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Asymptotic Behavior Near Zeros of Solutions of Elliptic and Parabolic Equations

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Abstract: This paper discusses two aspects of local behavior of solutions of elliptic and parabolic equations, namely, polynomial asymptotics and unique continuation. Particular stress is laid on describing a new approach to these problems, by connecting to the theory of Lyapunov exponents in dynamical systems via scaling transforms. New results obtained along this line are presented with a sketch of the key ingredients of the proofs.

Key words: Elliptic and parabolic equations, local asymptotics, unique continuation, dynamical systems, Lyapunov exponents

1 Local Asymptotics of Smooth Functions

1.1 Smooth Functions. Let $f(x)$ be a C^∞ function defined in a domain $\Omega \subset \mathbf{R}^N$. By the Taylor expansion, the local asymptotics of f near a point $x_0 \in \Omega$ can be classified into two types:

either (i) *Infinite Order Vanishing:* $f(x_0 + x) = O(|x|^h)$ as $x \to 0$ for any $h \geq 0$;

or else (ii) *Polynomial Local Asymptotics:* $f(x_0 + x) =$ a nontrivial homogeneous polynomial + a higher order remainder. Precisely,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-h} f(x_0 + \varepsilon x) = \Phi(x),$$

where $\Phi(x) \not\equiv 0$ is a homogeneous polynomial of degree $h \geq 0$.

1.2 Analytic Functions. If the function $f : \Omega \to \mathbf{R}$ was real analytic and was not identically zero, then we have only the second alternative: $f(x_0 + x) = \Phi(x) + o(|x|^h)$ where $\Phi(x) \not\equiv 0$ is a homogeneous polynomial of degree $h \geq 0$.

This follows from the *Unique Continuation Theorem:* if $f : \Omega \to \mathbf{R}$ is real analytic and $f(x_0 + x) = O(|x|^h)$ as $x \to 0$ for any $h \geq 0$ at some $x_0 \in \Omega$, then $f \equiv 0$ in Ω.

1.3 C^k Functions. If $f(x)$ was only a C^k function with $k < \infty$, the classification in Subsection 1.1 does not apply in general. For instance, $f(x) = |x|^{k+\alpha}$ with $k \in \mathbf{N}$ and $0 < \alpha < 1$ is a C^k function which satisfies neither 1.1 (i) nor 1.1 (ii) at $x_0 = 0$.

The aim of the present article is to discuss asymptotic properties similar to the above for solutions of elliptic and parabolic equations. Sections 2-5 describe the results, and Sections 6-8 give a rough sketch of the arguments. Throughout we have opted for illuminating new ideas rather than full generality. Most results are stated here under stronger conditions than are really needed in the papers [8, 9, 10].

2 Local Asymptotics for Elliptic PDEs

2.1 Earlier Results. In 1955 Lipman Bers [4] proved that near a zero point of finite order, a solution of an m-th order elliptic equation with Hölder continuous coefficients is asymptotic to a nontrivial homogeneous polynomial.

Under the Hölder condition on coefficients, the standard *a priori* estimates for elliptic equations guarantee that any solution is $C^{m+\alpha}$ and hence is a classical solution. Although solutions may not be C^∞ in general, Bers' theorem indicates that their local behavior at zero points is similar to that of C^∞ functions. Some generalizations to nonclassical solutions of second order elliptic equations with less smooth coefficients are obtained in [5, 24, 17].

2.2 New Result. In a recent work [10], based on a quite different approach, we extended the above theorem of Bers to a broader class of elliptic equations with very mild hypotheses on the coefficients.

Consider an m-th order elliptic equation:

$$\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = 0 \qquad x \in \Omega, \tag{2.1}$$

where $\Omega \subset \mathbf{R}^N$ is an open connected set. Assume the following conditions on coefficients:

for $|\alpha| = m$, $a_\alpha(x)$ are locally Hölder continuous and $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0$ for any $\xi \in \mathbf{R}^N \setminus \{0\}$ and any $x \in \Omega$;

and for $|\alpha| < m$, a_α are locally L_{p_α} integrable with $p_\alpha > \max\{N/(m-|\alpha|), 1\}$.

