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Modules of finite projective dimension over nice weakly triangular algebras

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Abstract: Let Λ be an artin algebra and let us consider its category of finitely generated, left modules, mod Λ. In this paper it is studied, for some classes of artin algebras, the full subcategory $\mathcal{P}^{<\infty}(\Lambda)$ of mod Λ defined by modules of finite projective dimension. Initially, it is assumed that Λ is an IIP-algebra, enlarging then the context to another class of weakly triangular algebras and, in the last section, to the so-called *nice* weakly triangular artin algebras. The classes of algebras that are studied have the property that $\mathcal{P}^{<\infty}(\Lambda)$ has Auslander-Reiten sequences and special consideration is given to the study of its relatively injective, indecomposable objects. The conducting line of this *resenha* is to focus on research done in the last few years by the members of the group of Representations of Algebras of the Instituto de Matemática e Estatística of the University of São Paulo. A brief review of the trajectory of the group is given in an appendix.

Key words: artin algebra, finite projective dimension, Auslander-Reiten sequence.

In this paper, unless otherwise specified, R is a commutative artin ring and k is an algebraically closed field. By a *module* we mean a left, finitely generated module, and by an *ideal* we mean a two-sided ideal. If Λ is a ring (resp. an (additive) category), mod Λ is the category of (finitely generated) Λ-modules and $\Lambda(M, N)$ the (group of) Λ-morphisms from the object (resp. module) M to the object (module) N.

For further developments and for terminologies and notations not explained in our text, we refer to [**ARS**] and [**R 1**].

1 Algebras with IIP condition.

Definition 1 *Let Λ be an artin R-algebra. It is said that Λ is an* IIP*-algebra if the following condition is satisfied.*

(IIP) *Every idempotent ideal of Λ is a projective Λ-module.*

IIP-algebras were first studied in [**P**], where Platzeck continued the research she began with M. Auslander and G. Todorov in [**APT**].

Example 1 *There are two trivial and, in a sense, extreme examples of* IIP-*algebras:* local algebras, *where the only idempotent ideal is the algebra itself, and* hereditary algebras, *where every ideal is projective.*

In order to simplify our exposition, we are going to assume in what remains of this section that R is an algebraically closed field k, and that Λ is basic, connected, and hence, being an artin algebra, of finite dimension over k. In other words, we suppose that $\Lambda = \mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_n$, with the $\mathbf{P}_i$ indecomposables and non isomorphic to one another, that Λ is indecomposable as a ring, and that

$$l_R(\Lambda) = \dim_k(\Lambda) \leq \infty$$

where, in general, $l_R(M)$ denotes the *length* of the R-module M.

Therefore, the $\mathbf{P}_i$'s represent all isoclasses of indecomposable, projective Λ-modules.

We will also keep throughout the following standard notations.

$\mathbf{S}_i$ denotes the simple Λ-module which is the *top* of $\mathbf{P}_i$, and $\mathbf{I}_i$ denotes the injective envelope of $\mathbf{S}_i$. Hence, the $\mathbf{S}_i$'s represent all the isoclasses of simple Λ-modules, and the $\mathbf{I}_i$'s, all the isoclasses of indecomposable, injective Λ-modules.

We know that each $\mathbf{P}_i$ is generated, as a left Λ-ideal, by an irreducible idempotent. In other words, there is a complete family of orthogonal, irreducible idempotents, e_i, such that each $\mathbf{P}_i$ is equal to Λe_i.

D denotes the usual duality in k-mod, $D = \mathrm{Hom}_k(-, k)$, so that we have the known identification

$$I_i = D(e_i\Lambda).$$

$\tau = \tau_\Lambda$ denotes the *Auslander-Reiten translation*, that is the map that kills all indecomposable projectives and associates each indecomposable, non-projective Λ-module N to the initial indecomposable of the Auslander-Reiten sequence ending at M: $0 \to \tau(N) \to M \to N \to 0$. Also, τ^- denotes the map that kills all indecomposable injectives and is the inverse of τ on the non-injective indecomposables.

