Resenhas IME-USP 1997, Vol. 3, No. 1, 1 - 12.

General Perturbation of the Exponential Dichotomy for Evolution Equations [1]

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Abstract: In this paper we prove that the exponential dichotomy for evolution equations in Banach spaces is not destroyed, if we perturb the equation by "small" unbounded linear operator. This is done by employing skew-product semiflow technique and a perturbation principle from linear operator Theory. Finally, we apply these results a partial parabolic equation.

Key words: evolution equations, skew-product semiflow, exponential dichotomy, perturbation.

1 Introduction

Many authors have been studying the existence and roughness(perturbation) of the exponential dichotomy (ED) for infinite dimensional evolution equations. For example, for partial differential equations one can find the work done by D. Henry [11], Kolesov [13] and X.-B. Lin [18]. In the case of functional differential equations we can see the work done by J. Hale [10], X.-B. Lin [17], and M. Lizana [19].

Roughly speaking these authors have studies the existence and roughness of the exponential dichotomy for the following abstract linear evolution equation in a Banach space Z

$$z' = (A + B(t))z, \quad t > 0, \tag{1.1}$$

where $t \to B(t) : I\!R \to L(Z)$ is bounded, continuous in the strong operator topology of $L(Z)$ and A is the infinitesimal generator of a C_0 -semigroup. For the existence of the exponential dichotomy we only have to put some gap condition on the spectrum of A and assume that $B(t)$ is small in the uniform topology of $L(Z)$, see for example Chow-Leiva [2], Rau [23] and Sacker-Sell [24].

The question of perturbation (roughness) for the exponential dichotomy can be formulated as follow: If the equation (1.1) has ED, then for which class of linear operators P on Z the equation

$$z' = (A + P)z + B(t)z, \quad t > 0, \tag{1.2}$$

has ED?. In this general setting, it is well known that: if P is a bounded linear operator, which is small enough in the uniform topology of $L(Z)$, then the

[1] Research supported by ULA.

AMS(MOS) subject classifications. primary: 34G10; secondary:35B40

equation (1.2) has ED. But, if P is unbounded this results is not true in general. Nevertheless, for some particular partial differential equations we can allow P to be unbounded.

All these problems can be treated in unifield setting of a linear Skew-Product Semiflow (LSPS), see for example Sacker-Sell [24], Latushkin -Stepin [14], [15], Latushkin, Smith and Randolph [16] and Chow-Leiva [1], [2] [3]. In [2] we give a necessary and sufficient conditions for the existence of exponential dichotomy for skew-product semiflow. Also, we prove that the ED for LSPS is not destroyed by small perturbation (roughness). But, the question of roughness for the equation (1.1) with unbounded perturbation P remains the same. In this paper we shall answer this quation for the more general class of unbounded operator P. That is to say, we will study the existence and roughness of the ED dichotomy for the following family of evolution equations in a Banach space Z.

$$z' = (A+P)z + B(\theta\cdot t)z, \quad t>0, \quad \theta\in\Theta, \quad P\in\mathcal{P}(A). \qquad (1.2)_P$$

Where the state $z \in Z$, A is the infinitesimal generator of a C_0-semigroup $\{T(t:A)\}_{t\geq 0}$, Θ is a compact Housdorff topological space which is invariantly connected under a flow $\sigma(\theta,t)=\theta\cdot t$, $B(\theta)$ is a bounded linear operator in Z and P is an unbounded linear operator in Z which belong to the set $\mathcal{P}(A)$ given in section 3. One of the goal in this work, is to prove the following statement:

If for some $P_0 \in \mathcal{P}(A)$ the equation $(1.2)_{P_0}$ has ED according to Definition 2.2, then there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for all $P\in\mathcal{N}(P_0)$ the equation $(1.2)_P$ has ED.

2 Notations and Preliminaries

In this section we shall present some definitions, notations and results about Linear linear skew-product semiflow in infinite dimensional Banach spaces.

