since $n > 2$, by [8, Theorem 2.1.10], $\overline{K} = M_n(F)$. It follows that for all $A_1, \dots, A_{2n-2} \in M_n^+(F)$,

$$M_n(F) \cdot (h(A_1, \dots, A_{2n-2}) - \alpha S_{2n-2}(A_1, \dots, A_{2n-2})) = 0.$$

Therefore, being $M_n(F)$ a prime ring,

$$h(s_1, \dots, s_{2n-2}) - \alpha S_{2n-2}(s_1, \dots, s_{2n-2})$$

must vanish in $M_n^+(F)$. Since $(M_n(F), t)$ satisfies no $*$-PI's in symmetric variables of degree lower than $2n$, it follows that $h = \alpha S_{2n-2}$ and we are done. □

Having solved problem 3.5 for the case $W_{m,1}$, it is natural to try to solve problem 3.5 for $W_{m,2}$. There is no existence theorem proved in this case. Nevertheless the following result holds ([7, Proposition 3])

Theorem 4.5 *Let $f(k_1, k_2, s_3, \dots, s_r)$ be a $*$-PI for $(M_n(F), t)$, $n > 2$, such that $deg_{k_1}(f) = deg_{k_2}(f) = 1$. Then $deg(f) \geq 2n - 1$.*

5 Symplectic Involution

In this section we study the space of multilinear $*$-PI's of $(M_n(F), s)$. Let us consider $*$-PI's of minimal degree for $(M_n(F), s)$ in symmetric variables. The polynomials $[s_1, s_2]$ and $[[s_1, s_2]^2, s_3]$ are $*$-PI's of minimal degree for $n = 2$ and $n = 4$ respectively. For $n > 4$ the best known lower bound is found in ([1, Theorem 2.4])

Theorem 5.1 *If char $F = 0$, $(M_n(F), s)$ does not satisfy $*$-PI's in symmetric variables of degree $n + 1$ for any $n > 4$.*

The proof is based on the following idea. If $n = 2m$, then every $*$-PI in symmetric variables for $(M_n(F), s)$ is an ordinary polynomial identity for the $m \times m$ matrix algebra $M_m(F)$. The polynomial identities of $M_n(F)$ of degree $m + 1$ have been described by Leron [9]. It turns out that they all follow from the standard identity $S_{2m}(x_1, \dots, x_{2m})$. Hence it suffices to show that no multilinear consequence of degree $n + 1 = 2m + 1$ of $S_{2m}(x_1, \dots, x_{2m})$ vanishes on the symmetric elements from $(M_n(F), s)$.

Inspired by this result is the following

Theorem 5.2 *([13, Theorem 3.1]) If char $F = 0$, $(M_6(F), s)$ does not satisfy $*$-PI's in symmetric variables of degree 8.*

Let now consider $*$-PI's in skew variables. What is the minimal possible degree of such a polynomial identity ? For $n = 2, 4$ the answer is as follows: $[k_1^2, k_2]$ is a $*$-PI in skew variables for $(M_2(F), s)$ of minimal degree; also, since $[s_1, k_2]^2$, when evaluated in $(M_4(F), s)$, takes values in the center (see [3, pag 203]), it follows that $[[k_1^2, k_2]^2, k_3]$ is a $*$-PI for $(M_4(F), s)$. It is also easy to see that this polynomial is of minimal degree among $*$-PI's in skew variables.

In the general case one could conjecture that if f is a $*$-PI for $(M_n(F), s)$ in skew variables of minimal degree, then $deg(f) = 2n - 1$. In [7] the authors constructed, for every n, a $*$-PI in skew variables of degree $2n - 1$.

Definition 5.3 *Let $r > 2$ and write $y = x_r$. Define the polynomial*

$$T_r(x_1, \ldots, x_{r-1}; y) = \sum_{i=1}^{[\frac{r+1}{2}]} P_r^{(2i-1)}(x_1, \ldots, x_{r-1}; y) + 2 \sum_{i=1}^{[\frac{r+3}{4}]} P_r^{(4i-2)}(x_1, \ldots, x_{r-1}; y).$$

Notice that in T_r the variable y never appears in the $4i$ position, $i = 1, \ldots, [r/4]$.

Using the technique of Theorem 4.2 one can prove (see [7]) the following

Theorem 5.4 *$T_{2n-1}(k_1, \ldots, k_{2n-2}; k_{2n-1})$ is a $*$-PI for $(M_n(F), s)$.*

Unfortunately we do not know if T_{2n-1} is of minimal degree among $*$-PI's for $(M_n(F), s)$ in skew variables. The known best bound is stated in the following

Theorem 5.5 *If $f(k_1, \ldots, k_r)$ is a $*$-PI for $(M_n(F), s)$, $n > 2$, then $deg(f) > n + n/2$.*

References

[1] V. S. Drensky and A. Giambruno, On the $*$-polynomial identities of minimal degree for matrices with involution, Boll. Un. Mat. It. (7) 9-A (1995), 471-482.

[2] V. S. Drensky and A. Giambruno, Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution, Canad. J. Math. 46 (1994), 718-733.

[3] A. Giambruno, Algebraic conditions on rings with involution, J. Algebra 50 (1978), 190-212.

[4] A. Giambruno and A. Regev, Wreath products and PI-algebras, J. Pure Applied Algebra 35 (1985), 133-149.

[5] A. Giambruno, $GL \times GL$ - representations and $*$-polynomial identities, Comm. Algebra 14 (1986), 787-796.

[6] A. Giambruno, On $*$-polynomial identities for $n \times n$ matrices, J. Algebra 133 (1990), 433-438.

[7] A. Giambruno and A. Valenti, On minimal $*$-identities of matrices, Linear and Multilinear Algebra, 39 (1995), 309-323.

[8] I. N. Herstein, Rings with Involution, U. of Chicago Press, Chicago, 1976.

[9] U. Leron, Multilinear identities of the matrix ring, Trans. Amer. Math. Soc. 183 (1973), 175-202.

[10] D. V.Levchenko, Finite basis of identities with involution for the second order matrix algebra, Serdica 8 (1982), 42-56 (Russian).

[11] B. Kostant, A theorem of Frobenius, a theorem of Amitsur-Levitzki and cohomology theory, Indiana J. Math. (and Mech.) 7 (1958), 237-264.

[12] Ma Wenxin and M. L. Racine, Minimal identities of symmetric matrices, Trans. Amer. Math. Soc. 320 (1990), 171-192.

[13] T. G. Rashkova, On the minimal degree of the $*$-polynomial Identities for the matrix algebra of order 6 with symplectic involution, Rend. Circ. Mat. Palermo (to appear).

[14] Y. P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero, Math USSR Izv. 8 (1974), 727-760.

[15] S. Rosset, A new proof of the Amitsur-Levitski identity, Israel J. Math. 23 (1976), 187-188.

[16] L. H. Rowen, Standard polynomials in matrix algebras, Trans. Amer. Math. Soc. 190 (1974), 253-284.

[17] L. Rowen, "Polynomial Identities in Ring Theory", Academic Press, New York, 1980.

[18] L. H. Rowen, A simple proof of Kostant's theorem, and an analogue for the symplectic involution, Contemp. Math., vol. 13, Amer. Math. Soc., Providence, R. I., 1982, pp. 207-215.

[19] R. Swan, An application of graph theory to algebra, Proc. Amer. Math. Soc. 14 (1963) 367-373.

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