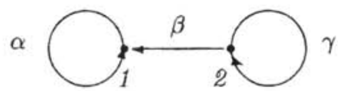
Finally, ind Λ will denote the full subcategory of modules determined by a family of representatives of the isoclasses of indecomposables.

It is well known that, with our simplifying hypotheses, there exists an, up to isomorphism *unique*, finite, connected *quiver* (*i. e. oriented graph*), $\mathbf{Q} = \mathbf{Q}(\Lambda)$, such that $\Lambda \cong k\mathbf{Q}/I$. Here, $k\mathbf{Q}$ denotes the *path algebra* of $\mathbf{Q}$, that is the set of formal k-linear combinations of paths of $\mathbf{Q}$ with the obvious multiplication given by the juxtaposition of paths (see [**R 1**]), and I stands for an *admissible ideal*, that is an ideal generated by paths of lengths ≥ 2 and that contains all paths of lengths $\geq m$ for some natural number m.

The usual procedure to define such an algebra is to give $\mathbf{Q}$ together with a finite set of equations linking paths or linear combinations of paths (the so-called *relations*), I being the ideal of $k\mathbf{Q}$ generated by them. If there is given just a family of (linear combinations of) paths, σ_λ, it is understood that the relations are $\sigma_\lambda = 0$.

Let us present a couple of examples for this, which will give also further examples of IIP-algebras.

Example 2 *Let* $\mathbf{Q}$ *be the following quiver.*



1. *Let* Λ_1 *be the* k*-algebra given by* $\mathbf{Q}$ *with the relations*

$$\alpha^3 = 0 \quad \text{and} \quad \gamma^2 = 0.$$

   Then, as shown in [**P**],(1.2), Λ_1 *is an* IIP*-algebra.*

2. *Let* Λ_2 *be the* k*-algebra given by* $\mathbf{Q}$ *with the relations*

$$\alpha^2 = 0, \quad \gamma^2 = 0 \quad \text{and} \quad \beta\alpha = 0.$$

   Then Λ_2 *is also* IIP *(see* [**P**],(1.2).*)*

We are going to present now some of the main results in [**CMMP**] which concern mainly the category $\mathcal{P}^{<\infty}(\Lambda)$ of finite projective dimension Λ-modules.

Proposition 1 *Let us assume that* Λ *is an* IIP*-algebra. Then, the following properties are satisfied.*

1. *If* $\Lambda(\mathbf{P}_i, \mathbf{P}_j) \neq 0$ *and if* $i \neq j$, $l_R(\mathbf{P}_i) < l_R(\mathbf{P}_j)$.

2. *It may be assumed that the ordering of the* $\mathbf{P}_i$*'s is such that*

$$\Lambda(\mathbf{P}_i, \mathbf{P}_j) = 0 \qquad \forall i < j.$$

3. *The (projective) finitistic dimension of* Λ *is less that or equal to 1.*

4. *Let us denote by* $\mathbf{A}_i$ *the quotient of* $\mathbf{P}_i$ *over the trace of* all other *indecomposable projectives in* $\mathbf{P}_i$*. Then,* $\mathcal{P}^{<\infty}(\Lambda)$ *is the subcategory of all modules which admit a filtration with factors in* $\mathbf{A}_1 \cdots, \mathbf{A}_n$ *(*see [**P**]*).*

5. *The subcategory $\mathcal{P}^{<\infty}(\Lambda)$ of Λ-modules of finite projective dimension is functorially finite and closed under extensions and, hence, has Auslander-Reiten sequences. (*See **[AS]**. *Cf. also Defs. 3, 4, 5 and Theorem 3 in section* **2** *below.)*

PROOF. We begin by observing that the idempotent ideals of Λ are the ideals generated by an idempotent, that is the ideals of the form $\Lambda e \Lambda$, with e an idempotent. But $\Lambda e \Lambda$ is precisely the *trace* of the projective Λe in Λ (where, in general, the trace of the module N in the module M is the submodule of M generated by all images of N into M). As an easy consequence, we deduce that Λ is an IIP-algebra if and only if all traces of projective modules in indecomposable projective modules are projective.