2.1 Linear Skew-Product Semiflow (LSPS)

We begin with the notion of LSPS on the trivial Banach bundle $\mathcal{E} = X\times\Theta$ where X is a fixed Banach space (the state space) and Θ is a compact Hausdorff space.

Definition 2.1 Suppose that $\sigma(\theta,t)=\theta\cdot t$ is a flow on Θ, i.e., the mapping $(\theta,t)\to\theta\cdot t$ is continuous, $\theta\cdot 0=\theta$ and $\theta\cdot(s+t)=(\theta\cdot s)\cdot t$, for all $s,t\in\mathbb{R}$.

A semiflow π on $\mathcal{E}=X\times\Theta$ is said to be **Linear Skew-Product Semiflow (LSPS)**, if it can be written as follows

$$\pi(x,\theta,t) = (\Phi(\theta,t)x, \theta\cdot t), \quad t\geq 0,$$

where $\Phi(\theta,t)\in L(X)$ has the following properties:

(1) $\Phi(\theta,0)=I$, the identity operator on X, for all $\theta\in\Theta$

(2) $\lim_{t\to 0^+} \Phi(\theta,t)x = x$, uniformly in θ. This means that for every $x \in X$ and every $\epsilon > 0$ there is a $\delta = \delta(x,\epsilon) > 0$ such that $\|\Phi(\theta,t)x - x\| \le \epsilon$, for all $\theta \in \Theta$ and $0 \le t \le \delta$.

(3) $\Phi(\theta,t)$ is a bounded linear operator from X into X that satisfies the cocycle identity:

$$\Phi(\theta,t+s) = \Phi(\theta\cdot t,s)\Phi(\theta,t) \quad \theta\in\Theta, \quad 0\le s,t. \tag{2.3}$$

(4) for all $t \ge 0$ the mapping from $\mathcal{E}$ into X given by

$$(x,\theta) \to \Phi(\theta,t)x$$

is continuous.

The properties (2) and (3) imply that for each $(x,\theta) \in \mathcal{E}$ the solution operator $t \to \Phi(\theta,t)x$ is right continuous for $t \ge 0$. In fact :

$$\|\Phi(\theta,t+h)x - \Phi(\theta,t)x\| = \|[\Phi(\theta\cdot t,h) - I]\Phi(\theta,t)x\|$$

which goes to 0 as h goes to 0^+.

2.2 Exponential Dichotomy (ED)

A mapping $\mathbf{P}: \mathcal{E} \to \mathcal{E}$ is said to be a projector if $\mathbf{P}$ is continuous and has the form $\mathbf{P}(x,\theta) = (P(\theta)x,\theta)$, where $P(\theta)$ is a bounded linear projection on the fiber $\mathcal{E}(\theta)$.

For any projector $\mathbf{P}$ we define the range and null space by

$$\mathcal{R} = \mathcal{R}(\mathbf{P}) = \{(x,\theta)\in\mathcal{E} : P(\theta)x = x\}, \quad \mathcal{N} = \mathcal{N}(\mathbf{P}) = \{(x,\theta)\in E : P(\theta)x = 0\}$$

The continuity of $\mathbf{P}$ implies that the fibers $\mathcal{R}(\theta)$ and $\mathcal{N}(\theta)$ vary continuously in θ. This also means that $P(\theta)$ varies continuously in the strong topology of $L(X)$.

A projector $\mathbf{P}$ on $\mathcal{E}$ is said to be **invariant** if it satisfies the following property

$$P(\theta\cdot t)\Phi(\theta,t) = \Phi(\theta,t)P(\theta) \quad t\ge 0, \quad \theta\in\Theta \tag{2.4}$$

Definition 2.2 We shall say that a linear skew-product semiflow π on $\mathcal{E}$ has an **exponential dichotomy(ED) over Θ** , if there are constants $k \ge 1$, $\beta > 0$ and invariant projector $\mathbf{P}$ such that for all $\theta \in \Theta$ we have the following:

(1) $\Phi(\theta,t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta\cdot t))$, $t \ge 0$ is an isomorphism with inverse:

$$\Phi(\theta\cdot t,-t) : \mathcal{N}(P(\theta\cdot t)) \to \mathcal{N}(P(\theta)), \quad t\ge 0$$

(2) $\|\Phi(\theta,t)P(\theta)\| \le ke^{-\beta t}, \quad t \ge 0$

(3) $\|\Phi(\theta,t)(I - P(\theta)\| \le ke^{\beta t}, \quad t \le 0.$

From $\mathcal{N}(P(\theta)) = \mathcal{R}(I - P(\theta))$ and the Open Mapping Theorem we have that $\Phi(\theta,t)(I - P(\theta))$ is a well defined linear and bounded operator for $t \le 0$.

The following Theorem says that the ED of the LSPS is not destroyed by small perturbation, it can be found in Chow-Leiva [2]. Also, for the case of linear skew-product flow(LSPF) there is a nice proof of this Theorem given by Latushkin, Montgomery-Smith and Randolph in [16] using evolutonary groups.

Theorem 2.1 *Suppose $\pi = (\Phi, \sigma)$ is a LSPS on $\mathcal{E}$ which has a ED (with exponent β and constant M). If*

$$L = \sup\{\|\Phi(\theta, t)\| : \quad 0 \leq t \leq 1, \quad \theta \in \Theta\}$$

and $Me^{-\beta} < e^{-\beta_1}$, $M_1 > M$, then there exists $\epsilon = \epsilon(\beta, \beta_1, M, M_1, L) > 0$ such that any linear skew-product semiflow $\tilde{\pi} = (\Psi, \sigma)$ on $\mathcal{E}$ satisfying

$$\sup\{\|\Phi(\theta, t) - \Psi(\theta, t)\| : \quad 0 \leq t \leq 1, \quad \theta \in \Theta\} \leq \epsilon$$

has ED with exponent β_1 and constant M_1.

3 Perturbation Principle

The results presented in this section follow from a combination of Theorem 19 in [9] pg. 31 and the chapter XIII of [12]. It is well known that, if A is the infinitesimal generator of a C_0- semigroup $\{T(t; A)\}_{t\geq 0}$ in the Banach space Z and P is a bounded linear operator in Z ($P \in L(Z)$), then $A+P$ is the infinitesimal generator of a C_0-semigroup $\{T(t; A+P)\}_{t\geq 0}$ which is given by the following formula

$$T(t; A+P)z = T(t; A)z + \int_0^t T(t-s; A)PT(s; A+P)zds, \quad z \in Z. \tag{3.5}$$

Now, we shall see that: if P is an unbounded linear operator which is not too irregular relative to A, then $A+P$ is the infinitesimal generator of a C_0-semigroup $\{T(t; A+P)\}_{t\geq 0}$, but, the formula 3.5 is not true in general.

We shall denote by $\mathcal{D}(S)$ the domain of an operator S in a Banach space W, $L(W)$ the space of bouded and linear operator defined on W and $\sigma(S)$ the spectrum of the linear operator S. With these notation in mind, we will consider the following class of unbounded linear operators: If A is the infinitesimal generator of a C_0-semigroup $\{T(t; A)\}_{t\geq 0}$ we denote $\mathcal{P}(A)$ the class of closed linear operators P satisfying the conditions

(I) $\mathcal{D}(A) \subseteq \mathcal{D}(P)$,

(II) for each $t > 0$, there exists a constant $h(t) \geq 0$ such that

$$\|PT(t, A)z\| \leq h(t)\|z\|, \quad \forall z \in \mathcal{D}(A),$$

(III) the integral $\int_0^1 h(t)dt$ exists.

Remark 3.1 A is bounded, if and only if $A \in \mathcal{P}(A)$.

The following Theroem can be found in [9]. pg 631.

