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On the Doubling Period Reversible Cusps in $\mathbb{R}^3$

Alain Jacquemard and Marco-Antonio Teixeira

Abstract: We deal with normal forms of germs at the origin of reversible diffeomorphisms of the space whose linear part is unipotent with two negative eigenvalues. The computing of these normal forms involves effective algebraic geometry algorithms. We also study generic one-paramater deformations.

Key words:[1] diffeomorphism, normal form, reversibility.

1 Introduction

Our goal in this paper is to give a qualitative analysis of reversible mappings around symmetric fixed points on $\mathbb{R}^3$. In [De], Devaney studies reversible vector fields in $\mathbb{R}^{2n}$. There, it is also studied the behavior of symmetric periodic orbits of such systems by means of the Poincaré first return mapping, which is itself a (germ of) reversible diffeomorphism of $\mathbb{R}^{2n-1}$. So, it is worth studying such diffeomorphisms as well as their unfoldings in the space of the reversible mappings.

We mention that the concept of reversibility is intrinsically linked to a given type of involution. We shall consider here (germs at 0 of) involutions of $\mathbb{R}^3$ such that their fixed points set admits a smooth manifold structure near the origin. In [JT1, JT2] we dealt with the case of involutions satisfying the condition $g - I_d$ non-singular at 0. In this paper we drop this condition, and simply assume that the codimension of this manifold of fixed points is 1. We denote by Γ^1_∞ the set of such (germs of) involutions. By Montgomery-Bochner's Theorem ([MZ]), it is well known that such a germ is $\mathcal{C}^\infty$ -conjugated to the germ of the involution

$$f_0 \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

We say that a mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is *f-reversible* if $\phi = g \circ f$ with $g \in \Gamma^1_\infty$. We denote by Ξ the space of all reversible mappings and by Ξ_0 the space of $\phi \in \Xi$ such that $\phi(0) = 0$.

We only consider here f_0-reversible mappings, but similar results hold when one allows f-reversible mappings with f near f_0 (see for instance [JT2]).

In this paper we deal with reversible mappings of $\mathbb{R}^3$, such that the linear part at

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a given fixed point is

$$U \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y-z \\ -z \end{pmatrix}$$

In the main result of this work, we classify and present normal forms of generic elements having such a linear part. The strategy of classification is the same as developed in [JT1, JT3], using the algorithmic properties of the ideals of polynomials as well as the polynomial factorization. As a consequence, unfoldings are also studied.

First, let us recall the classification of reversible mappings from their linear part. For any $\phi \in \Xi$ having a fixed point at $p \in \mathbb{R}^3$, let us denote by τ_p the translation such that $\tau_p(0) = p$. Observe that if p is near 0, then ${\tau_p}^{-1} \circ \phi \circ \tau_p \in \Xi_0$ and ${\tau_p}^{-1} \circ f_0 \circ \tau_p$ is near f_0. In the following, we denote by $\mathcal{V}_0$ a fundamental system of neighborhoods of $0 \in \mathbb{R}^3$.

Notation 1 *Let G_0 be the set of mappings $\phi \in \Xi_0$ such that the eigenvalues of $\ell_0(\phi)$ are all distinct. And let G be the set of mappings $\phi \in \Xi$ such that there exists $V \in \mathcal{V}_0$ such that for all $p \in Fix(\phi) \cap V \quad {\tau_p}^{-1} \circ \phi \circ \tau_p \in G_0$.*

When $\phi \in G$, we say that ϕ is *semi-elliptic* (*resp. semi-hyperbolic*) at $p \in Fix(\phi)$ if the eigenvalues of $\ell_p(\phi)$ are $1, \lambda, \lambda^{-1}$ with $\lambda \in S^1 \setminus \{-1, 1\}$ (*resp.* with $\lambda \in \mathbb{R} \setminus \{-1, 1\}$). We denote by $\mathrm{Trace}_x(\phi)$ the trace of $\ell_x(\phi)$. This function plays a great role towards the classification of mappings. Let us recall the following:

Theorem 1 **([JT1])**

1. *G is open in Ξ.*
2. *$\phi \in G_0 \iff |\mathit{Trace}_0(\phi) - 1| \neq 2$*
3. *there is a decomposition $G = G_s \cup G_e$ such that for all $\phi \in G$, there exists $V \in \mathcal{V}_0$ such that $Fix(\phi) \cap V$ is a smooth curve of points, and,*
   - *if $\phi \in G_s$, for all $p \in Fix(\phi) \cap V$, ϕ is semi-hyperbolic at p.*
   - *if $\phi \in G_e$, for all $p \in Fix(\phi) \cap V$, ϕ is semi-elliptic at p.*

