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Lines of Curvature, Umbilic Points and Carathéodory Conjecture

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1 Introduction

The study of umbilic points and lines of principal curvature go back to the works of Monge [Mon] and Dupin [Dup]. In these papers the first non-trivial example of the family of principal lines and umbilic points was considered. The global family of these lines was determined in the case of the ellipsoid with different axes and in all central quadric surfaces.

The configuration of principal lines around an umbilic point was studied by Cayley [Cay]. He was concerned with the number of principal lines approaching the umbilic point.

A crucial step in the study of lines of curvature near umbilic points, however, was taken by Darboux [Dar]. He described the whole local configuration of principal lines around the umbilic under generic hypotheses on the third order jet of the surface at the umbilic. This result can be regarded as an application of Poincaré's analysis and classification of solutions of differential equations in the plane near singular points.

A remarkable piece of research about umbilic points of more degenerate nature than those studied by Darboux was carried out by Gullstrand [Gul]. This work was motivated by applications to Ophthalmology.

The evolution of the interest of mathematicians in the lines of curvature near an umbilic point as well as in the evaluation of the index and the number of umbilic points in convex surfaces is not clearly recorded in the literature. One can only guess that it comes from Poincare's index formula for singularities of line fields and differential equations on surfaces. Papers on Carathéodory Conjecture constitute a landmark in this subject. Although no record of this conjecture by its author seems to appear in print. The earliest reference to it appears in Hamburger's papers [Ham].

Recently the subject of lines of curvature and umbilic points has be en renewed, both from the point of view of differential equations [SG1]-[SG7], [GGGT], [GS3], [Gui1]-[Gui2], [BF], [BT]; global analysis [SX1], [SX2] and Geometry [Sch], [Por]. The requirements of Computer Aided Design has also aroused pertinent work on umbilics and lines of curvature [MPW].

These notes contain a survey of results on umbilic points, motivated by ideas which have their origin in Differential Equations and Dynamical Systems.

A brief description of the contents of this survey follows.

Section 2 contains the basic definitions of umbilics and principal lines. Recent results concerning the structure defined by these elements on a surface are mentioned, including those of Gutierrez and Sotomayor [SG1], [SG2] and Garcia and Sotomayor [G-S2], among others.

Section 3 deals with new approaches to Carathéodory Conjecture and the analysis of equivalent conjectures. This part leans on contributions of Gutierrez and collaborators [GMS], [GS1], [GS2].

Section 4 includes a presentation of Klotz proof of Carathéodory Conjecture in the so called "regular case", which is the case most neatly understood.

2 Lines of curvature and umbilics

This section settles the basic definitions and terminology concerning lines of curvature and umbilic points on smooth surfaces.

The classical results are reviewed here and constitute an important background for the developments in this work.

The notion of principal configuration of a surface is introduced to synthesize the partition produced on the surface by its umbilic points and lines of curvature. It is the natural analog of the phase portrait of a vector field tangent to a surface.

The presentation of surfaces that follows focuses mainly the study of lines of curvature as related to the purpose of this work. Further details concerning geometric aspects of surfaces can be found in well known textbooks [Spi], [Str], [dCa], [One], among others.

2.1 Smooth Surfaces in Euclidean Space

The Euclidean 3-space $\mathbb{R}^3$ is endowed, by a once for all, with a fixed orientation defined by the volume form $dx \wedge dy \wedge dz$, where $\{x, y, z\}$ is an orthonormal chart relative to the inner product $\langle .,. \rangle$.

A *surface* $\mathbf{S}$ of *class* C^r in Euclidean space $\mathbb{R}^3$ is defined by a non empty set $\mathbf{S}$ which is *locally* defined in implicit form by an equation $\mathbf{f}(x, y, z) = 0$, where $\mathbf{f}$ is a real function of class C^r, $r \geq 1$, whose gradient $\nabla\mathbf{f} = (\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z)$ does not vanish on $\mathbf{S}$.

Locally, means here that every $\mathbf{p}$ in $\mathbf{S}$ has a neighborhood A in $\mathbb{R}^3$ where a C^r function $\mathbf{f}$ is defined, with $\nabla\mathbf{f} \neq 0$, and such that $A \cap \mathbf{S}$ is given by $\{\mathbf{f} = 0\}$.

The symbol r defines the class of $\mathbf{S}$; it may be a finite natural number ($\mathbf{f}$ is r times continuously differentiable), ∞ (infinity, $\mathbf{f}$ admits continuous partial derivatives of all orders) or ω (analytic, $\mathbf{f}$ is locally represented by its Taylor series).

A *surface* $\mathbf{S}$ of *class* C^r in Euclidean space $\mathbb{R}^3$ is equivalently defined by the requirement that every point $\mathbf{p}$ in $\mathbf{S}$ has a neighborhood A in $\mathbb{R}^3$ where a homeomorphism (u, v), from $U = A \cap S$ onto an open planar set V, is defined so that its inverse mapping $X = (u, v)^{-1}$, called a *parametrization*, is of class C^r and $X_u \wedge X_v \neq 0$. The map (u, v) is called a *chart* or *local coordinate* system on $\mathbf{S}$ with *domain* U; the unitary vector field on U defined by $N(u, v) = X_u \wedge X_v / |X_u \wedge X_v|$ is called the *associated* normal.

The surfaces of class C^∞ will be referred as *smooth*. The surfaces will be *oriented*. This means that $\mathbf{S}$ can be globally defined as $\mathbf{S} = \mathbf{f}^{-1}(0)$, where $\mathbf{f}$ is a C^r real valued function defined on an open set A of $\mathbb{R}^3$ such that $\mathbf{S} \subset A$ and $\nabla\mathbf{f}(\mathbf{p}) = (\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z)(\mathbf{p}) \neq 0$ at every point $\mathbf{p}$ in $\mathbf{S}$. The *positive orientation* on $\mathbf{S}$ is defined by the unit normal vector field $N = \nabla\mathbf{f}/|\nabla\mathbf{f}|$. The choice of another function $\mathbf{g}$ which also defines $\mathbf{S}$ implicitly as $\mathbf{S} = \mathbf{g}^{-1}(0)$, with $\nabla\mathbf{g} \neq 0$, forces $\mathbf{g} = a.\mathbf{f}$, with $a \neq 0$ on $\mathbf{S}$; the case $a < 0$, leads to the opposite normal $-N$ and, therefore, to the opposite orientation on $\mathbf{S}$. For $\mathbf{S}$ connected, which is the case of most surfaces studied here, only two orientations are possible.

In terms of charts, a surface is oriented if it can be covered by a collection of domains of charts, called *positive*, which on the overlap of two of them the correspondent associated normals coincide.

2.2 Fundamental Forms

The tangent bundle $T\mathbf{S} = \{(\mathbf{p}, \mathbf{v}); \mathbf{p} \in \mathbf{S}, \mathbf{v} \in \mathbb{R}^3, \langle N(\mathbf{p}), \mathbf{v}\rangle = 0\}$ will be endowed with the inner product induced from $\langle .,.\rangle$. This defines the bilinear form $I = \langle .,.\rangle$, on $T\mathbf{S}$, called the *First Fundamental Form* of $\mathbf{S}$.

In terms of a chart $(u, v) : U \to V$, defined on an open set U of $\mathbf{S}$, whose image is an open set V of $\mathbb{R}^2$, is defined a basis $\{du, dv\}$ in the space of linear forms. Here, $\{du, dv\}$ is the dual of the basis $\{\frac{\partial}{\partial u} = X_u, \frac{\partial}{\partial v} = X_v\}$ for the tangent space, where $X = (u, v)^{-1} : V \to U$ is the inverse map of the chart (u, v).

This induces the basis $\{du \otimes du,\ du \otimes dv,\ dv \otimes du,\ dv \otimes dv\}$ for the space of bilinear forms. The form I can be written as

$$I = E du \otimes du + F du \otimes dv + F dv \otimes du + G dv \otimes dv,$$

where $E = \langle X_u, X_u\rangle$, $F = \langle X_u, X_v\rangle, G = \langle X_v, X_v\rangle$.

The *Second Fundamental Form* of $\mathbf{S}$ is defined by the following bilinear form on $T\mathbf{S} : II = -\langle DN(\mathbf{p})\mathbf{v}, \mathbf{w}\rangle$, which is symmetric.

That is, $\langle DN(\mathbf{p})\mathbf{v}, \mathbf{w}\rangle = \langle DN(\mathbf{p})\mathbf{w}, \mathbf{v}\rangle$. This is equivalent to say that the *shape operator* $-DN$ of $\mathbf{S}$ is self adjoint with respect to I.

In fact, in terms of a chart $(u, v) : U \to V$, II is written as:

$$II = e du \otimes du + f du \otimes dv + f dv \otimes du + g dv \otimes dv,$$

where the coefficients of this form are given by

$$e = \langle X_{uu}, N\rangle = -\langle (N \circ X)_u, X_u\rangle, \quad g = \langle X_{vv}, N\rangle = -\langle (N \circ X)_v, X_v\rangle,$$

$$f = \langle X_{uv}, N\rangle = -\langle (N \circ X)_u, X_v\rangle = \langle X_{vu}, N\rangle = -\langle (N \circ X)_v, X_u\rangle.$$

The expression for f uses the invariance on the order of partial derivatives, which shows the symmetry for II.