Theorem 3.1 *Let A be the infinitesimal generator of a C_0semigroup $\{T(t;A)\}_{t\geq 0}$ in Z. If $P \in \mathcal{P}(A)$, then $A+P$ defined on $\mathcal{D}(A+P) = \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup $\{T(t;A+P)\}_{t\geq 0}$. Furthermore,*

$$T(t;A+P)z = \sum_{0}^{\infty} S_n(t), \quad t \geq 0, \tag{3.6}$$

where

$$S_0(t) = T(t;A) \quad \text{and} \quad S_n(t)z = \int_0^t T(t-s;A)PS_{n-1}(s)zds, \quad n \geq 1, \quad z \in Z,$$

and the serie (3.6) is absolutely convergent in the uniform norm of $L(Z)$, uniformly with respect t in each finite interval. For each n and z the function $S_n(t)z$ is continuous for $t \geq 0$.

The following facts can be found in [9].

(a) $\bigcup_{t>0} T(t;A)z \subseteq \mathcal{D}(P)$,

(b) the mapping $z \to PT(t;A)z, \quad z \in \mathcal{D}(A)$, has a unique extension to a bounded operator defined in on Z. In order to simplify the notation, we will call this extension $PT(t)$.

(c) $PT(t)z$ is continuous in $t > 0$ at each $z \in Z$. If $\omega_0 = \lim_{t\to\infty} \log \|T(t)\|/t$, then

$$\limsup_{t\to\infty} \frac{\log \|PT(t)\|}{t} \leq \omega_0.$$

(d) if $R(\lambda) > \omega_0$, then

$$P\mathcal{R}(\lambda;A)z = \int_0^\infty e^{-\lambda t} PT(t)zdt, \quad z \in Z;$$

where $\mathcal{R}(\lambda;A) = (A - \lambda I)^{-1}$.

(e) If $\omega > \omega_0$, then there exists $M_\omega < \infty$ such that

$$\|T(t)\| \leq M_\omega e^{\omega t}, \quad \text{and} \quad \|PT(t)\| \leq M_\omega e^{\omega t}, \quad t \geq 0.$$

(f) for all $\beta > o$

$$\int_0^\beta \|PT(t)\|dt < \infty.$$

(g) If $\quad \gamma = \int_0^\infty e^{-\omega t}\|PT(t)\|dt < 1$, then

$$\|S_n(t)\| \leq M_\omega e^{\omega t}\gamma^n, \quad n \geq 0.$$

Proposition 3.1 *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t;A)\}_{t\geq 0}$ of type ω_0. Define the function*

$$d_A(P_1,P_2)=\int_0^1 \|(P_1-P_2)T(t;A)\|dt, \quad P_1,P_2\in\mathcal{P}(A), \tag{3.7}$$

and for a fixed $\omega>\omega_0$ the function

$$\delta_A(P_1,P_2)=\int_0^\infty e^{-\omega t}\|(P_1-P_2)T(t;A)\|dt \quad P_1,P_2\in\mathcal{P}(A). \tag{3.8}$$

Then $\delta_A(P_1,P_2)$ and $d_A(P_1,P_2)$ are equivalent metrics on $\mathcal{P}(A)$. i.e., there exist constants M_A and m_A such that

$$m_A\delta_A(P_1,P_2)\leq d_A(P_1,P_2)\leq M_A\delta_A(P_1,P_2), \quad P_1,P_2\in\mathcal{P}(A).$$

Remark 3.2 *If P_1-P_2 is bounded, then*

$$d_A(P_1,P_2)\leq (\int_0^1\|T(t;A)\|dt)\|(P_1-P_2)\|.$$

Theorem 3.2 *The function $P\in\mathcal{P}(A)\to T(t;A+P)\in L(Z)$ is continuous. i.e.,*

$$\lim_{d_A(P,P_0)\to 0}\|T(t;A+P)-T(t;A+P_0)\|=0,$$

uniformly with respect to t in each interval of the form $[0,\beta]$, $\beta>0$.