Hence, if $i \neq j$ and $\Lambda(\mathbf{P}_i, \mathbf{P}_j) \neq 0$, the trace of $\mathbf{P}_i$ in $\mathbf{P}_j$ is not 0. But this trace, being a projective module, must be a direct sum of copies of $\mathbf{P}_i$, forcing $\dim_k(\mathbf{P}_i)$ to be strictly less then $\dim_k(\mathbf{P}_j)$. This proves 1.

After 1., 2. is immediate.

The proof of 3. can be found in **[P]**. We omit it here because this assertion is a particular case of Prop. 3 in section **2**.

Also, there is a proof of 4. in **[P]**. We do not write it here because this assertion is implied by Theorem 3 of section **2**.

A similar observation applies to the proof of 5 (see Theorem 3 and Prop. 4 in section **2** below). □

Remark 1 *In what follows, we are going to assume that $\Lambda(\mathbf{P}_i, \mathbf{P}_j) = 0$ for $i < j$. Hence, an* IIP*-algebra Λ can be thought as a* triangular matrix algebra *of the following form.*

$$\begin{bmatrix} \mathbf{A}_1 & 0 \\ M & \Lambda' \end{bmatrix}$$

where Λ' is the subalgebra generated by the idempotent $1 - e_1$, and where M is the projective Λ'-module which is the trace of the $\mathbf{P}_i$'s with $i > 1$ in $\mathbf{P}_1$.

This fact may be used to obtain some results or to simplify some proofs concerning IIP*-algebras.*

Let us write now some easy characterizations of the IIP-algebras that are hereditary algebras.

Proposition 2 *Let Λ be an* IIP*-algebra. Then, the following propositions are equivalent.*

1. *Λ is hereditary*
2. *There does not exist an indecomposable Λ-module of infinite projective dimension.*

3. $\mathrm{End}(\mathbf{P}_i)$ *is a division ring for all* $i = 1, \cdots, n$.

4. The quiver of Λ, $\mathbf{Q}$, *has no loops.*

PROOF. It is well known that 2., 3. and 4. are satisfied if Λ is hereditary.

If 2. is satisfied, the finitistic dimension of Λ coincides with its global dimension, so that 1. results from Prop. 1, 3.

If $\mathrm{End}(\mathbf{P}_i)$ is a division ring, $\mathrm{rad}(\mathbf{P}_i)$ coincides with the trace in $\mathbf{P}_i$ of the other indecomposable projectives, so that it is a projective module. So, $\mathrm{rad}(\Lambda)$ is projective and Λ is hereditary.

Finally, it is clear that 4. implies 3. □

We end this section with the main theorem of [**CMMP**].

Theorem 1 *Let* Λ *be an* IIP*-algebra which is of infinite representation type but is not hereditary. Then the category* $\mathcal{P}^{<\infty}(\Lambda)$ *of* Λ*-modules with finite projective dimension is also of infinite representation type.*

COMMENTS ON THE PROOF. The proof will not be written down here because it is rather technical and because it will occupy too much space. It depends on a careful analysis of the defining relations of $\Lambda = k\mathbf{Q}/I$ and the ad-hoc notion of a *suitable arrow*. On the other hand, the proof uses the concept of *glued algebras* which was presented and initially studied in [**AC**]. It is shown that, in case there are only a finite number of indecomposables of infinite projective dimension, the IIP-algebra Λ is a right glueing of hereditary algebras $B_1, \cdots, B_r$, all of infinite representation type, by an algebra C of finite representation type. It is shown then how a certain infinite family of indecomposable, say, B_1-modules can be extended to indecomposable Λ-modules that are not $B_1 \oplus \cdots \oplus B_r$-modules. This leads to a contradiction, because the $B_1 \oplus \cdots \oplus B_r$-indecomposables should form a cofinite family in ind Λ (*see* [**AC**]).

