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On Periodic Processes of Percolation Cellular Automata

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Abstract: Percolation cellular automata are a class or infinite Markov chains with local interaction, everyone of which has a parameter θ and a critical value $\theta^* \in (0,1)$ such that if $\theta > \theta^*$, the chain has only one invariant measure and if $\theta < \theta^*$, the chain has more than one invariant measure. Periodic processes are a generalization of invariant measures. We prove that if $\theta < 1/200$, the chain has at most two extremal periodic processes.

Key words: stochastic cellular automata, percolation, invariant measures, periodic processes, ergodicity.

§1. Introduction.

For any set V we may consider the set of its subsets, which we shall interpret as a configuration space $S = \{0,\ 1\}^V$. Components of a configuration are denoted $a(p),\ p \in V$. Call M_S the set of normed measures on S, i.e. on the σ-algebra generated by all its cylinder subsets. Denote δ_0 and δ_1 the measures concentrated at the configurations "all zeros" and "all ones" respectively. At the center of our attention is the configuration space $\{0,\ 1\}^{Z^{d+1}}$. Elements of Z^{d+1} are called integer points or integer vectors. They may be denoted $(s,\ t)$, where $s \in Z^d$ is the space coordinate and $t \in Z$ is the time coordinate. Now to define our cellular automata. Given a non-empty set of *parent vectors*

$$(\Delta s_1, \Delta t_1),\ \ldots,\ (\Delta s_n,\ \Delta t_n) \in Z^{d+1},$$

where all $\Delta s_i \in Z^d$ and all Δt_i are integer and positive. We assume that the linear combinations of parent vectors with integer coefficients cover all Z^{d+1}. (Otherwise Z^{d+1} breaks into parts which do not interact with each other and therefore can be treated separately.) Parent vectors are abbreviated as p.v. Values which depend on them may be called constants. Points $(s - \Delta s_1,\ t - \Delta t_1),\ \ldots,\ (s - \Delta s_n,\ t - \Delta t_n)$ are called *parents* of $(s,\ t)$. Call a sequence of points a *path* if everyone of them (except the last one) is a parent of the next one. Call one point an *ancestor* of another point if there is a path from the former to the latter.

Any function $\phi : \{0,\ 1\}^n \to [0,\ 1]$ may be used as a transition function. Given p.v. and a transition function, we say that we have a cellular automaton or CA. A measure μ on $\{0,\ 1\}^{Z^{d+1}}$ is called a process of this CA if the following "specification" holds: Given any t_0 and given components $a(s,\ t)$ for all $t < t_0$, the measure at t_0 is a product measure in which all components $a(s,\ t_0)$ are

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mutually independent and $a(s,\ t_0)$ equals 1 with a probability

$$\phi(a(s-\Delta s_1,\ t_0-\Delta t_1),\ \ldots,\ a(s-\Delta s_n,\ t_0-\Delta t_n)).$$

Call a process *extremal* if it cannot be represented as a linear combination of different processes with positive coefficients. We shall use the following limit construction: Choose a starting time t_0 and an arbitrary initial distribution of $a(s,\ t)$ for all $t < t_0$ and define a distribution of $a(s,\ t)$ for all $t \geq t_0$ in the inductive way, following our specification. All the limit points of these distributions when $t_0 \to -\infty$ with arbitrary initial distributions are processes. This shows that at least one process always exists.

Call a measure on $\{0,\ 1\}^{Z^{d+1}}$ *time-invariant* if the shift $t \to t+1$ turns it into itself and *space-invariant* if all the space shifts turn it into itself. Call a CA *monotonic* if the transition function is monotonic. For any monotonic CA we shall pay special attention to processes, which we denote $\mu_{\min}$ and $\mu_{\max}$, and which are obtained in the limit $t_0 \to -\infty$ if all the initial distributions are δ_0 and δ_1 respectively. It is easy to prove that in these cases the limit exists, both $\mu_{\min}$ and $\mu_{\max}$ are time- and space-invariant and extremal and that $\mu_{\min} \prec \mu \prec \mu_{\max}$ for any process μ , where $\prec$ means the partial order defined in [1, p.28].