Observe that the Trace_0 function is onto $\mathbb{R}$. It is clear that taking Trace_0 as a parameter, the orbit structure of ϕ can change drastically at the (bifurcation) values -1 and 3. The value 3 is studied in [JT2]. We study here the case where the linear part of diffeomorphism is of type U (with one block in the canonical Jordan's form). We denote by Ξ^- the set of mappings $\phi \in \Xi$ such that $\mathrm{Trace}_0(\phi) < 0$. We use the notations $[x,y,z]^n$ (*resp.* $[x]^n$) for terms belonging to the n-th power of the maximal ideal $< x,y,z >$ generated by the coordinate functions (*resp.* $< x >$).

We recall that if $\phi \in \Xi^-$, then $Fix(g)$ and $Fix(f_0)$ are transversal at 0. Hence, $Fix(\phi)$ is a smooth curve (refer to [JT1, JT2]). Assume that $\theta :]-\varepsilon, \varepsilon[\to \mathbb{R}^3$, $t \mapsto \theta(t)$ is a regular parametrization of $Fix(\phi)$ with $\theta(0) = p$. Let $V \in \mathcal{V}_0$ and $\varphi : V \subset \mathbb{R}^3 \to \mathbb{R}$ be any analytic map such that $\varphi(p) = 0$. We may develop $\phi(\theta(t)) = a_\alpha t^\alpha + \ldots$ with $a_\alpha \neq 0$. The number α does not depend of the choice of the regular parametrization, and we call it the *order* of φ at p on $Fix(\phi)$.
We give now this analytic definition:

Definition 1 *We say that $\phi \in \Xi$*

1. *is a degenerate doubling period cusp of order n at 0 if $\ell_0(\phi) = U$ and the mapping $p \mapsto (Trace_p(\phi) + 1)$ is of order $n+1$ at 0 on $Fix(\phi)$. We denote by Δ_0^n the set of such cusps.*

2. *is a degenerate doubling period cusp of order n if there exists $V \in \mathcal{V}_0$ and $p \in Fix(\phi) \cap \mathcal{V}_0$ such that $\tau_p^{-1} \circ \phi \circ \tau_p \in \Delta_0^n$. We denote by Δ^n the set of such cusps.*

Remark : of course the term cusp has to be interpreted as an abuse of terminology.

1.1 Statements of the Results

Theorem 2 *$\phi \in \Xi_0$ is a degenerate doubling period cusp of order 0 if and only if $j_0^2(\phi)$ presents the following normal form:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y - z + (2y+z)\,x \\ -z + 2\,(2y+z)\,x \end{pmatrix} \tag{1}$$

Theorem 3 *If $\phi \in \Xi^-$ is structurally stable then $\phi \in G \cup \Delta^0$*

Theorem 4 *$\phi \in \Xi_0$ is a degenerate doubling period cusp of order 1 if and only if $j_0^3(\phi)$ presents the following normal form:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y - z + (2y+z)\left(a\,x^2 + b\,(2y+z)\right) \\ -z + 2\,(2y+z)\left(a\,x^2 + b\,(2y+z)\right) \end{pmatrix} \tag{2}$$

with $(a,b) \in \mathbb{R}^2$, $a \neq 0$.

Remark : the normal forms (1), (2), are polynomial reversible mappings.

Theorem 5 *If ϕ_Λ, $\Lambda \in \mathbb{R}^k$, is a deformation of $\phi_0 \in \Delta_0^1$, in normal form (2), then $j_0^3(\phi_\Lambda)$ is $\mathcal{C}^\infty$ -equivalent to $j_0^3(\tilde{\phi}_\lambda)$ where $\tilde{\phi}_\lambda$ ($\lambda \in \mathbb{R}$) is a one-parameter deformation of the type:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + [x,y,z]^4 \\ -y - z + (2\,y + z)\left(\lambda + a\,x^2 + b\,(2\,y + z)\right) + [x,y,z]^4 \\ -z + 2\,(2\,y + z)\left(\lambda + a\,x^2 + b\,(2\,y + z)\right) + [x,y,z]^4 \end{pmatrix}$$

Hence, cusps of order 1 appear as codimension one reversible mappings.