2 Weakly triangular algebras.

We have remarked that if Λ is an IIP-algebra, then it can be viewed as a triangular matrix algebra as follows.

$$\begin{bmatrix} \mathbf{A}_1 & 0 \\ M & \Lambda' \end{bmatrix}$$

In [**MP**], M. I. Martins and J. A. de la Peña studied local extensions Λ of the following form.

$$\begin{bmatrix} \mathbf{A}_1 & 0 \\ P & \Lambda' \end{bmatrix}$$

Where P is the trace of all the $\mathbf{P}_i$'s with $i > 1$ in $\mathbf{P}_1$, and $\mathbf{A}_1 = \mathbf{P}_1/P$.

In the above mentioned paper, there is a study of the so-called *local extensions*, or triangular matrix algebras of the form

$$\begin{bmatrix} A & 0 \\ M & \Lambda \end{bmatrix}$$

where A is a local algebra. Clearly, our two ways of visualizing an IIP-algebra as a triangular matrix ring present it as a local extension.

In [**MP**] there are also results about the Auslander-Reiten quiver of a local extension. Some of these are based in formulas obtained in [**M 1**]. There is also a study about determining the representation type (finite or tame) by means of certain quadratic form, and, in the last section, there are given new proofs of the fact that, if Λ is IIP, then its finitistic projective dimension is ≤ 1 and of the fact that $\mathcal{P}^{<\infty}(\Lambda)$ has Auslander-Reiten sequences.

Another way of generalizing the notion of an IIP-algebra was considered in [**CP**], in the context of the so-called *weakly triangular algebras.*

We are going to present now the main results in that paper. We assume here that Λ is a right artin ring. We will denote by $(\mathbf{P}_i)_{i=1,\cdots,n}$ a set of representatives of the isoclasses of indecomposable projective Λ-modules.

Definition 2 *The artin ring Λ is said to be* weakly triangular *if the ordering of the $\mathbf{P}_i$'s can be chosen in such a way that*

$$i < j \Rightarrow \Lambda(\mathbf{P}_i, \mathbf{P}_j) = 0.$$

Note. Whenever we say that Λ is weakly triangular, we will assume that the ordering of the projectives is chosen as in this definition.

Remark 2 *It is clear that, if Λ is a finite dimensional k-algebra of the form $\Lambda = k\mathbf{Q}/I$, then Λ is weakly triangular if and only if the oriented cycles of $\mathbf{Q}$ are compositions of loops around the same vertex. It is also clear that any* IIP*-algebra is weakly triangular.*

More generally, a triangular matrix ring of the form

$$\begin{bmatrix} T & 0 \\ M & S \end{bmatrix}$$

is weakly triangular if and only if the artin rings T and S are weakly triangular.

Hence, any local extension of a weakly triangular ring is weakly triangular and, furthermore, any weakly triangular ring is the result of a finite number of local extensions of this form.

Lemma 1 *(see* [**CP**],(1.3)*) Let* Λ *be weakly triangular. If*

$$0 \to Q_r \to \cdots \to Q_0 \to M \to 0$$

is a minimal projective resolution of M *with* $Q_0 \in \operatorname{add}(\mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_l)$*, then for each* d*,* $Q_d \in \operatorname{add}(\mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_{l-d})$*. In particular,* $\operatorname{pd}(M) \leq l$*.*

PROOF. We proceed by induction on l, noting that the result is obvious for $l = 1$. In the general case, let N be the kernel of the epimorphism from Q_0 onto M. Then, it is not difficult to see that $\Lambda(\mathbf{P}_l, N) = 0$, because, if not, M would not be of finite projective dimension. But this means that $Q_1 \in \operatorname{add}(\mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_{l-1})$, so that the induction hypothesis applies. $\square$

As a first consequence, we obtain the following result (*see* [**CP**], Cor. 2.4).