Percolation cellular automata or PCA are a special kind of monotonic CA. Given p.v. and a number $\theta \in [0,\ 1]$, we define a transition function as follows:

$$\phi(x_1,\ \ldots,\ x_n) = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n = 0 \\ 1-\theta & \text{otherwise.} \end{cases}$$

We shall also use another way to define PCA. Let us associate with all points $(s,\ t),\ t \geq t_0$ i.i.d. random variables $b(s,\ t)$, everyone of which equals 0 with probability θ and 1 with probability $1-\theta$. Now, given all $a(s,\ t)$, $t < t_0$, we can define our distribution of $a(s,\ t),\ t \geq t_0$ inductively by the following rule:

$$a(s,\ t) = \max(a(s-\Delta s_1,\ t-\Delta t_1),\ \ldots,\ a(s-\Delta s_n,\ t-\Delta t_n)) \cdot b(s,\ t). \quad (1)$$

It is evident that any PCA is monotonic and its $\mu_{\min}$ is δ_0 .

It is known that for every set of p.v. under our assumptions there is a critical value $\theta^* \in (0,\ 1)$ such that if $\theta > \theta^*$, δ_0 is the only periodic process and that if $\theta < \theta^*$, there are at least two different periodic processes [3, 1]. Attention has been paid to time-invariant processes in the case when $d = 1$ and there are two parent vectors $(0,\ 1),\ (1,\ 1)$. In this case there are at most two linearly independent time-invariant processes: for all values of θ among space-invariant measures [2] and for small enough values of θ among all measures [4]. Here we use a different approach to obtain a new theorem in this vein. Call a measure on $\{0,\ 1\}^{Z^{d+1}}$ *periodic* if there are $d+1$ linearly independent vectors such that the corresponding shifts of Z^{d+1} turn it into itself.

Theorem. For every PCA with $\theta < 1/200$ there are at most two extremal periodic processes.

Comment. Of course, these two extremal periodic processes are $\mu_{\min}$ and $\mu_{\max}$. Therefore the set of all periodic processes is

$$\{x \cdot \mu_{\min} + (1 - x) \cdot \mu_{\max} \ : \ 0 \leq x \leq 1\}.$$

Since it is known that $\mu_{\min} \neq \mu_{\max}$ for small enough θ, we have a complete description of the set of all periodic processes in these cases.

§2. Proof of the theorem

The following will be used in the proof. It also explains why do we use the word 'percolation'. Let us say that the point $(s,\ t)$ is *blocked* if $b(s,\ t) = 0$ and *free* if $b(s,\ t) = 1$. Call a path *free* if all of its points, except perhaps the first one, are free. Then for any $t \geq t_0$ in the limit construction described above $a(s,\ t) = 1$ if and only if there is a free path to this point from some point $(s_1,\ t_1)$ where $a(s_1,\ t_1) = 1$ and $t_1 < t_0$.

We embed Z^{d+1} into R^{d+1} with the same coordinates, elements of which are called real points or real vectors. $|\cdot|$ means Euclidian norm. For any periodic measure μ and any state c we can speak of its *density* defined as

$$\lim \frac{1}{|B|} \sum_{p \in B} \mu(a(p) = c),$$

where $B \subset Z^{d+1}$ is a $(d+1)$-dimensional ball, $|B|$ is the number of points in B and the limit is taken when B grows to infinity.

For any PCA we shall use its *tripling*, which is an analog of coupling, but with three marginals. In this case for every point $(s,\ t)$ we have three variables $a_i(s,\ t),\ i = 1,\ 2,\ 3$. All the three are inductively defined by the rule

$$a_i(s,\ t) = \max\left(a_i(s - \Delta s_1,\ t - \Delta t_1),\ \ldots,\ a_i(s - \Delta s_n,\ t - \Delta t_n)\right) \cdot b(s,\ t), \quad (2)$$

where $b(s,\ t)$ are the same for all the three marginals and distributed as before. Here we use the same limit construction as before, where the initial conditions of the first and third marginals are δ_0 and δ_1 respectively for all t_0. Then the inequalities

$$a_1(s,\ t) \leq a_2(s,\ t) \leq a_3(s,\ t)$$

hold for every $(s,\ t)$ with probability 1. Therefore only combinations $(0,\ 0,\ 0)$, $(0,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ of values of $a_1(s,\ t),\ a_2(s,\ t)$, $a_3(s,\ t)$ are possible in the resulting process, that is in the limit $t_0 \to -\infty$.