2.3 Principal Curvatures and Line Fields, Umbilic Points.

The second fundamental form at a unitary tangent vector $\mathbf{v}$, $II(\mathbf{v}, \mathbf{v})$ gives the *normal curvature*, $k_n(\mathbf{v})$, of any curve γ on $\mathbf{S}$ tangent to $\mathbf{v}$ i.e. with $\gamma' = \mathbf{v}$; that is,

$$k_n(\mathbf{v}) = II(\mathbf{v}, \mathbf{v}) = \langle \gamma'', N\rangle = -\langle \gamma', (N \circ \gamma)'\rangle.$$

This allows the following geometric interpretation, which goes back to Euler's original approach to the study of curvature [Eul].

Consider the plane $\Pi = \Pi(\mathbf{v}, N)$, generated by $\mathbf{v}$ and N, oriented by declaring this a positive basis. Take as $\gamma = \gamma(\mathbf{v})$ the curve on $\mathbf{S}$ defined by the intersection of Π with $\mathbf{S}$. The (algebraic) curvature $\langle \gamma'', N \rangle$ of $\gamma = \gamma(\mathbf{v})$, as a planar curve on the oriented plane Π, gives precisely $k_n(\mathbf{v})$.

The extremal values of the normal curvatures $k_n(\mathbf{v})$ as $\mathbf{v}$ ranges on the unit circle of the tangent plane to $\mathbf{S}$ at a point $\mathbf{p}$, define the *minimal*, $k_1(\mathbf{p})$, and *maximal*, $k_2(\mathbf{p})$, principal curvatures at $\mathbf{p}$. On a chart (u, v) on which the coefficients of the fundamental forms are given in section 2.2 the principal curvatures are the roots of:

$$k^2(EG - F^2) - k(Eg + Ge - 2Ff) + (eg - f^2) = 0. \tag{1}$$

In fact, for $\mathbf{v} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$, $II(\mathbf{v}, \mathbf{v}) = ea^2 + 2fab + gb^2$ restricted to $I(\mathbf{v}, \mathbf{v}) = Ea^2 + 2Fab + Gb^2 = 1$, is stationary if and only if, for some (Lagrange multiplier) λ,

$$\frac{\partial}{\partial a}(II(\mathbf{v}, \mathbf{v})) = \lambda \frac{\partial}{\partial a}(I(\mathbf{v}, \mathbf{v})) \quad \text{and} \quad \frac{\partial}{\partial b}(II(\mathbf{v}, \mathbf{v})) = \lambda(\frac{\partial}{\partial b}(I(\mathbf{v}, \mathbf{v})) \tag{2}$$

Performing the derivation, get

$$ea + fb = \lambda(Ea + Fb) \text{ and } fa + gb = \lambda(Fa + Gb).$$

Eliminating a and b, get that λ satisfies equation (1). But $\lambda = k$, as follows by adding the first equation of (2), multiplied by a, to the second one, multiplied by b.

The unitary vectors $\pm\mathbf{e_1}(\mathbf{p})$ and $\pm\mathbf{e_2}(\mathbf{p})$ at which the extreme values $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ are attained, are called *minimal* and *maximal principal vectors* at $\mathbf{p}$. They are well defined and mutually orthogonal for $\mathbf{p}$ outside the set $\mathbf{U} = \mathbf{U_S}$ of *Umbilic Points* of $\mathbf{S}$, at which $k_1(\mathbf{p}) = k_2(\mathbf{p})$.

At the umbilic points, the forms II and I are proportional by a factor $k = k_1(\mathbf{p}) = k_2(\mathbf{p})$. Due to the way in which II is defined in terms of DN, this can also be stated by saying that the *shape operator* $-DN/T\mathbf{S}(\mathbf{p})$ is a multiple, by a factor $k = k_1(\mathbf{p}) = k_2(\mathbf{p})$, of the identity operator on $T\mathbf{S}(\mathbf{p})$. In fact, at an umbilic point, the surface is equally normally curved in all directions.

Eliminating λ in (2) gives that, in a chart, the components a, b of the principal vectors satisfy:

$$(Fg - Gf)b^2 + (Eg - Ge)ab + (Ef - Fe)a^2 = 0, \tag{3}$$

an equation that vanishes identically only at umbilic points.

The orthogonality of principal vectors stated above, away from the umbilical set, follows from the symmetry of II. This fact however can also be directly derived from (3).

While the principal vectors are defined up to sign, the lines $\mathbf{L_1}(\mathbf{p}) = \mathbb{R}(\pm\mathbf{e_1}(\mathbf{p}))$ and $\mathbf{L_2}(\mathbf{p}) = \mathbb{R}(\pm\mathbf{e_2}(\mathbf{p}))$ generated by them are well determined; they are called the *minimal* and *maximal principal directions* of $\mathbf{S}$ at $\mathbf{p}$. They define smooth line fields $\mathbf{L_1} : \mathbf{p} \to \mathbf{L_1}(\mathbf{p})$ and $\mathbf{L_2} : \mathbf{p} \to \mathbf{L_2}(\mathbf{p})$, mutually orthogonal, called *minimal* and *maximal principal line fields* of $\mathbf{S}$. In fact, (3) characterizes the slopes, b/a or a/b, of these line fields as smooth functions on the domain of (u, v) coordinates on $\mathbf{S} \setminus \mathbf{U}$.

The principal lines coincide with the eigenspaces of the shape operator $-DN/T\mathbf{S}(\mathbf{p})$, correspondent to its eigenvalues $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$. This fact follows also from the symmetry of II and its expression in terms of DN. It amounts to the classical *Rodrigues' Equation* [Spi], [Str]:

$$(DN + kI)\mathbf{v} = \mathbf{0}, \tag{4}$$

which characterizes principal vectors, $\mathbf{v}$, and curvature, k.

On $\mathbf{S} \setminus \mathbf{U}$ (resp. on $\mathbf{S}$), the principal curvatures k_1 and k_2 are smooth (resp. continuous). In fact, from their eigenvalue interpretation in (4), results that:

$$k_1 = \mathcal{H} - \sqrt{\mathcal{H}^2 - \mathcal{K}} \text{ and } k_2 = \mathcal{H} + \sqrt{\mathcal{H}^2 - \mathcal{K}},$$

where $\mathcal{H} = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\text{trace}(-DN)$ is the *Mean curvature* and $\mathcal{K} = k_1 \cdot k_2 = \det(-DN)$ is the *Gaussian curvature* of $\mathbf{S}$.

Since $\mathcal{H}^2 - \mathcal{K} = [(k_2 - k_1)/2]^2$, follows that the umbilic points can be also characterized by the zeros of this non negative function.

From (1) follows that, in terms of the coefficients of the fundamental forms, the Mean and Gaussian curvatures write:

$$\mathcal{H} = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \quad \mathcal{K} = \frac{eg - f^2}{EG - F^2}.$$

Remarkably, $eg - f^2$ and therefore $\mathcal{K}$ depend only on $\{E, F, G\}$ and their derivatives. This crucial fact for Intrinsic Geometry is established by *Gauss' Theorem Egregium* [Spi], [Str].

2.4 Principal Foliations and their Differential Equations

On $\mathbf{S} \setminus \mathbf{U}$, the *principal eigenvectors* $\mathbf{e_1}$ and $\mathbf{e_2}$ can be chosen so as to define local vector fields which are also smooth. In fact, (3) determines the components of these fields in the domain of a chart; they however cannot in general be globally defined on $\mathbf{S} \setminus \mathbf{U}$, due to the essential non orientability of principal line fields around most umbilic points; see Fig. 1. For this reason, global formulations are made in terms of the principal line fields $\mathbf{L_1}$ and $\mathbf{L_2}$. However the proofs of most results will be locally reduced to the principal vector fields $\mathbf{e_1}$ and $\mathbf{e_2}$ or directly to equation (3).

An integral curve of the line field $\mathbf{L_1}$ (resp. $\mathbf{L_2}$) is called a *minimal (resp. a maximal) principal curve* of $\mathbf{S}$. By this is meant a regular curve γ on $\mathbf{S} \setminus \mathbf{U}$ which

a) at each of its points is tangent to $\mathbf{L_1}$ (resp. $\mathbf{L_2}$) i.e. its tangent vector γ' generates $\mathbf{L_1}(\gamma)$ (resp. $\mathbf{L_2}(\gamma)$) and

b) it contains any regular curve with this property which intersects it.

The condition of tangency of a regular curve γ to a principal line field $\mathbf{L_1}$ (resp. $\mathbf{L_2}$) is also expressed by saying that, when parametrized by arc length, it is an integral curve of a principal local vector field $\mathbf{e_1}$ (resp. $\mathbf{e_2}$), i.e. it is a solution of the differential equation $\gamma' = \mathbf{e_1}(\gamma)$ (resp. $= \mathbf{e_2}(\gamma)$).