Proposition 3 *Let* Λ *be a weakly triangular artin ring. Then, the finitistic projective dimension of* Λ*, that is, the supremum of finite projective dimensions of* Λ*-modules, is less than or equal to the number of isoclasses of simple modules.*

Remark 3 *It also results easily from the lemma above that, when* $\operatorname{Ext}^1_\Lambda(S, S) \neq 0$ *for some simple module* S *of a weakly triangular artin ring* Λ*, then* S *has infinite projective dimension.*

One of the main results in the paper [**CP**] is the fact that, under certain hypothesis, the category $\mathcal{P}^{<\infty}(\Lambda)$ of modules with finite projective dimension of a weakly triangular artin ring Λ can be characterized as given by a *good filtration* in the sense of Ringel's paper [**R 2**] (*see* Theorem 2, below).

In the case that Λ is weakly triangular, let us denote by $\mathbf{Q}_i$ the direct sum of the indecomposable projectives up to the place i:

$$\mathbf{Q}_i = \mathbf{P}_1 \oplus \cdots \oplus \mathbf{P}_i,$$

and let $\mathbf{A}_i$ denote the quotient of $\mathbf{P}_i$ over the trace in it of the remaining projectives,

$$\mathbf{A}_i = \mathbf{P}_i / \tau_{\mathbf{Q}_{i-1}}(\mathbf{P}_i).$$

It is clear that $\mathbf{A}_i$ is also isomorphic to the endomorphism ring of $\mathbf{P}_i$, which is a local ring.

Before stating the theorem, let us recall that *the subcategory* $\mathcal{F}(\mathcal{B})$ *of the* Λ*-modules having a filtration with factors in a set of modules* $\mathcal{B}$ (see [**R 2**]) *is, by definition, the full subcategory of* Λ*-modules* M *such that there is a chain*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$$

such that all factors M_{j+1}/M_j $(j = 0, \cdots, s-1)$ *belong to* $\mathcal{B}$.

Theorem 2 (see [**CP**], Theorem 3.2) *Let Λ be a weakly triangular artin ring such that the traces of all the $\mathbf{Q}_i$'s in Λ have finite projective dimension. Then the subcategory $\mathcal{P}^{<\infty}(\Lambda)$ of modules with finite projective dimension coincides with the subcategory of modules having a filtration in $\mathcal{A} = \{\mathbf{A}_1, \cdots, \mathbf{A}_n\}$.*

PROOF. Obviously, since the $\mathbf{A}_i$'s have finite projective dimension, all $M \in \operatorname{ind}\Lambda$ admitting a filtration in $\mathcal{A}$ belong to $\mathcal{P}^{<\infty}(\Lambda)$.

For the converse, let us show first that, if $M \in \mathcal{P}^{<\infty}(\Lambda)$, then, for all $i = 1, \cdots, n$, $\tau_{\mathbf{Q}_i}(M)$ is also in $\mathcal{P}^{<\infty}(\Lambda)$ (*see* [**CP**], Lemma 2.2). Since, obviously, $M = \tau_{\mathbf{Q}_n}(M)$, it is enough to show that $\tau_{\mathbf{Q}_i}(M) \in \mathcal{P}^{<\infty}(\Lambda) \Rightarrow \tau_{\mathbf{Q}_{i-1}}(M) \in \mathcal{P}^{<\infty}(\Lambda)$.

Let us consider the exact sequence

$$0 \to K \to P \to \tau_{\mathbf{Q}_i}(M) \to 0,$$

where the epimorphism is a projective cover, and observe that, with our hypothesis, $K \in \mathcal{P}^{<\infty}(\Lambda)$.