1.2 Conclusion

Computer algebra methods are very useful in the theory of reversibility. Because of the presence of integral eigenvalues, the CFR algorithm is very efficient and provides easily a classification of the mappings, and simplifies the study of their unfoldings. The methods we develop here may be used to classify higher order cusps, but one should be aware of the fact that the algorithmic complexity increases rapidly.

2 Effective algebraic geometry and normal forms

Our strategy (see [L] for other strategies) for finding normal forms of reversible mappings is based on direct computations on their finite jets. Our method is a systematic one which uses some of the classical tools of Effective Algebraic Geometry and Computer Algebra.

2.1 Reversibility and algebraic geometry

We briefly recall here the method exposed in [JT1, JT3] in order to get normal forms for finite jets of reversible mappings. Notice that the multiplication on the right by f_0 induces a bijection $R\,:\,\Xi \longrightarrow \Gamma^1_\infty$ defined by $R(\phi) = \phi \circ f_0$.

1. By the bijection R we carry out classification problems concerning reversible mappings to classification problems concerning elements of Γ^1_∞.

2. We then consider finite jets of involutions. Let us fix an order $n \in \mathbb{N}$, $n \geq 2$. The involutivity condition $g \circ g = I_d$ upon a mapping g yields necessary conditions upon $j_0^n(g)$. These conditions are polynomial in terms of the (formal) coefficients of $j_0^n(g)$, denoted by $[C(g)]$. At this point we have to deal with a finite set of polynomial equations $f_i([C(g)]) = 0$, $i \in [1, k_n]$,

which defines an algebraic variety $V_n \subset \mathbb{R}^p$ with $p = card(C(g))$. It is worth considering the dual problem about the ideal $I_n = < f_1, f_2, \ldots, f_{k_n} >$.

3. To characterize the ideal I_n, we dispose of effective well-known algorithms such as the construction of Gröbner bases, and the integral factorization. One of the key point of our method is the association of these two algorithms. We develop this aspect in the following.

2.1.1 The CFR algorithm

Let us recall briefly some basic facts (the reader may consult [JT1] for further details). Let $\mathcal{I}$ be an ideal generated by polynomials in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$ and suppose that a generator f is factorized in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$ as $f = f'\, f''$. Then, $\mathcal{I}$ has the same zero set in $\mathbb{R}^n$ as the intersection $\mathcal{I}' \cap \mathcal{I}''$, replacing f by these factors. This idea can be applied during the construction of a Gröbner basis for the ideal $\mathcal{I}$ (see [D, MMN, GL, JT1]). We recall that this construction is done by recursively adding a new polynomial (a syzygy polynomial) to a set of generators (we refer to [CLO] for this classical algorithm). If this new polynomial splits in $\mathbb{Z}\,[X_1, X_2, \ldots, X_n]$, we replace the computation of the Gröbner basis by the computation of a list of Gröbner bases: the sygyzy polynomial is replaced by each of its square-free factors, and each of these factors yield a new basis. The complexity of the computation of the list of the new bases is much less than the initial complexity of the computation of the original basis, provided that factorizations occur. In our classification problem this is actually the case. Let us mention that the factorization may be seen as a kind of blowing-up of the (very singular) algebraic variety corresponding to the involutivity conditions.
However this process often leads to a great number of new ideals (or bases) and one has to process a reduction algorithm in order to eliminate redundant ideal (or bases). This elimination can also be done algorithmically (refer to [JT1]). In the following we refer to this algorithm in two steps (construction with factorization, then reduction) as the CFR algorithm.

2.2 Effective classification of involutions

We follow a recursive (depending upon the order of the jet) method in order to classify the reversible mappings (or, equivalently, the associated involutions). The generic classification of such reversible mappings is already made at order 1 (see [JT1]). Assume that a generic classification relying on the $(n-1)$-jet $(n \geq 2)$ is already achieved. We suppose that the reversible mappings which are not yet classified have a trivial $(n-1)$-jet, that is $j_0^{n-1}(\phi) = \ell_0(\phi)$. Equivalently, the involution $g = \phi \circ f_0$ is such that $j_0^{n-1}(g) = \ell_0(g)$. We then classify the n-jets of these involutions $\phi \circ f_0$:

- We begin by finding the necessary conditions on the n-jet for g to be an involution. This is done via the CFR algorithm.