In terms of a chart (u, v) on which the curve γ is locally written as $u = u \circ \gamma$, $v = v \circ \gamma$, the condition of being a principal curve is characterized by equation (3):

$$(Fg - Gf)(v')^2 + (Eg - Ge)u'v' + (Ef - Fe)(u')^2 = 0.$$

From the classical results on existence and uniqueness of solutions of ordinary differential equations follows that through each point $\mathbf{p}$ of $\mathbf{S} \setminus \mathbf{U}$ there are a unique minimal, $\gamma_1(\mathbf{p})$, and a unique maximal, $\gamma_2(\mathbf{p})$, principal curves of $\mathbf{S}$.

From the classical results on smooth dependence of solutions of ordinary differential equations on initial conditions and parameters, follows that if $\gamma_1 = \gamma_1(\mathbf{p}, s)$ is oriented and parametrized by arc length s, starting at $\mathbf{p}$, on its maximal interval of definition $\mathcal{I}(\mathbf{p}) = (w_-(\mathbf{p}), w_+(\mathbf{p}))$, then for any compact subinterval $\mathbf{K}$ of $\mathcal{I}(\mathbf{p})$, there is a neighborhood $\mathbf{V} = \mathbf{V}(\mathbf{p})$ of $\mathbf{p}$ in $\mathbf{S}$ such that the minimal principal curve $\gamma_1(\mathbf{q}) = \gamma_1(\mathbf{q}, s)$ through $\mathbf{q}$ in $\mathbf{V}$ can be oriented and parametrized by arc length starting at $\mathbf{q}$, so that its maximal interval of definition $\mathcal{I}(\mathbf{q}) = (w_-(\mathbf{q}), w_+(\mathbf{q}))$ contains the interval $\mathbf{K}$ and the mapping $\gamma_1 : \mathbf{V} \times \mathbf{K} \to \gamma_1(\mathbf{q}, \mathbf{s})$ is smooth. The same holds for maximal principal curves γ_2.

The families of minimal and maximal principal curves will be called respectively the *minimal and maximal principal foliations* of $\mathbf{S}$ and will be denoted respectively by $\mathbf{F_1}$ and $\mathbf{F_2}$. Notice that these foliations are defined only on $\mathbf{S} \setminus \mathbf{U}$. The umbilic points can be regarded as singular points for these foliations if one wishes to consider them defined on the whole $\mathbf{S}$.

2.5 Differential Equations for Principal Lines

For future reference, the most common forms of differential equations for principal lines are collected here.

a. *Implicit form:*

For $\mathbf{p} = (x, y, z)$ in $\mathbf{S}$ and $d\mathbf{p} = (dx, dy, dz)$, the eigenspace interpretation of principal lines (4) can be stated as

$$\langle N(\mathbf{p}), (DN(\mathbf{p})d\mathbf{p}) \wedge d\mathbf{p}\rangle = 0, \quad \langle N(\mathbf{p}), d\mathbf{p}\rangle = 0.$$

More explicitly, for a $\mathbf{S} = \{\mathbf{f} = 0\}$, this equation amounts to

$$\langle \nabla \mathbf{f}(\mathbf{p}), (D\nabla \mathbf{f}(\mathbf{p})) \cdot d\mathbf{p}) \wedge d\mathbf{p}\rangle = 0, \langle \nabla f(\mathbf{p}), d\mathbf{p}\rangle = 0.$$

b. *In a Local Chart.*

In terms of a local chart (u, v) on $\mathbf{S}$, it has been seen in (3) that the components du, dv of a principal direction are characterized by:

$$(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0.$$

c. *For a Monge Chart.*

For the Monge chart (x, y), on which the surface can be expressed as the graph of a function $z = f(x, y)$, the equations for principal lines becomes

$$Adx^2 + Bdxdy + Cdy^2 = 0, \text{ where}$$

$$A = (1 + f_x^2)f_{xy} - f_x f_y f_{xx}, \quad B = (1 + f_x^2)f_{yy} - (1 + f_y^2)f_{xx},$$
$$C = -(1 + f_y^2)f_{xy} + f_x f_y f_{yy}.$$

Further simplification in the expression for the differential equations for principal lines achieved in terms of Bonnet coordinates and functions, as developed in Section 3.

2.6 Principal Configurations

The *principal configuration* on a surface $\mathbf{S}$ is defined by the triple $\mathbf{P_S} = \mathbf{P}(\mathbf{U_S}, \mathbf{F_1}, \mathbf{F_2})$, where $\mathbf{U_S}$ is the set of umbilic points and $\mathbf{F_1}$ and $\mathbf{F_2}$ denote respectively the minimal and maximal principal foliations on $\mathbf{S} \setminus \mathbf{U}$.

It synthesizes the qualitative properties of the principal foliations on a surface and represents the way their lines approach the umbilic set. It fact, it is the natural analog of the *phase portrait* of a vector field on a surface.

The most trivial principal configurations are those of the round sphere and the plane, which both reduce to $\mathbf{U}$. Conversely, a connected compact (resp. non compact) surface all whose points are umbilic is a round sphere (resp. an open subset of a sphere or a plane) [Spi], [Str].

Two surfaces $\mathbf{S_1}$ and $\mathbf{S_2}$ with principal configurations

$$\mathbf{P_S} = \mathbf{P}(\mathbf{U(1)}, \mathbf{F_1(1)}, \mathbf{F_2(1)}) \qquad \textbf{and} \qquad \mathbf{P_S} = \mathbf{P}(\mathbf{U(2)}, \mathbf{F_1(2)}, \mathbf{F_2(2)})$$

are regarded as having the same qualitative structure, with respect to principal foliations, if they are *principally equivalent*: there is a homeomorphism H from the first to the second, which maps $\mathbf{U(1)}$ onto $\mathbf{U(2)}$ and maps lines of $\mathbf{F_1(1)}$ onto those of $\mathbf{F_1(2)}$, and the lines of $\mathbf{F_2(1)}$) onto those of $\mathbf{F_2(2)}$. A homeomorphism such as H is called a *principal equivalence* between $\mathbf{S_1}$ and $\mathbf{S_2}$.

The following are classical examples of principal equivalences [Spi], [Str], [dCa].

Rigid motions.- An orientation preserving (resp. reversing) rigid motion $M = T + R$ in $\mathbb{R}^3$, where T is a translation and R a rotation represented by an orthogonal matrix with positive (resp. negative) determinant, induces a principal equivalence from any oriented surface $\mathbf{S}$, with positive normal N, onto its image $M(\mathbf{S})$, endowed with the induced orientation, with positive normal $R \cdot N$, (resp. endowed with the opposite to the induced orientation, with positive normal $-R \cdot N$). This follows from Rodrigues Equation, which shows also that the principal curvatures are preserved in the orientation preserving case.

Homoteties.- A map such as $F(\mathbf{p}) = c\mathbf{p}$, also preserves principal configurations, but not principal curvatures, for $c > 0$.

Parallel Translations.- For any oriented surface $\mathbf{S}$ with positive normal N. For r small, the map $T_r = I + rN$ induces a principal equivalence between $\mathbf{S}$ and its image $\mathbf{S_r}$. In fact both have the same positive normal at corresponding points. In this case the principal curvatures are not preserved.

Inversions.- $G(\mathbf{p}) = \mathbf{p}/\langle\mathbf{p},\mathbf{p}\rangle$ is called an *inversion with respect to the unit sphere* centered at $\mathbf{0}$. These mappings also preserve principal lines (see [Spi] for a proof); however, since they reverse orientation, they exchange principal foliations.

2.7 Further Developments

The dependence of Principal Configurations on small deformations of the surface has been studied by Gutierrez and Sotomayor. In particular, a dense class of compact smooth surfaces whose principal configuration remains topologically equivalent to that of the perturbed ones (structural stability), as well as the simplest patterns of topological changes (bifurcations), have been described in [SG1]-[SG7]. This study includes the analysis of structurally stable umbilic points (through the Darbouxian conditions reviewed in section 2.2), periodic principal lines (through the hyperbolicity condition on the derivative of the return map).

The elimination (as well as explicit examples of) recurrent principal lines, including dense lines of curvature, have been studied in [SG2] and [SG7].

The pattern of the expressions for the successive derivatives of the return map in terms of the curvature functions has been determined by Gutierrez and Sotomayor [SG5] and Garcia and Sotomayor [G-S1], [G-S5] and applied to the study of the bifurcations of periodic principal lines.

Garcia and Sotomayor have partially extended the results on structural stability of Principal Configurations on compact smooth surfaces to the class of Algebraic Surfaces [G-S2]. This includes the analysis of principal lines tending to infinity. This study was preceded by the analysis of the stable patterns of principal lines around a conic singularity of an implicit surface, carried out by Garcia and Sotomayor in [G-S3]. On this line of work it is pertinent to mention the description of the principal configuration around a Whitney Umbrella singularity of a surface mapped into $\mathbb{R}^3$, published firstly in [SG4] and corrected later in [SGG1].

Integro-differential properties of the holonomy maps defined by the principal foliations have been isolated, abstracted, and related to Codazzi compatibility equations of Surface Theory [So1]. This paper was preceded by the study of the holonomy properties of principal foliations on surfaces with Constant Mean Curvature [SG3], [So2].

Expository presentations concerning the structural stability and bifurcations of principal configurations can be found in [So3], [SG8].