By taking up traces of $\mathbf{Q}_{i-1}$ in the terms of this sequence, and noting that the trace of $\mathbf{Q}_{i-1}$ in $\tau_{\mathbf{Q}_i}(M)$ is equal to $\tau_{\mathbf{Q}_{i-1}}(M)$, we obtain another exact sequence

$$0 \to \tau_{\mathbf{Q}_{i-1}}(K) = K \to \tau_{\mathbf{Q}_{i-1}}(P) \to \tau_{\mathbf{Q}_{i-1}}(M) \to 0.$$

And from this, noting that the first two terms are in $\mathcal{P}^{<\infty}(\Lambda)$, we get the desired fact, that $\tau_{\mathbf{Q}_{i-1}}(M) \in \mathcal{P}^{<\infty}(\Lambda)$.

Now we use this to show that the filtration

$$0 \subset \tau_{\mathbf{Q}_1}(M) \subset \cdots \subset \tau_{\mathbf{Q}_n}(M) \subset M$$

leads to a filtration with factors in $\mathcal{A}$.

This is easy, because each quotient $X_i = \tau_{\mathbf{Q}_i}(M)/\tau_{\mathbf{Q}_{i-1}}(M)$ is a projective $\mathbf{A}_i$-module or, equivalently, a direct sum of copies of $\mathbf{A}_i$. In fact, if P is the projective cover of X_i and K the kernel of it, it is clear that $K = \tau_{\mathbf{Q}_{i-1}}(P)$ so that $X_i = P/\tau_{\mathbf{Q}_{i-1}}(P)$. □

In the classic study [**AS**], M. Auslander and S. Smalø solved the problem of finding conditions for an additive subcategory of mod Λ to have (relative) Auslander-Reiten sequences. It turned out that two conditions are crucial: That the subcategory is closed under extensions and that it is *functorially finite*. Let us explain these conditions.

Let us denote by $\mathbf{A}$ any R-artin algebra, and let $\mathcal{C}$ be a full subcategory of mod $\mathbf{A}$ which we assume to be closed under finite direct sums and under taking direct summands. As usual, ind $\mathcal{C}$ denotes a full subcategory of $\mathcal{C}$ defined by representatives of the isoclasses of indecomposable modules in $\mathcal{C}$. We say that $\mathcal{C}$ is of *finite type* if ind $\mathcal{C}$ is finite. Otherwise, we say that $\mathcal{C}$ is of *infinite type*.

Given $M \in \mathrm{mod}\Lambda$, we are going to consider the contravariant and the covariant representable functors associated to it, that is the functors $\mathcal{C}(-, M)$ and $\mathcal{C}(M, -)$, but restricted to $\mathcal{C}$.

Definition 3 *Let* $M \in \mathrm{mod}\Lambda$.

1. *We say that a morphism* $M_r \to M$, *with* $M_r \in \mathcal{P}^{<\infty}(\Lambda)$, *is a* right approximation *to* M *in* $\mathcal{C}$, *if it induces an epimorphism*

$$\mathcal{C}(X, M_r) \to \mathcal{C}(X, M)$$

for each $X \in \mathcal{C}$.

2. *We say that a morphism* $M \to M_l$, *with* $M_l \in \mathcal{C}$ *is a* left approximation *to* M *in* $\mathcal{C}$, *if it induces an epimorphism*

$$\mathcal{C}(M_l, X) \to \mathcal{C}(M, X)$$

for each $X \in \mathcal{C}$.

In other words, a right approximation to M in $\mathcal{C}$, is a morphism $\rho : M_r \to M$, with M_r in $\mathcal{C}$, with the property that every morphism $f : X \to M$ with X in $\mathcal{C}$ factors in the form $f = \rho f'$, where $f' : X \to M_r$. And, a left approximation is a morphism $\lambda : M \to M_l$, with M_l in $\mathcal{C}$, with the property that every morphism $g : M \to X$ with X in $\mathcal{C}$ factors in the form $g = g'\lambda$, where $g' : M_l \to X$.

These approximations, if they exist, are not unique. A right (resp. left) approximation is called *minimal* if no proper direct summand of it induces also a right (resp. left) approximation. It is not hard to show that minimal approximations are always unique up to isomorphism.