THEOREM. *Under the above assumptions, let $u \in W^m_{p;loc}(\Omega)$ be a generalized solution of equation* (2.1) *with*

$$1 < p < \min_{m-N<|\alpha|<m} p_\alpha, \qquad p \le \min_{|\alpha|\le m-N} p_\alpha.$$

If $x_0 \in \Omega$ and $u(x_0) = 0$, then one of the following alternatives holds true:

either (i) *u vanishes of infinite order at x_0; that is,*

$$\varepsilon^{-h} u(x_0 + \varepsilon x) \to 0 \ \textit{in}\ W^m_{p;loc}(\mathbf{R}^N) \ \textit{as}\ \varepsilon \downarrow 0 \tag{2.2}$$

for all $h > 0$;

or else (ii) *there exists an integer* $h \geq 1$ *such that*

$$\varepsilon^{-h} u(x_0 + \varepsilon x) \to \Phi(x) \text{ in } W^m_{p;loc}(\mathbf{R}^N) \text{ as } \varepsilon \downarrow 0 \tag{2.3}$$

where $\Phi(x)$ *is a homogeneous polynomial of degree* h *satisfying the osculating equation*

$$\sum_{|\alpha|=m} a_\alpha(x_0) D^\alpha \Phi(x) = 0 \qquad x \in \mathbf{R}^N, \tag{2.4}$$

and is not identically zero.

The proof of the above theorem combines a homothety scaling argument with some elements of the theory of Lyapunov exponents (see later Sections 6 and 7).

2.3 Hausdorff Dimension of Zero Sets. The *vanishing order* of u at x_0, by definition, is ∞ in the case of Theorem 2.2 (i), and is the positive integer h in the case of Theorem 2.2 (ii). For $h \in \mathbf{N} \cup \{\infty\}$, denote by $Z_h(u)$ the set of all zero points $x_0 \in \Omega$ of u with vanishing order $\geq h$. By Theorem 2.2, the set of all zero points of u is equal to $Z_1(u)$. The following can be derived from the theorem together with a geometric measure theoretic argument (see [8]).

COROLLARY. *Let* $u \not\equiv 0$ *be a solution of* m*-th order elliptic equation* (2.1) *in* $\Omega \subset \mathbf{R}^N$ *and assume all conditions in Theorem 2.2.*

(i) *If* $p > N/m$, *then the Hausdorff dimension of* $Z_1(u) \setminus \overline{Z_\infty(u)}$ *is not larger than* $N - 1$.

(ii) *If* $p > N$, *then the Hausdorff dimension of* $Z_m(u) \setminus \overline{Z_\infty(u)}$ *is not larger than* $N - 2$.

2.4 Heuristic Discussion. Let u be as in Theorem 2.2. We first note that the rescaled solution $u_\varepsilon(x) := u(x_0 + \varepsilon x)$ satisfies

$$A_\varepsilon u_\varepsilon = \sum_{|\alpha| \leq m} \varepsilon^{m-|\alpha|} a_\alpha(x_0 + \varepsilon x) D^\alpha_x u_\varepsilon(x) = 0.$$

In the limit as $\varepsilon \downarrow 0$, the rescaled operator A_ε reduces to the "osculating operator"

$$A_0 = \sum_{|\alpha|=m} a_\alpha(x_0) D^\alpha_x$$

(at least at the formal level).

From this observation combined with the use of Taylor series, the polynomial asymptotic result, Theorem 2.2, is immediately seen to hold for C^∞ solutions. However, it was not easy to demonstrate the same result for solutions which are not C^∞. The arguments of Bers for classical solutions made explicit use of power series expansion of fundamental solutions at crucial steps. The generalizations to weak solutions of second order elliptic equations in [5, 24, 17] were even more difficult and needed more detailed computations of some special functions such as fundamental solutions, spherical harmonics, and etc.

A different approach is taken in [10]. Quite unexpectedly, the theory of dynamical systems sheds light on this problem. As will be described in Section 7, a correct splitting of rescaled solutions puts us in a framework of iterative dynamics. Our analysis becomes notably transparent, appealing to the dynamic Lemma 6.1.

3 Local Asymptotics for Parabolic PDEs

In [8] we extended the results in the previous section to parabolic equations as well. See also [1, 2, 9, 11, 15] for related results for the second order case.