- We search all polynomial conjugations χ of the type

$$\chi \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \sum\limits_{i+j+k=n} \alpha_{ijk}\, x^i\, y^j\, z^k \\ y + \sum\limits_{i+j+k=n} \beta_{ijk}\, x^i\, y^j\, z^k \\ z + \sum\limits_{i+j+k=n} \gamma_{ijk}\, x^i\, y^j\, z^k \end{pmatrix}$$

such that $j_0^n(\chi^{-1} \circ g \circ \chi) \in \Gamma_\infty^1$. We proceed as follows:
we compute firstly the formal conjugation (up to order n terms). In this way, we get the polynomial conditions corresponding to the involutivity conditions. Then, we apply again the CFR algorithm to express the $\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}$, $(i,j,k) \in [0,n]\ /\ i+j+k=n$, in terms of the coefficients $C(g)$.

The computings were processed on the servers of the GDR MEDICIS from a code written in Axiom. The results are given in the next section.

3 Normal forms

3.1 2-jets and quadratic involutions

Assume $\phi \in \Xi_0$, such that $\ell_0(\phi) = U$, and let $g = \phi \circ f_0$. This mapping has the form

$$g \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + \sum\limits_{i+j+k=2} a_{ijk}\, x^i\, y^j\, z^k \\ -y + z + \sum\limits_{i+j+k=2} b_{ijk}\, x^i\, y^j\, z^k \\ -z + \sum\limits_{i+j+k=2} c_{ijk}\, x^i\, y^j\, z^k \end{pmatrix}$$

The conditions of involutivity are:

$$c_{200} = 0,\ c_{110} - 2\,b_{110} = 0,\ c_{101} + b_{110} = 0,\ c_{020} = 0,\ c_{011} - 2\,b_{020} - 2\,b_{011} = 0,$$

$$c_{002} + b_{020} + b_{011} = 0,\ a_{200} = 0,\ a_{110} + 2\,a_{101} = 0,\ a_{020} = 0,\ a_{011} + 2\ a_{002} = 0$$

and we get a corresponding normal form for the 2-jet. The CFR algorithm provides us the following conjugation:

$$\chi_2 \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + a_{101}\, x\, y + a_{002}\, y^2 \\ y + \frac{1}{2}\, b_{200}\, x^2 + \frac{1}{2}\, b_{020}\, y^2 - \frac{1}{4}\, (b_{020} - 2\, b_{002})\ z^2 \\ z - \frac{1}{2}\, (b_{110} + 2\, b_{101})\, x\, z - (b_{020} + b_{011})\, y\, z \end{pmatrix}$$

which yields the following normal form for the 2-jet:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y+z+\frac{1}{2}\,b_{110}\,x\,(2\,y-z) \\ z+b_{110}\,x\,(2\,y-z) \end{pmatrix}$$

(which is a polynomial involution). By considerations upon the codimension of the fixed points manifold, the first coordinate function of g has to be of the form $x \mapsto x+[2\,y-z]^3$. It is obvious to get the corresponding normal forms for the reversible mappings, just by changing z into $-z$. The trace function of ϕ is given by $-1+2\,b_{110}\,x+[x,y,z]^2$, while the fixed points manifold is given by $y=[x]^3, z=[x]^3$. So the trace is linear if and only if $b_{110}\neq 0$. We then conjugate by $b_{110}^{-1}\,I_d$ and get the normal form (1). This proves Theorem 2. ∎

3.2 Proof of Theorem 3

Assume that $\phi\in\Xi^-$. The property $\ell_0(\phi)$ has distinct eigenvalues is stable under perturbation of ϕ. On the other hand, the property $\mathrm{Trace}_0(\phi)=-1$, and $\ell_0(\phi)$ has no Jordan block, is not stable. Assume now that $\phi\in\Delta_0^0$. By Jordan's block theory, $\ell_0(\phi)$ is stable, and we may assume up to $\mathcal{C}^\infty$ -conjugation that $\ell_0(\phi')=U$ for ϕ' sufficiently near ϕ, and that $\phi'(0)=0$. Applying then the conjugation χ_2, ϕ' has a normal form of type (1). And the coefficient b'_{110} remains different of zero. ∎