Definition 4 1. *It is said that the subcategory* $\mathcal{C}$ *is* contravariantly finite *if every* Λ*-module has a right approximation in* $\mathcal{C}$.

2. *It is said the* $\mathcal{C}$ *is* covariantly finite *if every* Λ*-module has a left approximation in* $\mathcal{C}$.

3. *It is said that* $\mathcal{C}$ *is* functorially finite *if it is both contra- and covariantly finite.*

Definition 5 *The subcategory* $\mathcal{C}$ *is* closed under extensions *if every short exact sequence* $0 \to A \to B \to C \to 0$ *with* $A, C \in \mathcal{C}$ *has also* $B \in \mathcal{C}$.

It is said that a subcategory of modules as $\mathcal{C}$ *has Auslander-Reiten sequences* if for every $C \in \mathrm{ind}\mathcal{C}$ which is not ext-projective, there is an Auslander-Reiten sequence of $\mathcal{C}$ ending up at C, and, for every $A \in \mathrm{ind}\mathcal{C}$ that is not ext-injective, there is an Auslander-Reiten sequence of $\mathcal{C}$ beginning at A.

The main result of M. Auslander and S. Smalø in [**AS**] is the following.

Theorem 3 *In order that $\mathcal{C}$ has Auslander-Reiten sequences it is sufficient that the following two conditions are satisfied.*

1. *(i) $\mathcal{C}$ is functorially finite.*
2. *(ii) $\mathcal{C}$ is closed under extensions.*

In the famous paper [**R 2**], C. M. Ringel showed the following important result.

Theorem 4 *If $\mathcal{A} = (\mathbf{A}_1, \cdots, \mathbf{A}_n)$ is a finite sequence of Λ modules verifying*

$$i \geq j \Rightarrow \mathrm{Ext}^1_\Lambda(\mathbf{A}_i, \mathbf{A}_j) = 0,$$

then the category $\mathcal{F}(\mathcal{A})$ is functorially finite.

Since such a category is obviously closed under extensions, the main consequence is that, in this case, $\mathcal{F}(\mathcal{A})$ has Auslander-Reiten sequences. In the same paper, C. M. Ringel showed that this result applies to the category of modules filtered by standard modules of a quasi-hereditary algebra.

In [**P**], M. I. Platzeck proved that this is also the case for IIP-algebras, and her proof can be easily adapted to our special weakly triangular algebras as stated in the following proposition (*see* [**CP**], Prop. 3.1).

Proposition 4 *Let Λ be a weakly triangular algebra such that all $\tau_{\mathbf{Q}_i}(\Lambda)$ have finite projective dimension. Then*

$$i \geq j \Rightarrow \mathrm{Ext}^1_\Lambda(\mathbf{A}_i, \mathbf{A}_j) = 0.$$

PROOF. Let us consider the following exact sequence.

$$0 \to \tau_{\mathbf{Q}_{i-1}}(\mathbf{P}_i) \to \mathbf{P}_i \to \mathbf{A}_i \to 0,$$

and let us apply to it the functor $\Lambda(-, \mathbf{A}_j)$, with $j \geq i$. We obtain

$$\cdots \to \Lambda(\tau_{\mathbf{Q}_{i-1}}(\mathbf{P}_i), \mathbf{A}_j) \to \mathrm{Ext}^1_\Lambda(\mathbf{A}_i, \mathbf{A}_j) \to \mathrm{Ext}^1_\Lambda(\mathbf{P}_i, \mathbf{A}_j) = 0.$$

But it is not difficult to show, using our hypothesis, that the first term is also 0, implying the desired conclusion, that $\mathrm{Ext}^1_\Lambda(\mathbf{A}_i, \mathbf{A}_j) = 0$. □

Corollary 1 *Let Λ be a weakly triangular algebra such that all $\tau_{\mathbf{Q}_i}(\Lambda)$ have finite projective dimension. Then $\mathcal{F}(\mathcal{A})$ has Auslander-Reiten sequences.*

3 $\mathcal{P}^{<\infty}(\Lambda)$ in the case of *nice* weakly triangular algebras: the relative injectives

In this section we will present the latter developments in connection with the results we introduced above, especially the contents of [**MMP**].