3.1 Polynomial Asymptotics. Consider

$$u_t = \sum_{|\alpha| \leq m} a_\alpha(x,t) D_x^\alpha u \qquad (x,t) \in \Omega, \tag{3.1}$$

where $\Omega \subset \mathbf{R}^N \times \mathbf{R}$ is an open connected set. Assume the following conditions on coefficients:

for $|\alpha| = m$, a_α are locally Hölder continuous and $\sum_{|\alpha|=m} a_\alpha(x,t)\xi^\alpha \neq 0$ for any $\xi \in \mathbf{R}^N \setminus \{0\}$ and any $(x,t) \in \Omega$;

and for $|\alpha| < m$, a_α are locally L_{p_α} integrable with $p_\alpha > (N+m)/(m-|\alpha|)$.

THEOREM. *Under the above assumptions, let* $u \in W^{m,1}_{p;loc}(\Omega)$ *be a generalized solution of equation* (2.1) *with*

$$1 < p < \min_{m-N<|\alpha|<m} p_\alpha, \qquad p \leq \min_{|\alpha| \leq m-N} p_\alpha.$$

If $u(x_0,t_0) = 0$ *at a point* $(x_0,t_0) \in \Omega$, *then one of the following alternatives holds true:*

either (i) *u vanishes of infinite order at* (x_0,t_0); *that is,*

$$\varepsilon^{-h} u(x_0 + \varepsilon x, t_0 + \varepsilon^m t) \to 0 \text{ in } W^{m,1}_{p;loc}(\mathbf{R}^N \times \mathbf{R}) \text{ as } \varepsilon \downarrow 0 \tag{3.2}$$

for all $h > 0$;

or else (ii) *there exists an integer* $h \geq 1$ *such that*

$$\varepsilon^{-h} u(x_0 + \varepsilon x, t_0 + \varepsilon^m t) \to \Phi(x,t) \text{ in } W^{m,1}_{p;loc}(\mathbf{R}^N \times \mathbf{R}) \text{ as } \varepsilon \downarrow 0 \tag{3.3}$$

where $\Phi(x,t)$ *is a polynomial of parabolic homogeneity of degree* h:

$$\Phi(\lambda x, \lambda^m t) = \lambda^h \Phi(x,t) \qquad (x,t) \in \mathbf{R}^N \times \mathbf{R}, \lambda > 0, \tag{3.4}$$

satisfies the osculating equation

$$\Phi_t(x,t) = \sum_{|\alpha|=m} a_\alpha(x_0,t_0) D_x^\alpha \Phi(x,t) \qquad (x,t) \in \mathbf{R}^N \times \mathbf{R}, \tag{3.5}$$

and is not identically zero.

3.2 Hausdorff Dimension of Zero Sets. Let $u \not\equiv 0$ be a solution of equation (3.1) in $\Omega \subset \mathbf{R}^N \times \mathbf{R}$. For a positive integer h, denote by $Z_h(u)$ the set of all zero points $(x_0,t_0) \in \Omega$ of u of vanishing order $\geq h$. Let $Z_\infty(u)$ be the set of all zero points $(x_0,t_0) \in \Omega$ of infinite vanishing order. Note that $\Omega \supset Z_1(u) \supset Z_2(u) \supset \cdots \supset Z_\infty(u)$ and that $Z_1(u)$ is exactly the zero set of u in Ω.

COROLLARY. *Assume all conditions in Theorem 3.1.*

(i) *If $p > (N+m)/m$, then the (Euclidean) Hausdorff dimension of $Z_1(u) \cap (\mathbf{R}^N \times \{t\}) \setminus \overline{Z_\infty(u)}$ is not larger than $N-1$ for every t and the parabolic Hausdorff dimension of $Z_1(u) \setminus \overline{Z_\infty(u)}$ is not larger than $N+m-1$.*

(ii) *If $p > N+m$, then the parabolic Hausdorff dimension of $Z_m(u) \setminus \overline{Z_\infty(u)}$ is not larger than $N+m-2$.*

Recall that the standard Hausdorff measure is built on the Euclidean distance $|x-x'|$, whereas the parabolic Hausdorff measure and the parabolic Hausdorff dimension are defined using the parabolic distance $|x-x'| + |t-t'|^{1/m}$ in the space-time $\mathbf{R}^N \times \mathbf{R}$. If E is a k-dimensional vector subspace of $\mathbf{R}^N$, the parabolic Hausdorff dimension of $E \times \mathbf{R}$ is $k+m$. In particular, the parabolic Hausdorff dimension of the whole space-time $\mathbf{R}^N \times \mathbf{R}$ is equal to $N+m$. The dimension estimates in Corollaries 2.3 and 3.2 are optimal, as one can immediately see from simple examples.