Now let us prove our theorem by contradiction. If a PCA has three extremal periodic processes, at least one of them μ is neither $\mu_{\min}$ nor $\mu_{\max}$. Then, taking $(\delta_0,\ \mu,\ \delta_1)$ as the initial condition in the tripling for all t_0 and letting $t_0 \to -\infty$, we obtain a process in which all the three marginals are extremal and both (0, 0, 1) and (0, 1, 1) have positive densities. Let us denote (0, 0, 0), (0, 0, 1) and (0, 1, 1) by 0, 1 and 2 respectively. Thus we represent our tripling as a cellular automaton with $\{0,\ 1,\ 2\}^{Z^{d+1}}$ as the configuration space and the transition function which is also defined by (1). All we need to prove is the following.

Main lemma. Consider a cellular automaton on $\{0,\ 1,\ 2\}^{Z^{d+1}}$ with (1) as the transition rule. If $\theta < 1/200$, such a CA cannot have an extremal periodic process in which both densities of 1-s and 2-s are positive.

Before proving the main lemma let us make some notes.

Note 1. Given two non-empty homothetic polytops $P_1 \subset P_2 \subset R^d$. Then at least one vertex of P_2 is at a distance at least $(\mathrm{Diam}(P_2) - \mathrm{Diam}(P_1))/2$ from P_1. Here $\mathrm{Diam}(\cdot)$ means diameter.

Proof. Denote O the center of homothety. At least one vertex of P_2 is at a distance at least $\mathrm{Diam}(P_2)/2$ from O. This vertex is at a distance at least $(\mathrm{Diam}(P_2) - \mathrm{Diam}(P_1))/2$ from P_1. □

Call the *parent cone* the set of linear combinations of parent vectors with non-negative real coefficients.

Note 2. There is a constant R such that whenever a ball with the center $c \in Z^{d+1}$ and radius R belongs to the parent cone, c is an ancestor of (0, 0).

Proof is based on two facts:

A. There is a constant R_0 such that any real point in the parent cone is at a distance less or equal to R_0 from some ancestor of (0, 0).

Proof of A: The parent cone is a union of several sets, generated by $(d+1)$-tuples of linearly independent parent vectors. All these sets are affine images of the quadrant "all coordinates of R^{d+1} are non-negative". For each of these sets our statement is true because the set of linear combinations of these $(d+1)$ parent vectors with non-negative integer coefficients comes close enough to any real point in the corresponding set. Therefore our statement is also true for the union of these sets.

B. For any $R > 0$ there is a ball with the radius R, all integer points of which can be represented as linear combinations of parent vectors with non-negative integer coefficients.

Proof of B: Since the parent vectors generate all Z^{d+1}, they generate all the integer points in the ball with the radius R and the center (0, 0). By shifting this ball at a vector $K \cdot ((\Delta s_1,\ \Delta t_1) + \cdots + (\Delta s_n,\ \Delta t_n))$ with a large enough integer K, we obtain the ball we need. □

Note 3. Consider the site percolation on the oriented graph, which has Z^2 as the set of sites and from every site $(x,\ y)$ oriented bonds go to $(x+1,\ y)$ and $(x,\ y+1)$. Suppose that every site of this graph is blocked with probability θ and free with probability $1-\theta$ independently from other sites. The edges are always free in the direction of orientation and always blocked in the opposite direction. Then the probability that there is no percolation from (1, 1) to $(x,\ y)$ is not greater than $150\,\theta$ for all θ and all positive integer $x,\ y$ such that $x \leq 2^{y-1}$

and $y \leq 2^{x-1}$.

Proof. Using that version of the contour method which is explained in [3, 1], it is easy to prove that the probability of no percolation from $(1,\ 1)$ to $(x,\ y)$ does not exceed

$$2 \cdot \sum_{k=1}^{\infty} (27\,\theta)^k + x \cdot \sum_{k=y}^{\infty} (27\,\theta)^k + y \cdot \sum_{k=x}^{\infty} (27\,\theta)^k =$$

$$\frac{2 \cdot 27\,\theta + x \cdot (27\,\theta)^y + y \cdot (27\,\theta)^x}{1 - 27\,\theta}.$$

This is less than $150\,\theta$ under our assumptions. □

Now let us prove the main lemma. In fact we assume that the density of 2-s is positive and prove that the density of 1-s is less than any given $\varepsilon > 0$. Let us choose some $\theta < 1/200$ and describe our construction. Call a *disk* with the center $(s_0,\ t_0)$ and radius r the set of points $(s,\ t_0) \in R^{d+1}$ where $|s - s_0| \leq r$. Choose U so large that $(150\,\theta)^U < \varepsilon/2$. Then choose R so large that for any disk of radius R the probability of having no point in a state 2 is less than $\varepsilon/(2U)$. (We can do this because the process is extremal periodic and the density of 2-s is positive.) Thus chosen values of R and U are not yet final: we shall increase them at a later stage.