Garcia [Ga1]-[Ga3] has extended crucial aspects of the study of pincipal configurations on surfaces in $\mathbb{R}^3$ to hypersurfaces of $\mathbb{R}^4$.

Gutierrez and his collaborators have extended some of the aspects of principal lines and umbilic points on surfaces in $\mathbb{R}^3$ to surfaces in $\mathbb{R}^4$; see [GGGT], [GS3] and [GGST].

Some of the ideas and results presented above for principal lines have been also established for asymptotic lines. See Garcia-Sotomayor,[G-S4], and Sotomayor-Gutierrez-Garcia [SGG2].

3 Carathéodory Conjecture

The classical Carathéodory Conjecture states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$, i.e. an ovaloid, must have at least two umbilics. A well known approach to the problem is based on a "semi-local" argument. As already mentioned in Section 2, for any surface in $\mathbb{R}^3$, the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the umbilics. To each isolated umbilic we can attach the index of either one of the two fields, which is half an integer, and the sum of those indexes is the Euler-Poincaré characteristic of the surface, if the surface is compact and all umbilics are isolated. So, if an ovaloid has only one umbilic, this must have index two. We just observe that, up to an inversion in $\mathbb{R}^3$, we can always suppose that the curvature at a given umbilic is positive and therefore the convexity hypothesis is not relevant for this argument.

Examples of umbilics of index j are known for all $j \leq 1$. A local conjecture stronger than Carathéodory's, known as the Bol-Loewner conjecture, states that there are no umbilics of index greater than one. This conjecture has been asserted to be true for analytic surfaces by several authors, among whom are H. Hamburger [Ham], G. Bol [Bol], T. Klotz [Klo], C. J. Titus [Tit] and Scherbel [Sch], implying therefore Carathéodory Conjecture for analytic surfaces. Nevertheless, Klotz [Klo] pointed out a gap in Bol's proof; also, Scherbel [Sch] claims that there are gaps in the works of Klotz and Titus. Other references challenging these proofs are those of [Lan], [Yau] and [SX1]. All existing proofs are difficult and we believe that it would be worth to find a simpler proof.

3.1 Bonnet function

Orient the sphere $S^2 \subset \mathbb{R}^3$ so that the positive unitary normal vector at $p \in S^2$ is p itself. Let $\mathcal{S} \subset \mathbb{R}^3$ be a C^r-*ovaloid.* By this we mean that $\mathcal{S}$ is an oriented C^r-embedded surface such that its Gauss map $N : \mathcal{S} \to S^2$ is an orientation preserving diffeomorphism. This definition implies that $\mathcal{S}$ is convex, compact and that its Gaussian curvature is positive everywhere. We define the *support function* *of* $\mathcal{S}$ as the map $\sigma : S^2 \to I\!R$ given by

$$\sigma(p) = p \cdot N^{-1}(p),$$

where the dot stands for the usual inner product, some times also denoted by $<,>$.

Given $\delta \in \{-,+\}$, let $\Pi^\delta : I\!R^2 \to S^2 \setminus \{(0,0,\delta(1))\}$ be the diffeomorphism given by

$$\Pi^\delta(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{\delta(x^2+y^2-1)}{1+x^2+y^2}).$$

That is, Π^δ is the inverse map of the corresponding stereographic projection. The map

$$\Phi^\delta(x,y) = (X^\delta(x,y), Y^\delta(x,y), Z^\delta(x,y)) = N^{-1} \circ \Pi^\delta(x,y),$$

defined in $I\!R^2$, provides a global C^{r-1} parametrization of $\mathcal{S} \setminus \{N^{-1}(0,0,\delta(1))\}$ called *Bonnet chart associated to* $(\mathcal{S}, \Pi^\delta)$. Given the support function σ of $\mathcal{S}$, associated to $(\mathcal{S}, \Pi^\delta)$ we define the *Bonnet function*

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y)).$$

Proposition 3.1 *Let $\mathcal{S} \subset I\!R^3$ be a C^r ovaloid, $r \geq 3$. Then the support function σ of $\mathcal{S}$ is of class C^r and the differential equation of the principal lines of curvature of $\mathcal{S}$ in its Bonnet chart, associated to $(\mathcal{S}, \Pi^\delta)$, is given by $\omega^\delta = 0$ where*

$$\omega^\delta \;=\; \beta^\delta_{xy} dx^2 + (\beta^\delta_{yy} - \beta^\delta_{xx}) dx dy - \beta^\delta_{xy} dy^2 \tag{5}$$

and

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y)).$$

Proof:

Let $\Lambda^\delta(x,y) = (1+x^2+y^2)\Pi^\delta(x,y)$; that is,

$$\Lambda^\delta(x,y) = (2x, 2y, \delta(x^2+y^2-1)).$$

As $\Lambda^\delta \cdot \Phi^\delta_x = \Lambda^\delta \cdot \Phi^\delta_y = 0$, where the subindex means the partial derivative with respect to this variable, we have that $\Lambda^\delta_x \cdot \Phi^\delta = \beta^\delta_x$ and $\Lambda^\delta_y \cdot \Phi^\delta = \beta^\delta_y$. This together with $\Lambda^\delta \cdot \Phi^\delta = \beta^\delta$ can be written in matrix notation as $M^\delta \cdot \Phi^\delta = \mathcal{B}^\delta$, where

$$M^\delta = \begin{pmatrix} 2x & 2y & \delta(x^2+y^2-1) \\ 2 & 0 & \delta(2x) \\ 0 & 2 & \delta(2y) \end{pmatrix}, \quad \Phi^\delta = \begin{pmatrix} X^\delta \\ Y^\delta \\ Z^\delta \end{pmatrix}, \quad \mathcal{B}^\delta = \begin{pmatrix} \beta^\delta \\ \beta^\delta_x \\ \beta^\delta_y \end{pmatrix}. \tag{6}$$

As N is of class C^{r-1}, Φ^δ is also of class C^{r-1}. Therefore, $M^\delta \cdot \Phi^\delta = \mathcal{B}^\delta$ implies that β^δ is of class C^r. Since, for all $(x,y) \in I\!R^2$, the determinant of M^δ is $-(\delta)4(1+x^2+y^2) \neq 0$, we may write $\Phi^\delta = (M^\delta)^{-1} \cdot \mathcal{B}^\delta$. From this, one can obtain the first and second fundamental forms of Φ^δ and therefore the proof of the proposition. ∎

Remark 3.2 *There are several proofs of equation (5) (see [Bon], [Dar, pp 285-300] and [Bla, pp 283-289]). Our direct approach has the merit that it can be easily checked using a symbolic computer system. Appendix 3.6 is the program, written in Mathematica, we used to perform the calculations.*

Remark 3.3 *Notice that the proposition above can be reformulated locally. By this we mean that if $\mathcal{S} \subset \mathbb{R}^3$ is an oriented C^r-embedded surface and $p \in \mathcal{S}$, then we may define the support and the Bonnet functions in a neighborhood of p and obtain the differential equation of the principal lines of curvature for the considered neighborhood of p. Next proposition has a local character.*

Proposition 3.4 *Let $\beta : \mathbb{R}^2 \to \mathbb{R}$ be a C^r function, with $r \geq 3$. Suppose that the 2-jet $j^2\beta_{(0,0)}$ of β at $(0,0)$ has the form*

$$j^2\beta_{(0,0)}(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2 \tag{7}$$

and that

$$a_{00}^2 + a_{00}(a_{02} + a_{20}) - a_{11}^2 + a_{02}a_{20} \neq 0. \tag{8}$$

Then there exists an open neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $\beta|_U$ is the Bonnet function of an oriented C^r surface embedded in $\mathbb{R}^3$.

Proof The function $\beta^+ = \beta(x,y)$ determines the function $\mathcal{B} = \mathcal{B}^+$ as in (6); let $\Phi = (M^+)^{-1} \cdot \mathcal{B}$, where $M = M^+$. Using (8), we can check that

$$\det \begin{pmatrix} \frac{\partial \Phi_1}{\partial x}(0,0) & \frac{\partial \Phi_1}{\partial y}(0,0) \\ \frac{\partial \Phi_2}{\partial x}(0,0) & \frac{\partial \Phi_2}{\partial y}(0,0) \end{pmatrix} \neq 0$$

and therefore (as $\mathcal{B}$ is of class C^{r-1}) there exists an open neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $\Phi : U \to \mathbb{R}^3$ is a C^{r-1} regular parametrization of $S = \Phi(U)$. We want to show that there exists an open neighborhood of $\Phi(0,0)$ in S which is the inverse image of a regular value of a C^r function. Let $\varphi : U \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ be given by

$$\varphi(x,y,s) = \Phi(x,y) + s\Pi^+(x,y).$$

Then, by the Inverse Mapping Theorem, φ is a C^{r-1} local diffeomorphism around $(0,0,0)$. Let $V = \varphi(U \times (-\epsilon, \epsilon))$. By taking $U \times (-\epsilon, \epsilon)$ smaller if necessary, we shall proceed assuming that $\varphi : U \times (-\epsilon, \epsilon) \to V$ is a diffeomorphism. Let $g = \pi_3 \circ \varphi^{-1} : V \to \mathbb{R}$, where $\pi_3(x,y,s) \equiv s$. Then $S = g^{-1}(0)$. Since $(d\varphi^{-1})(\text{grad}.g) = (0,0,1)$, $\text{grad}.g = \Pi^+ \circ \pi_{12} \circ \varphi^{-1}$, where $\pi_{12}(x,y,s) \equiv (x,y)$. Therefore $\text{grad}.g$ is of class C^{r-1} so that g is of class C^r and 0 is a regular value. Finally, because $\Lambda \cdot \Phi = \Lambda \cdot M^{-1} \cdot \mathcal{B} = \beta$, we get that β is the Bonnet function of $\Phi(U)$. ∎