4 Unique Continuation for Elliptic PDEs

The understanding of the unique continuation property of elliptic equations is much better than that of parabolic equations. The first work dates back to 1930s. Carleman proved in [6] the strong unique continuation for second order elliptic equations in two dimensions and introduced a basic technical tool, the so-called "Carleman inequality". The result was later generalized to second order elliptic equations in arbitrary dimensions by Aronszajn [3] and Cordes [12]. There have been new developments recently in relaxing the smoothness assumptions on coefficients (see review articles [19, 27]).

The unique continuation theorem does not necessarily hold for elliptic equations of fourth or higher order (see [22]).

5 Unique Continuation for Parabolic PDEs

5.1 Examples of Infinite Order Vanishing. The fundamental solution

$$H(x,t) = \begin{cases} (4\pi t)^{-N/2} \exp\left\{-\frac{|x|^2}{4t}\right\} & (x,t) \in \mathbf{R}^N \times (0,\infty), \\ 0 & (x,t) \in \mathbf{R}^N \times (-\infty, 0] \setminus \{(0,0)\}, \end{cases}$$

of the classical heat equation $u_t = \Delta u$ vanishes of infinite order at any point $(x, 0)$ with $x \neq 0$. One can even construct a nontrivial solution of the heat equation on $\mathbf{R}^N \times \mathbf{R}$, which is smooth everywhere and vanishes on $\mathbf{R}^N \times (-\infty, 0]$ (see [13]).

These examples show that the unique continuation property in the usual sense fails for (second order) parabolic equations. In order to get positive results, some careful reformulations are needed.

5.2 Autonomous Equations. For second order parabolic equations with time-independent coefficients: weak unique continuation theorems were proved decades ago (see [18, 28]); more recently, a strong unique continuation theorem is obtained by F.-H. Lin ([21]). A basic observation is that solutions of this class of equations can be represented by a Fourier series in terms of eigenfunctions of the corresponding elliptic operator. This, combined with some nontrivial arguments, makes it possible to reduce the analysis to the elliptic unique continuation theorem.

5.3 Nonautonomous Equations: Weak Unique Continuation. Weak unique continuation results are proved in [25, 26] for parabolic equations with time-dependent coefficients. The reduction technique mentioned in Subsection 5.2 is no longer effective for this class. Instead, the arguments [25, 26] are based on intricate derivations of some parabolic variants of the Carleman inequality.

5.4 Nonautonomous Equations: Strong Unique Continuation. In applications to nonlinear parabolic problems, most often what we need is strong unique continuation theorem for parabolic equations with time-dependent coefficients. The understanding in this direction is far from complete. The reduction technique in Subsection 5.2 does not work. Moreover, the Carleman inequalities used in the proofs of weak unique continuation in [25, 26] are not enough to derive strong unique continuation.

The following theorem was established in [9] by a quite different idea. The key steps were to recast parabolic equations under the self-similar variables and to derive appropriate energy estimates (see Section 8).

THEOREM. *Let $u(x,t)$ satisfy*

$$u_t = \Delta u + \sum_{j=1}^{N} b_j(x,t)\partial_j u + c(x,t)u \qquad x \in \mathbf{R}^N, t \in (T_1, T_2), \tag{5.1}$$

$$|u(x,t)| \leq C_1 \exp\left\{C_2|x|^2\right\} \qquad x \in \mathbf{R}^N, t \in (T_1, T_2), \tag{5.2}$$

where the coefficients $b_j(x,t)$ *and* $c(x,t)$ *are bounded measurable functions, and* C_1 *and* C_2 *are constants. If* $(x_0,t_0) \in \mathbf{R}^N \times (T_1,T_2)$ *and* $u(x,t_0) = O(|x-x_0|^h)$ *for all* $h>0$ *as* $x \to x_0$, *then* $u \equiv 0$ *on* $\mathbf{R}^N \times (T_1,T_2)$.