3.3 First order cusps

Assume that $\phi\notin\Delta_0^0$, with $\ell_0(\phi)=U$. Applying the same process as before, we find conditions on the 3-jet of $\phi\circ f_0$. Then, we conjugate it by the mapping (provided by the CFR algorithm)

$$\chi_3 \quad : \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+\chi_y(x,y,z) \\ z+\chi_z(x,y,z) \end{pmatrix}$$

with:

$$\begin{cases} \chi_y(x,y,z)=\frac{1}{2}\,b_{300}\,x^3+\frac{1}{2}\,b_{120}\,x\,y^2-\left(\frac{1}{4}\,b_{120}-\frac{1}{2}\,b_{102}\right)x\,z^2+\left(\frac{1}{4}\,b_{030}-\frac{1}{3}\,b_{012}\right)y^3 \\ \chi_z(x,y,z)=\left(\frac{1}{2}\,b_{210}+b_{201}\right)x^2\,z-\left(b_{120}+b_{111}\right)x\,y\,z-\left(\frac{3}{4}\,b_{030}+b_{021}+b_{012}\right)y^2\,z \\ \qquad\qquad +\left(\frac{1}{8}\,b_{030}-\frac{1}{3}\,b_{012}-b_{003}\right)z^3 \end{cases}$$

and get finally the normal form for the 3-jet:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y+z+\varphi(x,y,z) \\ z+2\,\varphi(x,y,z) \end{pmatrix}$$

with

$$\varphi(x,y,z) = (2\,y - z)\left(b_{210}x^2 + b_{110}\;x + \frac{1}{4}\,b_{030}\,(2\,y - z)^2\right)$$

Moreover the trace function of $\phi = g \circ f_0$ is:

$$2\;b_{210}\;x^2 + b_{030}\;\frac{3}{2}\,(2\;y + z)^2 - 1$$

so, the mapping $p \mapsto \mathrm{Trace}_p(\phi) + 1$ is of order 2 on the fixed points manifold (defined by $y = [x]^4, z = [x]^4$) provided that $b_{210} \neq 0$. As seen previously, the first coordinate function of ϕ has the form $x \mapsto x + [2\,y + z]^4$, and we get the normal form (2) just by changing z into $-z$ in the normal form of g. This proves Theorem 4. ∎

4 Unfoldings of first order doubling period cusps

In this section we study the deformations of any $\phi_0 \in \Delta_0^1$. It arises, due to the non-existence of a differentiable structure in Ξ, obstructions in considering the usual concept of codimension of a singularity. We choose an alternative way which is developed in the sequel.
We first study the reversible mappings in a neighborhood of a degenerate cusp of first order :

Proposition 4.1 *Let $\phi_0 \in \Delta_0^1$ be in normal form (2). There exists a neighborhood $\mathcal{B}$ of ϕ_0 in Ξ such that any $\phi \in \mathcal{B}$ is $\mathcal{C}^\infty$ -conjugated (near I_d) to a mapping whose 3-jet is written (in a unique way):*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y - z + (2\,y + z)\,(\mu + a\,x^2 + b\;(2\,y + z)) \\ -z + 2\;(2\,y + z)\,(\mu + a\,x^2 + b\;(2\,y + z)) \end{pmatrix}$$

where μ is a $\mathcal{C}^\infty$ -function of ϕ. Let us denote by $\omega : \mathcal{B} \mapsto \mathbb{R}$ the mapping defined by $\omega(\phi) = \mu$.

Proof:
A straightforward calculation shows that the *affine* part of $g = \phi \circ f_0$ must have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ 2\,\alpha - y + z + \mu\;(2\,y - z) \\ z + 2\,\mu\;(2\,y - z) \end{pmatrix}$$

By Jordan's block stability theory, if ϕ is sufficiently near ϕ_0 (say, in a neighborhood $\mathcal{B}$), we can conjugate the linear part by a linear mapping θ_μ near I_d which

$\mathcal{C}^\infty$ depends upon μ such that ${\theta_\mu}^{-1} \circ \phi \circ \theta_\mu$ has still U as linear part. So, for a while, we restrict ourselves to ϕ with $\mu = 0$. Now, conjugating by the translation t_α: $(x, y, z) \mapsto (x, y + \alpha, z)$, we may consider mappings ϕ such that $\phi(0) = 0$. We then conjugate $\phi \circ f_0$ by χ_3, and we get the following form of the 3-jet for ϕ :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y - z + \varphi(x, y, z) \\ -z + 2\,\varphi(x, y, z) \end{pmatrix}$$

with

$$\varphi(x, y, z) = (2\,y + z)\left(c\,x + a\,x^2 + b\,(2\,y + z)\right)$$

We remark that up to rescaling, the coefficients a and b remain invariant. It suffices now to perform a translation on the variable x, to make the term $c\,x$ disappear ($a \neq 0$). Doing this, the terms of degree 1 in $(2\,y + z)$ reappear. Notice that all transformations defined above are $\mathcal{C}^\infty$, and they just depend upon the coefficients of the 3-jet. ■

Let us study now the generic deformations of $\phi_0 \in \Delta_0^1$.