Remark 3.5 *With respect to the proof above, we notice that the parametrization φ is essentially defined in terms of what will be the unit normal vector so it has a class of differentiability one less than the (natural) class of differentiability of S.*

Remark 3.6 *Fix $\delta \in \{-,+\}$. Under the conditions of Proposition 3.1, if the Bonnet function β is defined in a neighborhood of $(0,0)$ and the 2-jet of $\beta = \beta^\delta$ at $(0,0)$ is written as in the statement of Proposition 3.4, then (since $\Phi_x(0,0)$ and $\Phi_y(0,0)$ are linearly independent) the inequality (8) of Proposition 3.4 is satisfied.*

The following will be needed later

Lemma 3.7 *Let $\delta \in \{-,+\}$ and let*

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y))$$

be the functions determined by σ as above. If $\mathcal{I}\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is the inversion

$$\mathcal{I}(u,v) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}),$$

then, for all $(x,y) \in \mathbb{R}^2 \setminus \{0\}$,

$$(x^2+y^2)\beta^- \circ \mathcal{I}(x,y) = \beta^+(x,y). \tag{9}$$

Proof:
The result follows immediately from the identity

$$\sigma(\Pi^+(x,y)) = \sigma(\Pi^- \circ \mathcal{I}(x,y)). \qquad \blacksquare$$

3.2 Umbilics and their indexes

Let $\mathcal{F}$ be a one dimensional C^1-foliation defined on a neighborhood U of $0 \in \mathbb{R}^2$. Suppose that $\mathcal{F}$ has exactly one singularity which is 0. The index of 0 (i.e. of $\mathcal{F}$ at 0) is one-half of the degree of the map that takes each point q of a small circle centered at 0 to the element, of the projective line $\mathbb{R}P$, which is tangent at q to the leaf of $\mathcal{F}$. Here, we identify $\mathbb{R}^2$ with the tangent space of U at q; also, the projective line $\mathbb{R}P$ is the well known quotient space obtained from $\mathbb{R}^2 \setminus \{0\}$. If $\mathcal{F}$ is orientable, this definition coincides with the usual Hopf-Poincaré index.

If p is an isolated umbilic point of a C^r oriented surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, the index of p is defined by using local coordinates and either one of the two foliations induced by the principal lines of curvature of $\mathcal{S}$. The umbilics are precisely the singularities of these foliations.

Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin, k be a positive real number and let $\beta : U \to R$ be given by

$$\beta(u,v) = \frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + 0[(u^2+v^2)^2]$$

By Proposition 3.4, β is locally, around $(0,0)$, the Bonnet function of an oriented surface in $\mathbb{R}^3$ which has, in its associated Bonnet chart, an umbilic point p corresponding to $(0,0)$. The umbilic p is called *Darbouxian* provided, in the above expression, the following two conditions T and D hold:

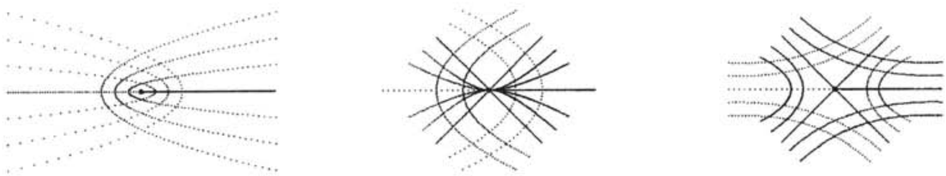
T) $b(b-a) \neq 0$ and

D) either

$D_1 : a/b > (c/2b)^2 + 2$,

$D_2 : (c/2b)^2 + 2 > a/b > 1$, $a \neq 2b$,

$D_3 : a/b < 1$



Type D_1 Type D_2 Type D_3

Figure 1

The independence of this definition on the chart used to write the coefficients (a, b, c), as well as the justification for the pictures in Fig. 1, which illustrate the principal configurations near Darbouxian umbilics can be found in [SG1].

The suffixes refer to the number of *umbilical separatrices* which are the curves, drawn in heavy lines, tending to the umbilic point and separating regions with principal lines having different patterns of approach to it.

The index of p is $1/2$ (resp. $-1/2$) if p is either D_1 or D_2 (resp. D_3).

3.3 Analyticity versus smoothness

We say that a C^r vector field ξ on $\mathbb{R}^2$ fulfills a *Lojasiewicz–inequality at* $(0,0)$ if there exist $k \in \mathbb{N}^*$ and $\delta > 0$ such that $||\xi(x,y)|| \geq \delta ||(x,y)||^k$ on some neighborhood of $(0,0)$. Under these circumstances, we will also say that ξ satisfies a *Lojasiewicz–inequality of order k (with associated constant δ) at* $(0,0)$. Suppose that a C^r oriented surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, has an isolated umbilic point $p \in \mathcal{S}$. We will say that *p is an umbilic of Lojasiewicz–type* (of order k with $1 \leq k \leq r-2$) if there is a local C^r diffeomorphism φ of a neighborhood of $p \in \mathbb{R}^3$ onto an open set of $\mathbb{R}^3$ such that the image surface $\varphi(\mathcal{S}) = \tilde{\mathcal{S}}$ satisfies the following properties:

a) *$\tilde{p} = \varphi(p)$ is an isolated umbilic of $\tilde{\mathcal{S}}$ with the same index of p and $\tilde{\mathcal{S}}$ has positive curvature in $\tilde{p}$ and unit normal vector* $(0,0,-1)$.

b) *The Bonnet function $\tilde{\beta}$ of $\tilde{\mathcal{S}}$ is such that the vector field $\tilde{\xi}(x,y) = (\tilde{\beta}_{xx} - \tilde{\beta}_{yy}, 2\tilde{\beta}_{xy})$ satisfies a Lojasiewicz–inequality of order k at* $(0,0)$.

Remark 3.8

(a) *The composition of an appropriate rigid translation and the inversion* $\mathcal{I}(p) = \frac{p}{||p||^2}$ *preserves the principal lines of curvature, hence umbilics and their indexes as well. Thus an inversion may be used to transform a flat umbilic into an umbilic of positive curvature. Therefore, up to a conformal diffeomorphism, the first condition is always satisfied.*

(b) *With the notation right above, the index of* ξ *at* $(0,0)$ *is twice the index of the umbilic point* p *[SX1]. In fact, at a given point, the element* (dx, dy) *solves the equation* $\beta_{xy}(dx^2 - dy^2) + (\beta_{yy} - \beta_{xx})dxdy = 0$, *if and only if (at the given point) the vector field* $(2\beta_{xy}, \beta_{yy} - \beta_{xx})$ *is orthogonal to* $(dx^2 - dy^2, 2dxdy)$; *or, equivalently (using complex notation), the vector field* $(\beta_{xx} - \beta_{yy}, 2\beta_{xy})$ *is collinear with*

$$(dx^2 - dy^2, 2dxdy) = dx^2 - dy^2 + 2idxdy = (dx + idy)^2.$$

This proves the claim because the argument of $(dx + idy)^2$ *is twice the argument of* $dx + idy = (dx, dy)$.

(c) *If a vector field on* $\mathbb{R}^2$ *satisfies a Lojasiewicz-inequality at the singular point* $(0,0)$, *then* $(0,0)$ *is an isolated singularity of the vector field.*

(d) *Suppose that* $Y : (U, (0,0)) \to (\mathbb{R}^2, (0,0))$ *is an analytic vector field defined in an open set* $U \subset \mathbb{R}^2$. *Then* $(0,0)$ *is an isolated singular point of* Y *if, and only if,* Y *satisfies a Lojasiewicz-inequality at* $(0,0)$ *[Loj]. Therefore, using (a) above, an analytic surface immersed in* $\mathbb{R}^3$ *always satisfies a Lojasiewiecz-inequality at an isolated umbilic point.*

Lemma 3.9 *If* $\xi : (U, (0,0)) \to (\mathbb{R}^2, (0,0))$ *is a* C^r *vector field,* $r \geq 1$, *defined in a neighborhood* U *of* $(0,0)$ *and satisfying a Lojasiewicz-inequality of order* $k, 1 \leq k \leq r$ *at* $(0,0)$, *then*

(a) *the k-jet* $j^k\xi_0$ *of* ξ *at* $(0,0)$ *satisfies a Lojasiewicz-inequality of order* k *at* $(0,0)$;

(b) *both* ξ *and its k-jet* $j^k\xi_0$ *at* $(0,0)$ *have the same index at their common isolated singularity* $(0,0)$.