If u is a nontrivial solution of (5.1)-(5.2), the above result states that $Z_\infty(u)$ is empty. This readily yields the polynomial asymptotics of u at any zero point, in light of Theorem 3.1. Moreover we can subsequently prove the Hausdorff dimension estimates, as we did in Corollary 3.2. (The derivation of the polynomial asymptotics we used in paper [9] was not exactly done in this way; it actually used a little more complicated argument.)

6 Lyapunov Exponents and Asymptotics

The proofs of Theorems 2.2, 3.1, and 5.4 are all related to a dynamic result of perturbed operator iterations.

Let $\{u_k\}_{k\geq 0}$ be a sequence in a Banach space X satisfying

$$u_{k+1} = Ku_k + \xi_k \qquad k \geq 0, \tag{6.1}$$

where K is a compact linear operator in X and $\{\xi_k\} \subset X$ is "small" in the sense that

$$u_k \neq 0 \ (k \geq 0) \quad \text{and} \quad \lim_{k\to\infty} \frac{\|\xi_k\|}{\|u_k\|} = 0. \tag{6.2}$$

The next result quoted from [10] describes the growth/decay rate and the asymptotic direction of $\{u_k\}$, namely, the limits of $\|u_k\|^{1/k}$ and $u_k/\|u_k\|$ as $k \to \infty$. The limits of $k^{-1}\log\|u_k\|$ are called *Lyapunov exponents* in the theory of dynamical systems. For more details, see [16, 7, 10].

6.1 LEMMA. (i) Lyapunov Exponent: *The limit* $\lim_{k\to\infty}\|u_k\|^{1/k} = \lambda$ *exists and is either* 0 *or else equal to the modulus of a nonzero eigenvalue of* K*;*

(ii) Asymptotic Direction: *If the limit* λ *in Part* (i) *is positive, then* $\hat{u}_k := u_k/\|u_k\|$ *is relatively compact and its limit set*

$$\omega(\hat{u}) := \{\phi \in X \,|\, \hat{u}_{k_n} \to \phi \text{ for some subsequence } k_n \to \infty\}$$

consists of unit vectors in X_λ, *which is the direct sum of generalized eigenspaces of* K *of eigenvalues on the circle* $\{|z| = \lambda\}$*;*

(iii) Finer Asymptotics: *If* $\|\xi_k\|/\|u_k\| = O(\mu^k)$ *for some* $\mu \in (0,1)$ *and if* λ *is positive in Part* (i), *then there exist a* $\phi \in X_\lambda \setminus \{0\}$ *and a* $\nu \in (0,1)$ *such that*

$$u_k = K^k\phi + O(\lambda^k\nu^k) \quad \text{as } k \to \infty.$$

In the applications to partial differential equations, the case where the limit λ in Part (i) equals 0 corresponds to the case of the infinite order vanishing in Theorems 2.2 and 3.1. The result in Part (iii) is used to obtain the polynomial asymptotic results.

7 Rescaling Elliptic PDEs

The main step of the proof of Theorem 2.2 is to split the rescaled solution family into two parts to which the dynamic lemma in the last section can be applied.

Let u be as in Theorem 2.2. Without loss of generality, we may assume that $x_0 = 0$, $\overline{B_1} \subset \Omega$, and $u \not\equiv 0$ on any B_ε with $0 < \varepsilon < 1$. Let

$$A_0 = \sum_{|\alpha|=m} a_\alpha(0) D_x^\alpha,$$

and choose a fundamental solution $G(x)$ of A_0; that is, $A_0 G(x) = \delta(x)$. Denote $X = W_p^m(B_1)$ and define operators S and T in X by

$$(Sv)(x) := v(x/2),$$

$$(Tv)(x) := \int_{B_1} G(x-y)(A_0 v)(y)dy.$$

The standard interior L_p estimates for elliptic equations imply that $K := S - ST$ is a compact operator in X. By some elementary analysis, we find that the eigenvalues K are $\{2^{-h}\}_{h\geq 0}$ and that the corresponding eigenfunctions are given by nontrivial homogeneous polynomials Φ satisfying $A_0\Phi(x) = 0$ (see [10, Section5]).