Call a parent vector *mixed* if it can be represented as a linear combination of some non-collinear parent vectors with positive coefficients. For any point $(s,\ t)$, where $t > 0$, call the *base* of $(s,\ t)$ and denote $B(s,\ t)$ the intersection of the parent cone shifted at $(s,\ t)$ with the real plane $t = 0$. Notice that bases of all points $(s,\ t)$, $t > 0$, are homothetic to each other and there is a constant L such that $\mathrm{Diam}(B(s,\ t)) = L \cdot t$. Notice also that the vertices of $B(s, t)$ are the points of intersection of the plane $t = 0$ with those rays that go through (s, t) and are parallel to the non-mixed parent vectors. Now choose T in the range $(6U \cdot R/L,\ 8U \cdot R/L)$ and estimate the probability that $a(0,\ T) = 1$. Since $a(0,\ T) = 1$, there is a free path from some point in $B(0,\ T)$ to $(0,\ T)$. Choose such a path and choose a sequence of points $p_0 = (s_0,\ t_0),\ \ldots,\ p_U = (s_U,\ t_U)$ along this path, so that $t_0 = 0$, $p_U = (0,\ T)$ and every difference $t_k - t_{k-1}$ is in the range $(5R/L,\ 9R/L)$.

Due to Note 1 we can find a vertex V_k of $B(p_k)$, whose distance from $B(p_{k-1})$ is not less that half of the difference of their diameters, which equals $(t_k - t_{k-1}) \cdot L/2$, which is in the range $(4R,\ 10R)$. Let us denote $\overline{V}_k$ a corresponding non-mixed parent vector. Thus we can choose a disk $D_k \subset B(p_k) - B(p_{k-1})$ with the radius R, the distance of whose points from V_k is in the range $(2R,\ 12R)$.

Let us consider two cases:

Either there is $k \in \{1,\ \ldots,\ U\}$ such that there is no point in D_k in the state 2. The probability of this event is less than $\varepsilon/2$.

Or for every k there is a point $q_k \in D_k$ such that $a(q_k) = 2$. In this case for all k there must be no percolation from q_k to p_k. Let us estimate the probability of this. Since $q_k \in D_k \subset B(p_k)$, from Note 2 there is some path from q_k to p_k. Let us consider the vectors of which this path consists and denote by $G_1, \ldots, G_x$ those of them which are parallel to $\overline{V}_k$ and by $H_1, \ldots, H_y$ all the others. Now consider the following graph Γ_k. Its vertices are points

$$Q_{i,\,j} = q_k + (G_1 + \cdots + G_i) + (H_1 + \cdots + H_j), \tag{3}$$

where $0 \leq i \leq x$ and $0 \leq j \leq y$. ¿¿¿¿From every $Q_{i,\,j}$ two bonds (or less) go: to $Q_{i+1,\,j}$ if $i < x$ and to $Q_{i,\,j+1}$ if $j < y$. It is evident that this graph is isomorphic to the relevant part of the graph considered in Note 3.

Let us write $A \asymp B$ to denote the fact that there are positive constants C_1 and C_2 such that $C_1 \cdot A \leq B$ and $C_2 \cdot B \leq A$. Notice that $x + y \asymp T \asymp UR$ and that $y \asymp R$. Hence it is sufficient to choose R and U large enough and equal to each other to make x and y satisfy conditions of Note 3 for all $k \in \{1, \ldots, U\}$. Therefore the probability that there is no percolation from q_k to p_k in this graph is not greater than $150\,\theta$. Since the graphs $\Gamma_1, \ldots, \Gamma_U$ have no common vertices, the probability that none of them percolates is less than $(150\,\theta)^U$. Thus the probability that there is no percolation from q_k to p_k for all $k \in \{1, \ldots, U\}$ is less than $\varepsilon/2$. Finally,

$$\text{Prob}\,(a(0,\, T) = 1) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The main lemma is proved and our theorem is also proved. □

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