Proof. Let $\xi = j^k\xi_0 + \varphi$. By assumption there exists a constant $\delta > 0$ such that $||\xi(x,y)|| \geq \delta||(x,y)||^k$, for all (x,y) in a neighborhood V of $(0,0)$. As φ is the Taylor remainder of order k, by shrinking V if necessary, we may find $\frac{\delta}{4} > \epsilon > 0$ so that, for all $(x,y) \in V$, $||\varphi(x,y)|| \leq \epsilon||(x,y)||^k$. This implies that, when restricted to V, each element of the family $\xi_\mu = j^k\xi_0 + \mu\varphi$, with parameter $\mu \in [0,1]$, satisfies a Lojasiewicz-inequality of order k with associated constant $\delta - \epsilon$ at $(0,0)$. It is easy to see that the family ξ_μ provides an Homotopy between ξ and $j^k\xi_0$ such that each $\xi_\mu|_V$ has a unique singularity: $(0,0)$. This implies, by Index Theory, the lemma [G-P]. ■

We are now in condition to prove the result announced.

Theorem 3.10 *If p is an umbilic point of a C^r surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, that satisfies a Lojasiewicz–inequality of order $1 \leq k \leq r-2$, then there is an analytic surface $\tilde{\mathcal{S}}$ such that p is an isolated umbilic point of $\tilde{\mathcal{S}}$ and $\mathcal{S}$ having, in both cases, the same index.*

Proof. By definition, we may assume that the Gaussian curvature of a surface $\mathcal{S}$ at p is positive and that the unitary normal vector to $\mathcal{S}$ at p is $(0,0,-1)$. Let $\beta = \beta(x,y)$ be the C^r Bonnet function associated to $\mathcal{S}$ at a neighborhood of p. By assumption the vector field

$$\xi(x,y) = (\beta_{xx} - \beta_{yy}, 2\beta_{xy})$$

satisfies a Lojasiewicz–inequality of order k at the singularity $(0,0)$. Let $\gamma = \gamma(x,y)$ be the $k+2$ jet of β at $(0,0)$. As β satisfies the properties mentioned in Remark 3.6 we can apply Proposition 3.4 to make a Bonnet function of γ with associated surface $\tilde{\mathcal{S}}$. Let

$$Y(x,y) = (\gamma_{xx} - \gamma_{yy}, 2\gamma_{xy}).$$

As Y and the k–jet of ξ at $(0,0)$ coincide, it follows from the lemma above that p is an umbilic point of both $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfying the conditions of the lemma. ∎

Theorem 3.11 *Assuming the truth of the Bol-Loewner Conjecture for isolated umbilics on analytic surfaces, if a C^r surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, satisfies a Lojasiewicz–inequality at an umbilic point p, then the index of p is at most 1. Therefore if a C^r immersion of a sphere has one umbilic of Lojasiewicz type, it must have at least one more umbilic.*

Remark 3.12 *Inspired by the work [GMS], Sotomayor and de Mello [dMS] have obtained an equivalent formulation some of the results in this section, using Ribaucour instead of Bonnet coordinates. In doing this, some calculations seem to be more direct.*

3.4 Representing ovaloids

Theorem 3.13 *Let $\mathcal{S} \subset \mathbb{R}^3$ be a C^r ovaloid, $r \geq 3$. Then the inverse N^{-1} : $S^2 \to \mathcal{S}$, of the Gauss map N, can be written as follows:*

$$N^{-1}(u,v,w) = f(u,v,w) \cdot (u,v,w) + \mathcal{A}(u,v,w) \cdot \nabla f(u,v,w)$$

where $f : S^2 \to \mathbb{R}$ denotes the support function of $\mathcal{S}$, ∇f its gradient vector field and

$$\mathcal{A}(u,v,w) = \begin{pmatrix} v^2+w^2 & -uv & -uw \\ -uv & u^2+w^2 & -vw \\ -uw & -vw & u^2+v^2 \end{pmatrix}. \tag{10}$$

Conversely, given a C^r function $f : S^2 \to \mathbb{R}$, $r \geq 3$, there exists a constant $c > 0$ such that $f + c$ is the support function of an ovaloid of class C^r.

Proof: Let $\mathcal{S} \subset \mathbb{R}^3$ be a C^r ovaloid, $r \geq 3$ and let $f : S^2 \to \mathbb{R}$ be its support function. Consider an arbitrary extension $F = F(u, v, w)$ of f defined in a neighborhood of S^2.

Given $\delta \in \{-, +\}$, the function f determines the function

$$\beta^\delta(x, y) = (1 + x^2 + y^2) f(\Pi^\delta(x, y))$$

and so the column vector $\mathcal{B}^\delta$ as in (6).

Observe that, for all $(u, v, w) \in S^2 \setminus \{(0, 0, \delta(1))\}$,

$$(x, y) = (\Pi^\delta)^{-1}(u, v, w) = (\frac{u}{1 - \delta w}, \frac{v}{1 - \delta w}).$$

Therefore, if we denote by f, β^δ, $F_u \cdots$, the functions $f = f(u, v, w)$, $\beta^\delta \circ (\Pi^\delta)^{-1}(u, v, w)$, $F_u(u, v, w), \cdots$, respectively, we will have that for all $(u, v, w) \in S^2 \setminus \{(0, 0, \delta(1))\}$,

$$\mathcal{B}^\delta = \begin{pmatrix} \beta^\delta \\ \beta^\delta_x \\ \beta^\delta_y \end{pmatrix} = \frac{2}{1 - \delta w} \begin{pmatrix} f \\ uf + (1 - w - u^2)F_u - uvF_v + u(1 - \delta w)F_w \\ vf - uvF_u + (1 - w - v^2)F_v + v(1 - \delta w)F_w \end{pmatrix}$$

Moreover,

$$M^\delta \circ (\Pi^\delta)^{-1} = \frac{2}{1 - \delta w} \begin{pmatrix} u & v & \delta w \\ 1 - \delta w & 0 & u \\ 0 & 1 - \delta w & v \end{pmatrix} \qquad \textit{and}$$

$$(M^\delta)^{-1} \circ (\Pi^\delta)^{-1} = \frac{1}{2(1 - \delta w)} \begin{pmatrix} u(1 - \delta w) & 1 - u^2 - w^2 & uv \\ v(1 - \delta w) & uv & 1 - u^2 - w^2 \\ -(1 - \delta w)^2 & u(1 - \delta w) & v(1 - \delta w) \end{pmatrix}.$$

Recall that $\Phi^\delta = N^{-1} \circ \Pi^\delta$ and that $\Phi^\delta = (M^\delta)^{-1} \cdot \mathcal{B}^\delta$. It follows that

$$N^{-1} = \Phi^\delta \circ (\Pi^\delta)^{-1} = ((M^\delta)^{-1} \cdot \mathcal{B}^\delta) \circ (\Pi^\delta)^{-1}.$$

Under these conditions, it may be seen that, when restricted to S^2, N^{-1} has the required form (see the remark following this proof).

Conversely, Let $f : S^2 \to \mathbb{R}$ be an arbitrary C^r function and let $F : \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary C^r extension of f. Given $\sigma \in (0, \infty)$, we define $H : \mathbb{R}^3 \to \mathbb{R}^3$ and $G : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$H(u, v, w) = \sigma \cdot (u, v, w) + G(u, v, w)$$

and

$$G(u, v, w) = F(u, v, w) \cdot (u, v, w) + \mathcal{A}(u, v, w) \cdot \nabla F(u, v, w).$$

Let $Id : I\!R^3 \to I\!R^3$ be the identity and let DG denote the derivative of G. As S^2 is compact, we obtain that if σ is large enough, for all $(u, v, w) \in S^2$,

$$Id + \frac{1}{\sigma} DG(u, v, w)$$

is invertible. This implies that $H|_{S^2} : S^2 \to H(S^2)$ is an immersion. By definition of H and the first part of this proof, $H \circ \Pi^\delta = \Phi^\delta$ and therefore, $N^{-1} = H$. This implies that the Gauss map must be not only a local diffeomorphism but, in fact, a global one. These conditions imply that $H(S^2)$ is an ovaloid. For more details of this argument and information about ovaloids see [Hop].■

Remark 3.14 *Let $f : S^2 \to I\!R$ be a C^r function, $r \geq 3$, and let $F = F(u, v, w)$ be an arbitrary extension of f to a neighborhood of S^2. As $A(u, v, w) \cdot (u, v, w) \equiv 0$, it follows that*

$$\mathcal{A}(u, v, w) \cdot \nabla f(u, v, w) = \mathcal{A}(u, v, w) \cdot \nabla F(u, v, w)$$

where ∇f and ∇F are the gradient vector fields of f and of F, respectively.

Example 2.4.3

If $f : S^2 \to I\!R$ is identically the constant $r > 0$, then the ovaloid associated to f is the sphere $r \cdot S^2$.