Let $u_k := S^k u$ and $\xi_k := STu_k$. We then have

$$u_{k+1} = Ku_k + \xi_k.$$

The coefficients conditions guarantee that $\|\xi_k\|/\|u_k\|$ decays like $O(\mu^k)$ for some $0 < \mu < 1$. Applying now Lemma 6.1, we obtain Theorem 2.2.

The above proof is strikingly simple and natural. So it comes a surprise to us that this approach has not been adapted previously. We feel that the method has even wider applications than have been described here. To illustrate its strength and effectiveness, let us mention that a similar argument leads to a simple new proof of the well-known L^p and Schauder estimates for elliptic and parabolic equations of arbitrary order. The traditional proofs of these results needed masterful use of a great number of functional inequalities. (I confess that although I have applied these estimates on many occasions, the original proofs in the literature are so complicated and long that I have never tried to read them until very recently.) In contrast, our method only uses the dynamic Lemma 6.1, the homothety scaling, and the corresponding estimates for the osculating equations. As a matter of fact, the L^p estimates are almost corollaries of our Theorems 2.2 and 3.1. The Schauder estimates are obtained if we choose C^δ as the basic function space instead of L^p. Both interior and boundary estimates can be treated in a unified fashion. The details will be reported somewhere else.

8 Parabolic Self-Similar Variables

In this section we shall discuss the proof of Theorem 5.4. We first show how to convert (5.1) into a perturbation problem and next give an energy estimate excluding the super-exponential decay of solutions.

8.1 Self-Similar Variables. Let

$$y = (x - x_0)/\sqrt{t_0 - t}, \qquad t_0 - t = e^{-s}$$

and set

$$u(x,t) = v(y,s).$$

The asymptotics of $u(x,t)$ as $x \to x_0$ and $t \uparrow t_0$ can be read from the behavior of $v(y,s)$ as $s \to \infty$. A computation shows that under the new coordinates (y,s), equation (5.1) is transformed into

$$v_s - \Delta v + \frac{1}{2} y \cdot \nabla v = e^{-s/2} \sum b_j \partial_j v + e^{-s} cv, \tag{8.1}$$

$$y \in \mathbf{R}^N, s > -\log(t_0 - T_1).$$

When $s \to \infty$, the right side is an exponentially decaying perturbation. This allows us to relate the long-term behavior of $v(\cdot, s)$ to that of solutions of the unperturbed equation.

The complete detail of the arguments can be found in my recent paper [9]. The analogous ideas have been applied to blow-up problems by Giga and Kohn [14]. There were also backward uniqueness results obtained along similar lines [2, 11, 20, 23].

8.2 Energy Estimates. In order to prove Theorem 5.4, we need to show that any nontrivial solution v of (8.1) cannot decay faster than certain exponential rate. To be more precise, consider the Hilbert space

$$X = L^2(\mathbf{R}^N, \exp(-|y|^2/4)dy).$$

It follows from (5.2) that $v(\cdot, s) \in X$ for sufficiently large s. The elliptic operator appearing in the left side of equation (8.1),

$$-\Delta + \frac{1}{2} y \cdot \nabla,$$

can be extended to a self-adjoint operator A in X. The following provides the required estimates.

LEMMA. *Consider the Dirichlet quotient:*

$$Q(s) := 1 + \frac{\langle Av(\cdot, s), v(\cdot, s)\rangle_X}{\|v(\cdot, s)\|_X^2}.$$

Then,

(i) $Q(s) \leq M_0 Q(s_0)$ *for* $s \geq s_0$, *where* $M_0 > 0$ *is a constant depending only on the coefficients* b_j *and* c.

(ii) $\|v(\cdot,s)\|_X \geq \|v(\cdot,s_0)\|_X e^{-M_1(s-s_0)}$ *for* $s \geq s_0$, *where* $M_1 > 0$ *is a constant depending only on* M_0 *and* $Q(s_0)$.

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