Furthermore. If $\delta_A(P,P_0)<1$, then there exists a constant $M=M(P_0)$ such that

$$\|T(t;A+P)-T(t;A+P_0)\|\leq\frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)}Me^{\omega t}, \quad t\geq 0.$$

4 Main Results

From the foregoing section we have that $(\mathcal{P}(A),d_A)$ is a metric space endow with the metric d_A. Now, we are ready to study the following family of evolution equations.

$$z'=(A+P)z+B(\theta\cdot t)z, \quad t>0, \quad \theta\in\Theta, \quad P\in(\mathcal{P}(A),d_A). \tag{4.9$_P$}$$

where the mapping $\theta\in\Theta\to B(\theta)z$ is continuous in θ for $z\in Z$ fixed. The equation $(4.9)_P$ generates a linear linear skew-product semiflow $\pi_P=(\Phi_P,\sigma)$ on $z\times\Theta$ according to Definition 2.1 given by

$$\pi_P(z,\theta,t)=(\Phi_P(\theta,t)z,\theta\cdot t), \quad \theta\in\Theta, \quad P\in(\mathcal{P}(A),d_A). \tag{4.9}$$

Where $\Phi_P(\theta,t)$ is the evolution operator associated with the equation $(4.9)_P$ which is given by the formula

$$\Phi_P(\theta,t)z = T(t;A+P)z + \int_0^t T(t-s;A+P)B(\theta\cdot s)\Phi_P(\theta,s)zds, \quad z\in Z. \quad (4.10)$$

Theorem 4.1 *If for some $P_0 \in (\mathcal{P}(A), d_A)$ the linear linear skew-product semiflow π_{P_0} generated by $(4.9)_{P_0}$ has exponential dichotomy over Θ, then there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for each $P \in \mathcal{N}(P_0)$ the LSPS π_P generated by $(4.9)_P$ has ED over Θ.*

Proof We shall apply Theorem 2.1. So, we need to prove that, there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for each $P \in \mathcal{N}(P_0)$ we have the following estimate

$$\sup\{\|\Phi_P(\theta,t) - \Phi_{P_0}(\theta,t)\| : \ 0\le t\le 1, \ \theta\in\Theta\} \le \epsilon$$

where ϵ is given in Theorem 2.1. In fact, from Theorem 3.2 we have that

$$\|T(t;A+P) - T(t;A+P_0)\| \le \frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)} Me^{\omega t}, \quad t\ge 0.$$

Now, from formula (4.10) we get that

$$\begin{aligned}
\Phi_P(\theta,t) - \Phi_{P_0}(\theta,t) &= T(t;A+P) - T(t;A+P_0) \\
&+ \int_0^t (T(t-s;A+P) - T(t-s;A+P_0))\mathcal{B}(\theta\cdot s)\Phi_{P_0}(\theta,s)ds \\
&+ \int_0^t T(t-s;A+P)\mathcal{B}(\theta\cdot s)(\Phi_P(\theta,s) - \Phi_{P_0}(\theta,s))ds
\end{aligned}$$

So,

$$\begin{aligned}
\|\Phi_P(\theta,t) - \Phi_{P_0}(\theta,t)\| &\le \frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)} Me^{\omega t} \\
&+ \int_0^t \frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)} Me^{\omega(t-s)} R\|\Phi_{P_0}(\theta,s)\|ds \\
&+ \int_0^t \|T(t-s;A+P)\|R\|(\Phi_P(\theta,s) - \Phi_{P_0}(\theta,s))\|ds.
\end{aligned}$$

Where $\|\mathcal{B}(\theta)\| \le R$ for all $\theta\in\Theta$. Clearly, if $t\in[0,1]$ then there are constants M_1 and N_1 such that

$$\begin{aligned}
\|\Phi_P(\theta,t) - \Phi_{P_0}(\theta,t)\| &\le M_1\frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)} \\
&+ N_1\int_0^t \|(\Phi_P(\theta,s) - \Phi_{P_0}(\theta,s))\|ds, \quad 0\le t\le 1.
\end{aligned}$$

Now, applying Gronwall's inequality we get

$$\|\Phi_P(\theta,t)-\Phi_{P_0}(\theta,t)\| \leq M_1\frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)}e^{N_1t}, \quad 0\leq t\leq 1.$$