Example 2.4.4

Let $\epsilon > 0$, $\varphi : I\!R \to (-\epsilon, \epsilon)$ be an orientation preserving smooth diffeomorphism and $f : S^2 \setminus \{(0, 0, 1)\} \to I\!R$ be given by

$$f(u, v, w) = (1 - w)\varphi(\frac{u}{1 - w})\varphi(\frac{v}{1 - w}).$$

Observe that f is smooth and extends continuously to $\{(0, 0, 1)\}$ by defining $f(0, 0, 1) = 0$. Let

$$\beta(x, y) = (1 + x^2 + y^2)f(\Pi(x, y)) = \varphi(x) \cdot \varphi(y).$$

We have that, for all $(x, y) \in I\!R^2$,

$$\beta_{xy}(x, y) = \varphi'(x) \cdot \varphi'(y) \neq 0$$

It follows from Proposition 1.1 that if $g : S^2 \to I\!R$ is a smooth map which is a perturbation of f supported in an arbitrarily small neighborhood W of $\{(0, 0, 1)\}$ and $c > 0$ is an appropriate constant, we will have that the smooth ovaloid associated to $g + c$ has no umbilics in the complement of $N^{-1}(W)$, where N is the Gauss map of the ovaloid.

Example 2.4.5

Let $f : S^2 \to I\!R$ be given by

$$f(u, v, w) = w^2.$$

Using the notation of Section 3.1 and that of Theorem above, we may easily see that

$$\beta(x, y) := \beta^+(x, y) = \beta^-(x, y) = \frac{(x^2 + y^2 - 1)^2}{1 + x^2 + y^2}.$$

Hence, for all $(x, y) \in I\!R^2$,

$$\beta_{xy}(x, y) = 32\frac{x^2 - y^2}{(1 + x^2 + y^2)^3}$$

and

$$\beta_{xx} - \beta_{yy} = 32\frac{xy}{(1 + x^2 + y^2)^3}$$

Therefore, the ovaloid associated to $f + c$, where $c > 0$ is an appropriate positive constant, has exactly two umbilics: $N^{-1}(0, 0, 1)$ and $N^{-1}(0, 0, -1)$.

3.5 Equivalent conjectures

Let $r = 3, 4, \cdots, \infty, \omega$. The conjectures that we are interested in are the following ones:

C^r Bol-Loewner Conjecture (i.e. C^r-LC)
The index of an umbilic, of a surface C^r embedded in $I\!R^3$, is at most one.

C^r Bol-Loewner Conjecture* (i.e. C^r-LC*)
Let $\beta : U \subset I\!R^2 \to I\!R$, be a map of class C^r defined in a neighborhood U of $(0, 0) \in I\!R^2$. If $(0, 0)$ is an isolated singularity of the vector field

$$X : (x, y) \to (\beta_{xx} - \beta_{yy}, 2\beta_{xy}),$$

then the index of X at $(0, 0)$ is less or equal than 2.

C^r Bol-Loewner Conjecture with Lojasiewicz condition (i.e. C^r-LC with LC)
The index of an umbilic of Lojasiewicz–type, of a surface C^r embedded in $I\!R^3$, is at most one.

C^r Bol-Loewner Conjecture* with Lojasiewicz condition (i.e. C^r-LC* with LC)

Let β and X be as in C^r-LC*. If X satisfies a Lojasiewicz Condition at the singularity 0, then the index of X at 0 is less or equal than 2.

C^r-Carathéodory Conjecture (i.e. C^r-CC)
Every C^r convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics.

C^r-Carathéodory Conjecture* (i.e. C^r-CC*)
Let $\rho > 0$ and $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}$, be of class C^r, where U is a neighborhood of the compact disc $D(0,\rho) \subset \mathbb{R}^2$ of radius ρ centered at 0. If β restricted to a neighborhood of the circle $\partial D(0,\rho)$ has the form

$$\beta(x,y) = \frac{ax^2 + by^2}{x^2 + y^2}$$

where $a < b < 0$, then the vector field (defined in U)

$$X : (x,y) \to (\beta_{xx} - \beta_{yy}, 2\beta_{xy})$$

has at least two singularities in $D(0,\rho)$.

Theorem 3.15 *Let $r = 3, 4, \cdots, \infty, \omega$.*

*(a) C^r-LC is equivalent to C^r-LC**

(b) C^r-LC with LC is equivalent to C^r-LC with LC*

(c) The following conjectures are equivalent:

(c1) C^ω-LC,*

(c2) Polynomial-LC,*

(c3) C^r-LC with LC.*

*(d) If $r \neq \omega$, C^r-CC is equivalent to C^r-CC**

Proof:
The proofs of (a) and (b) are the same; they follow from Propositions 3.1 and 3.4 and Remark 3.8(b).

The proof of (c) is similar to that of Theorem 3.10. It follows from Lemma 3.9 and Remark 3.8(d).

Let us proceed to prove (d). We shall use the notations introduced in Section 3.1.

First, we shall see that if C^r-CC* is true then C^r-CC is true.

Without lost of generality, we may assume that the ovaloid is tangent to the xy-plane at $\bar{0} = (0,0,0)$, that $\bar{0}$ is not an umbilic point and that $N(\bar{0}) = (0,0,1)$.

By using the formula $M \cdot \Phi^- = \mathcal{B}^-$ and the assumption that $\Phi(0,0) = \bar{0}$, we conclude that $\beta^-(0,0) = \beta^-_x(0,0) = \beta^-_y(0,0) = 0$. Therefore, by rotating the ovaloid around the z-axis if necessary, we may assume that

$$\beta^-(x,y) = ax^2 + by^2 + \text{higher order terms},$$

and that $|a| \leq |b|$. It is easy to see that the assumptions imply that β^- has a local maximum at $\bar{0}$. By the convexity of the ovaloid and as $\bar{0}$ is not umbilic, we obtain that $a < b < 0$.

By a small C^2-perturbation of the ovaloid, around $\bar{0}$, we may assume that, around $\bar{0}$,

$$\beta^-(x,y) = ax^2 + by^2.$$

As, by Lemma 3.7,

$$\beta^+(u,v) = (u^2+v^2)\beta^-(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2})$$

we get that, there exists $\rho > 0$ such that, for all (u,v) in an neighborhood of $\{(u,v) : u^2+v^2 = \rho^2\}$,

$$\beta^+(u,v) = \frac{au^2+bv^2}{u^2+v^2}.$$

By the assumption this implies that the vector field

$$(u,v) \to (\beta^+_{uu} - \beta^+_{vv}, 2\beta^+_{uv})$$

has at least two singularities in $\{(u,v) : u^2+v^2 \leq \rho^2\}$. Each of these singularities is taken by the parametrization Φ^+ to an umbilic of the ovaloid. This proves that C^r-CC is true.

Now, we shall see that if C^r-CC is true then C^r-CC* is true. Suppose that we have a function $\beta = \beta^+ = \beta^+(u,v)$, and real numbers ρ, a, b, with $a < b < 0$, as in the assumptions of C^r-CC*. In particular, we are assuming that β^+ restricted to a neighborhood of the circle $\partial D(0,\rho)$ has the form

$$\beta^+(u,v) = \frac{au^2+bv^2}{u^2+v^2}$$

Without lost of generality, we may assume that β^+ is defined in the whole $\mathbb{R}^2$ and that, when restricted to a neighborhood of the set $\{(u,v) \in \mathbb{R}^2 : u^2+v^2 \geq \rho\}$, is given by the expression right above. Define $\beta^- = \beta^-(x,y)$, in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the expression

$$\beta^-(x,y) = (x^2+y^2)\beta^+(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}).$$

Let $\beta^-(0,0) = 0$. It can be seen that β^- restricted to a neighborhood of

$$S = \{(x,y) : x^2+y^2 \leq \frac{1}{\rho^2}\}$$

has the form $\beta^-(x,y) = ax^2 + by^2$. By construction, β^+ and β^- can be made the Bonnet functions of the support function $\sigma : S^2 \to \mathbb{R}$ of class C^r. By Theorem 3.13, there exists $c \in \mathbb{R}$ such that we have an ovaloid associated to $\sigma + c$. By the assumptions, the ovaloid has at least two umbilics. These umbilics must be contained in $\Phi^+(D(0,\rho))$. Each of these umbilics is taken by $(\Phi^+)^{-1}$ to a singularity (contained in $D(0,\rho)$) of the vector field

$$(u,v) \to (\beta^+_{uu} - \beta^+_{vv}, 2\beta^+_{uv}).$$

This proves that if C^r-CC is true then C^r-CC* is also true. ∎

3.6 Appendix

```
(*
   This program is devised to obtain the differential equation of the principal lines
of curvature as required in Proposition 2.1. The expressions "first", "second" and
"third" that will be found are to form the required differential equation: first dy^2
+ second dx dy + third dx^2 =0
*)
u = 2x
v = 2y
w = x^2 + y^2 - 1
lambda = {u , v, w }
lambdax = D[lambda, x]
lambday = D[lambda, y]
beta = q[x,y]
betax = D[beta, x]
betay = D[beta, y]
betaxx = D[betax, x]
betaxy = D[betax, y]
betayy = D[betay, y]
calB = { beta , betax, betay }
matriz = { lambda , lambdax, lambday }
matrizinv = Inverse[matriz]
phi = matrizinv . calB
phix = D[phi, x]
phiy = D[phi, y]
EE = Together[ (phix . phix ) * 4 * (1 + x^2 + y^2)^2 ]
FF = Together[ (phix . phiy ) * 4 * (1 + x^2 + y^2)^2 ]
GG = Together[ (phiy . phiy ) * 4 * (1 + x^2 + y^2)^2 ]
phixx = D[phix, x] * (1 + x^2 + y^2)^3
phixy = D[phix, y] * (1 + x^2 + y^2)^3
phiyy = D[phiy, y] * (1 + x^2 + y^2)^3
ee = Simplify[lambda . phixx]
```

```
ff = Simplify[lambda . phixy]
gg = Simplify[ lambda . phiyy]
primero = Simplify[FF * gg - GG * ff]
segundo = Simplify[EE * gg - GG * ee]
tercero = Simplify[EE * ff - FF * ee]
secondmatriz = { lambda, phix, phiy }
volume = 4 * (1 + x^2 + y^2)^4 * Det[secondmatriz]
first = Simplify[primero / volume]
second = Simplify[segundo / volume]
third = Simplify[tercero / volume]
```

4 An Introduction to Klotz proof

4.1 Preliminaries

All existing proofs deal with Bol-Loewner Conjecture for the case of analytic surfaces. The proofs given in [Ham], [Bol], [Klo] and [Sch] have similar structure. We base this presentation in Klotz paper [Klo].