Definition 2 *Let $\phi_0 \in \Xi$.*

1. *We say that two C^r deformations (with $r \geq 3$) ϕ'_Λ and ϕ_Λ of ϕ_0 are 3-equivalent if there exists a deformation θ_Λ of I_d such that $j_0^3\phi'_\Lambda = j_0^3({\theta_\Lambda}^{-1} \circ \phi_\Lambda \circ \theta_\Lambda)$.*

2. *We say that a C^r (with $r \geq 3$) deformation of $\phi_0 \in \Delta_0^1$ is k-generic (or generic) if the mapping $(\Lambda, x, y, z) \mapsto (x, y, z, j^k_{(x,y,z)}(\phi_\Lambda))$ is transversal at $(0, 0, 0, \ldots, 0, 0) \in \mathbb{R}^3 \times \mathbb{R}^{n_k}$ to $(\phi - I_d, j^k(\phi))^{-1}((0, 0, 0) \times \Delta^1)$.*

Using Proposition 4.1, up to 3-equivalence, we may restrict us to deformations of ϕ_0 whose 3-jet has the form :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y - z + (2\,y + z)\left(\mu + a\,x^2 + b\,(2\,y + z)\right) \\ -z + 2\,(2\,y + z)\left(\mu + a\,x^2 + b\,(2\,y + z)\right) \end{pmatrix}$$

where μ is a $\mathcal{C}^\infty$ -function of Λ vanishing for $\Lambda = 0$.

Lemma 4.2 *Let ϕ_Λ be a deformation (with $r \geq 3$) of $\phi_0 \in \Delta_0^1$.*

1. *$\phi_\Lambda \in \Delta^1$ if and only if $\omega(\phi_\Lambda) = 0$.*

2. *ϕ_Λ is 3-generic if and only if the mapping $\Lambda \mapsto \omega \circ \phi_\Lambda$ is a submersion at $\Lambda = 0$.*

Proof:
A simple elimination shows that if $(x, y, z) \in Fix(\phi_\Lambda)$, then $y = [x, y, z]^4$, and $z = [x, y, z]^4$. Moreover,

$$\mathrm{Trace}_p(\phi_\Lambda) = -1 + 4\,\mu + 2\,a\,x^2 + \frac{3}{2}\,b\,(2\,y + z)^2 + [x, y, z]^3$$

Hence, at a fixed point $p_0 = (x_0, y_0, z_0)$, we have:

$$\mathrm{Trace}_{p_0}(\phi_\Lambda) = -1 + 4\,\mu + 2\,a\,x_0^2 + [x_0]^3$$

Let $p_0 \in Fix(\phi_\Lambda)$. Then ${\tau_{p_0}}^{-1} \circ \phi_\Lambda \circ \tau_{p_0} \in \Delta_0^1$ if and only if $\mu = 0$. Hence $\phi_\Lambda \in \Delta^1$ if and only if $\omega(\phi_\Lambda) = 0$ and $p_0 = 0$. Moreover ϕ_Λ is generic if and only if the Jacobian matrix of the mapping $(\Lambda, x, y, z) \mapsto (\omega \circ \phi_\Lambda, x, y, z)$ is surjective at $(0, 0, 0, 0)$. This is equivalent to the fact that the mapping $\Lambda \mapsto \omega \circ \phi_\Lambda$ is a submersion. ∎

Let us consider a generic deformation of ϕ_0 with one parameter. By the rank theorem, we can assume that the mapping $\lambda \mapsto \omega \circ \phi_\lambda$ is the identity mapping and this finishes the proof of Theorem 5. ∎

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Alain Jacquemard
UMR 5584 CNRS
Laboratoire de Topologie
Université de Bourgogne
BP 138 21004 DIJON cedex
jacmar@medicis.polytechnique.fr
France

Marco-Antonio Teixeira
Departamento de Matemática
IMECC
Universidade Estadual de Campinas
13081-970, Campinas, SP
teixeira@ime.unicamp.br
Brasil

GDR CNRS 1026 MEDICIS
http://medicis.polytechnique.fr