Therefore, by the continuity of δ_A there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for each $P\in\mathcal{N}(P_0)$ we have that

$$\sup\{\|\Phi_P(\theta,t)-\Phi_{P_0}(\theta,t)\|: \quad 0\leq t\leq 1, \quad \theta\in\Theta\}\leq\epsilon$$

□

Next, we shall consider a particular case of the family of equations $(4.9)_P$. Let us study the family of equations

$$z' = \mathcal{A}_\lambda z + \mathcal{B}(\theta\cdot t)z, \quad t\geq 0, \quad \theta\in\Theta, \quad \lambda\in\Lambda. \qquad (4.10)_\lambda$$

Where Λ is a topological space, $\mathcal{A}_\lambda$ is the infinitesimal generator of a C_0-semigroup $\{T_\lambda(t)\}_{t\geq 0} = \{T(t;\mathcal{A}_\lambda)\}_{t\geq 0}$ and for all $\lambda,\lambda_0\in\Lambda$ we have that $\mathcal{A}_\lambda-\mathcal{A}_{\lambda_0}\in(\mathcal{P}(\mathcal{A}_{\lambda_0}),d_{\mathcal{A}_{\lambda_0}})$.

Moreover, the mapping

$$\lambda\in\Lambda\rightarrow\mathcal{A}_\lambda-\mathcal{A}_{\lambda_0}\in(\mathcal{P}(\mathcal{A}_{\lambda_0}),d_{\mathcal{A}_{\lambda_0}}),$$

is continuous. Under the above conditions the equation $(4.10)_\lambda$ generates a linear skew-product semiflow $\pi_\lambda=(\Phi_\lambda,\sigma)$ on $Z\times\Theta$ given by

$$\pi_\lambda(z,\theta,t) = (\Phi_\lambda(\theta,t)z,\theta\cdot t), \quad \theta\in\Theta, \quad t\leq 0. \qquad (4.11)$$

Where $\Phi_\lambda(\theta,t)$ is the evolution operator associated with the equation $(4.10)_\lambda$ which is given by the formula

$$\Phi_\lambda(\theta,t)z = T_\lambda(t)z+\int_0^t T_\lambda(t-s)B(\theta\cdot s)\Phi_\lambda(\theta,s)zds, \quad z\in Z. \qquad (4.12)$$

Corollary 4.1 *If for some $\lambda_0\in\Lambda$ the LSPS π_{λ_0} generated by $(4.10)_{\lambda_0}$ has ED over Θ, then there exists a neighborhood $\mathcal{N}(\lambda_0)$ of λ_0 such that for each $\lambda\in\mathcal{N}(\lambda_0)$ the LSPS π_λ generated by $(4.10)_\lambda$ has ED over Θ.*

4.1 Example of Parabolic Equations

Consider the following parabolic equation

$$u_t = u_{xx}+a(x)u_x+b(\theta\cdot t,x)u, \quad t>0, \quad \theta\in\Theta, \qquad (4.13)$$

with the initial conditions

$$\lim_{t\rightarrow 0}u(t,x) = u_0(x), \quad \text{uniformly in } x\in\mathbb{R}. \qquad (4.14)$$

Where $a : I\!R \to I\!R$ is uniformly continuous and bounded function, $b(\cdot,\cdot) : \Theta \times I\!R \to I\!R$ is a continuous and bounded function. Let $Z = C_{ub}(I\!R)$ the space of uniformly continuous and bounded functions with the sup-nom and consider the operator $A = \frac{\partial^2}{\partial x^2}$ whose domain $\mathcal{D}(A)$ consists of all $u \in Z$ such that u_x and u_{xx} belong to Z. It is well known that A generates a C_0-semigroup $\{T(t; A)\}_{t\geq 0}$ on Z. Moreover,

$$\sigma(A) \subset (-\infty, 0) \text{ and } \|T(t; A)\| \leq 1, \quad t \geq 0.$$

Furthermore,

$$T(t; A)u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} u(x+s)ds, \quad u \in Z, \quad t > 0. \tag{4.15}$$

Next, let P_a be the closed unbounded operator defined by:

(a) the domain of P_a consits of all $u \in Z$ such that u has a continuous derivative in a neighborhood of each x_0 for which $a(x_0) \neq 0$ and $a(x)u \in Z$.