Consider a closed oriented smooth curve C in the x, y-plane. Suppose that C does not pass through the origin. Then we define the winding number $N[C]$ of C with respect to the origin by

$$N[C] = N^+[C] - N^-[C]$$

where $N^+[C]$ is the number of times C crosses the positive x-axis with increasing y, and where $N^-[C]$ is the number of times C crosses the positive x-axis with decreasing y. We note that

$$N[C] = \frac{1}{2\pi}\nabla \arg\, C,$$

where the change in argument is taken as we transverse C once in the direction of its orientation.

Now we consider the index of the vector field

$$X = (\beta_{xx} - \beta_{yy}, -2\beta_{xy})$$

associated to the real valued C^2-function β defined in a neighborhood of $(0,0)$. To that end, we shall assume that X has an isolated singularity at $(0,0)$. Notice that, if the index of X at $(0,0)$ is n, then the index of the vector field

$$\overline{X} = (\beta_{xx} - \beta_{yy}, 2\beta_{xy}),$$

considered in Remark 3.8(b), at $(0,0)$ is $-n$. In particular the index of the umbilic that corresponds to $(0,0)$ (in terms of Proposition 3.4) is $-n/2$.

From now on, we shall suppose that ρ is so small that all considered curves are well defined.

Let

$$\Gamma_\rho = \Gamma_\rho(\theta) = X(\rho\cos\theta, \rho\sin\theta).$$

If $\omega = \beta(\rho\cos\theta, \rho\sin\theta)$ then, for ρ fixed,

$$\Gamma_\rho(\theta) = -\frac{1}{\rho^2} R_{2\theta}(x(\theta), y(\theta)) \tag{11}$$

where

$$R_{2\theta} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & cos 2\theta \end{pmatrix}$$

and

$$x(\theta) = \rho^2\, \omega_{\rho\rho} + \rho\, \omega_\rho + \omega_{\theta\theta} \tag{12}$$

$$y(\theta) = \rho\, \omega_{\rho\theta} - \omega_\theta. \tag{13}$$

To check (11) we have used the program, in Mathematica, of Appendix 4.4.

As the origin is an isolated singularity of X, there is a $\rho_0 > 0$ such that $C_\rho := (x(\theta), y(\theta))$ does not touch $(0,0)$ for $\rho < \rho_0$. But then $N[C_\rho]$ is defined when $\rho < \rho_0$ and so

$$N[\Gamma_\rho] = -2 + N[C_\rho].$$

This number - 2 is the contribution of the rotation $R_{2\theta}$. Therefore, Bol-Loewner Conjecture can be restated as follows:

Theorem 4.1 *Let ω, and C_ρ be defined as above. If C_ρ does not touch $(0,0)$ for $\rho < \rho_0$, then for all $\rho < \rho_0$,*

$$N[C_\rho] \geq 0$$

Remark 4.2 *For each $\rho > 0$ let $K_\rho(\theta)$ be a closed curve which is described as θ goes from 0 to 2π. Suppose there is a closed curve $K(\theta)$ such that for any $\sigma > 0$ we can choose ρ so small that*

$$|K_\rho(\theta) - K(\theta)| < \sigma.$$

If $N[K]$ is defined then, for all sufficiently small ρ, $N[K_\rho]$ is defined and

$$N[K_\rho] = N[K].$$

4.2 The winding number of a special curve

Lemma 4.3 *Let C be the curve*

$$x(t) = f'' - af$$

$$y(t) = bf',$$

where $a > 0, b > 0$ and where $f(t) \in C^2$ is periodic in t, while $f'(t) \neq 0$.

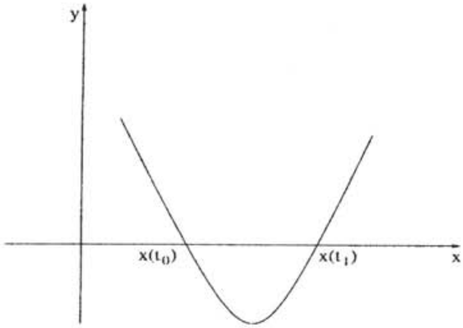


Fig. 2

(a) If C leaves a point $(x(t_0), 0)$ on the x-axis with $y(t)$ decreasing, then C next touches the x-axis to the right of $(x(t_0), 0)$, and therefore it next crosses the x-axis to the right of $(x(t_0), 0)$.

(b) If C does not pass through $(0, 0)$, $N[C] \geq 0$.

Proof:

Let us prove (a). Suppose C next touches the x-axis at the point $(x(t_1), 0)$, with $t_1 > t_0$. (see Fig. 2). In the interval

$$t_0 \leq t \leq t_1, \quad y(t) \leq 0 \quad \text{holds}$$

and therefore for $t \in [t_0, t_1]$,

$$f'(t) \leq 0.$$

Since, for some $t^* \in [t_0, t_1]$, $y(t^*) < 0$, we obtain, by integrating $f'(t)$

$$f(t_1) - f(t_0) < 0.$$

But,

$$f'(t_0) = f'(t_1) = 0, \quad f''(t_0) \leq 0, \quad f''(t_1) \geq 0.$$

Therefore

$$x(t_1) - x(t_0) = f''(t_1) - f''(t_0) - a[f(t_1) - f(t_0)] > 0.$$

This proves (a)

By (a), every contribution to $N^-[C]$ is followed by a contribution to $N^+[C]$. Therefore $N[C] = N^+[C] - N^-[C] \geq 0$. This proves (b) ■

4.3 Proof of Theorem 4.1 in the simplest case

Let β, X, ω and $C_\rho(\theta) = (x(\theta), y(\theta))$ be as in Section 4.1. As we are assuming that $(0,0)$ is a singularity of X, we may suppose that

$$\beta(0,0) = \beta_x(0,0) = \beta_y(0,0) = \beta_{xy}(0,0) = 0 \quad \text{and} \quad \beta_{xx}(0,0) = \beta_{yy}(0,0)$$

Hence, as we are assuming ω analytic,

$$w(\rho, \theta) = \sum_{m=k}^{\infty} \rho^m \, w_m(\theta), \tag{14}$$

where $w_m(\theta)$ is a trigonometric polynomial of degree m in θ and $k \geq 3$.

If we substitute (14) in both (12) and (13) and then divide both $x(\theta)$ and $y(\theta)$ by ρ^k, we obtain the curve I_ρ given by

$$\begin{aligned} x &= \sum_{m=k}^{\infty} \rho^{m-k} \{w_m'' - m(m-2)w_n\}, \\ y &= \sum_{m=k}^{\infty} \rho^{m-k} \{2(m-1)w_m'\}. \end{aligned}$$

By definition of winding number,

$$N[I_\rho] = N[C_\rho]$$

for $0 < \rho < \rho_0$.

But I_ρ , unlike C_ρ , does not shrink to a point as $\rho \to 0$. For sufficiently small ρ, I_ρ comes arbitrarily close to the curve M given by

$$\begin{aligned} x &= w_k'' - k(k-2)w_k \, , \\ y &= 2(k-1)w_k' \, , \end{aligned}$$

where M, unlike I_ρ , does not depend upon ρ.

If M does not touch $(0,0)$, then, by Lemma 4.3(b), $N[M] \geq 0$, and, by the property of winding number stated in Remark 4.2,

$$N[I_\rho] = N[M] \geq 0. \tag{15}$$

■

Klotz [Klo] goes on with the proof by claiming that –even when M above touches $(0,0)$– it can always be found a curve $\mathcal{J}_\rho$ for which $N[\mathcal{J}_\rho] \geq 0$ is well defined and such that

$$N[C_\rho] = N[\mathcal{J}_\rho]$$

4.4 Appendix

```
g = f[x, y]
x = r Cos[t]
y = r Sin[t]
gr = D[g, r]
gt = D[g, t]
grr = D[gr, r]
grt = D[gr, t]
gtt = D[gt, t]
ere = (- 1/r^2) { {Cos[2 t], Sin[2 t]}, {Sin[-2 t], Cos[2 t]} }
resul = Simplify[ere. {gtt + r gr - r^2 grr, -2 gt + 2 r grt} ]
Save["anexo2.res", resul]
```

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