(b) for $u \in \mathcal{D}(P_a)$ we put $P_a u(x) = a(x)u_x$.

Now, we define the family of operators $B(\theta) \in L(C_{ub}(I\!R))$, $\theta \in \Theta$ as follow

$$B(\theta)u(x) = b(\theta, x)u(x), \quad \theta \in \Theta, \quad x \in I\!R.$$

Therefore, the equation (4.13) can be written as follow

$$u' = (A + P_a)u + B(\theta \cdot t)u, \quad t > 0, \quad \theta \in \Theta, \quad a \in Z. \tag*{$(4.15)_a$}$$

Also, we shall consider the unperturbed equation

$$u' = Au + B(\theta \cdot t)u, \quad t > 0, \quad \theta \in \Theta. \tag{4.16}$$

The equation $(4.15)_a$ will be well defined if we verify that P_a belong to $(\mathcal{P}(A), d_A)$, which imply by Theorem 3.1 that $A + P_a$ with domain $\mathcal{D}(A)$ generates a C_0-semigroup. In fact, clearly $\mathcal{D}(A) \subset \mathcal{D}(P_a)$. If $u \in \mathcal{D}(A)$ and $t > 0$, then

$$\begin{aligned}
\|P_a T(t; A)u\| &\leq \|a\| \sup_{x \in I\!R} \left|\frac{d}{dx} T(t; A)u(x)\right| \\
&= \|a\| \sup_{x \in I\!R} \left|\frac{-1}{4t\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(\xi - x)^2/4t} u(\xi) d\xi\right| \\
&\leq \frac{\|a\|\|u\|}{4t\sqrt{\pi t}} \sup_{x \in I\!R} \int_{-\infty}^{\infty} |\xi - x| e^{-(\xi - x)^2/4t} d\xi \\
&= \frac{\|a\|\|u\|}{2t\sqrt{\pi t}} \int_0^{\infty} \xi e^{-\xi^2/4t} d\xi \\
&= \frac{\|a\|\|u\|}{\sqrt{\pi t}}.
\end{aligned}$$

Hence,

$$\|P_a T(t;A)u\| \leq \frac{\|a\|}{\sqrt{\pi t}}\|u\|, \quad u \in \mathcal{D}(A). \tag{4.17}$$

Therefore, P_a belong to $(\mathcal{P}(A), d_A)$.

Remark 4.1 *We know that $\sigma(A) \subset (-\infty, 0)$. If the funcion $b(\cdot,\cdot)$ is one of the following type, then the equation (4.13) has exponential dichotomy.*

(a) for any $\beta < 0$, $b(\theta, x) = \beta$.

(b) for a function b is independent of x and the dynamical spectrum of the ODE $z' = b(\theta \cdot t)z$ is $[\alpha, \beta]$ with $\beta < 0$.

(c) for any function $b(\theta, x)$ such that $|b(\theta, x) - \bar{b}(\theta)|$ is small enough uniformly on x and $\bar{b}$ is given in (b).

Proposition 4.1 *If the equation (4.16) has ED over Θ, then there exists a neighborhood $\mathcal{N}(0) \subset C_{ub}(I\!R)$ of 0 such that for each $a \in \mathcal{N}(0)$ the equation $(4.15)_a$ has ED over Θ.*

Proof We only need to prove that the mapping $a \in C_{ub}(I\!R) \to P_a \in (\mathcal{P}(A), d_A)$ is continuous at zero. In fact, from (4.17) we get that

$$d_A(P_a, 0) = \int_0^1 \|P_a T(t;A)\| dt \leq \|a\| \int_0^1 \frac{dt}{\sqrt{\pi t}} = \frac{2}{\sqrt{\pi}}\|a\|.$$

□

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