Initially, we got a nice result (*see* [**MMP**], Theorem 1): that the requirement that all traces $\tau_{\mathbf{Q}_i}(\Lambda)$ have finite projective dimension is equivalent to the requirement that all modules $\mathbf{A}_i$ have finite projective dimension. This prompt us to introduce the following definition.

Definition 6 *Let Λ be a weakly triangular artin algebra. We will say that Λ is a* nice weakly triangular algebra *if the following condition is satisfied.*

$$(N) \qquad \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n \in \mathcal{P}^{<\infty}(\Lambda).$$

It turns out then that these *nice* algebras are a particular case of the *standardly stratified algebras* introduced by I. Agoston, V. Dlab and E. Lukács (*see* [**ADL**]) and recently studied more deeply by I. Agoston, D. Happel, E. Lukács and L. Unger (*see* [**AHLU 1**] and [**AHLU 2**]). It is interesting to mention that our main results in [**MMP**] were extended to these algebras by M. I. Platzeck and I. Reiten (*see* [**PR**]).

Let us begin with the following result, stated as Theorem 1 in [**MMP**].

Theorem 5 . *Let Λ be a weakly triangular artin algebra. Then, the following conditions are equivalent.*

$$(i)\ \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n \in \mathcal{P}^{<\infty}(\Lambda)$$

$$(ii)\ \tau_{\mathbf{Q}_i}(\Lambda) \in \mathcal{P}^{<\infty}(\Lambda)\ \forall i = 1, 2, \cdots, n$$

In order to prove it, we need first the three following lemmas and a corollary of the first.

Let us recall the following results of [**APT**].

Let us consider for a projective Λ-module Q the full subcategories $\mathbf{C}_o^Q$, $\mathbf{C}_1^Q$ and $\mathbf{C}_\infty^Q$ of mod Λ defined in the following way.

The modules in $\mathbf{C}_o^Q$ are those having their projective cover in add$Q$, the full subcategory of mod Λ consisting of direct summands of finite sums of Q. The modules in $\mathbf{C}_1^Q$ are those having a projective presentation in add$Q$, and, finally $\mathbf{C}_\infty^Q$ consists of the modules with a projective resolution in add$Q$.

Let Q be a projective Λ-module and $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$. It is known that the functor $\Lambda(Q, \ ) : \mathrm{mod}\Lambda \to \mathrm{mod}\Gamma$ induces an equivalence of categories between $\mathbf{C}_1^Q$ and mod Γ. Moreover, it is proven in [**APT**] that this equivalence carries projective resolutions of modules in $\mathbf{C}_\infty^Q$ into projective resolutions of Γ-modules, proving in particular the following result.

Lemma 2 . [**APT**, Cor. 3.3 a)]. *Let Λ be an artin algebra, Q a projective Λ-module and $M \in \mathbf{C}_{\infty}^{Q}$. Then* $\mathrm{pd}_{\Lambda} M = \mathrm{pd}_{\Gamma} \Lambda(Q, M)$.

When the artin algebra Λ is weakly triangular we obtain the following corollary.

Corollary 2 . *Let Λ be weakly triangular, $\mathbf{Q}_i = \oplus_{j \leq i} \mathbf{P}_j$, $\Gamma_i = \mathrm{End}_{\Lambda}(\mathbf{Q}_i)^{op}$. Let $M \in mod\ \Lambda$ be such that $\tau_{\mathbf{Q}_i}(M) = M$. Then* $\mathrm{pd}_{\Lambda} M = \mathrm{pd}_{\Gamma_i} \Lambda(\mathbf{Q}_i, M)$.

Proof. Since Λ is weakly triangular the subcategories $\mathbf{C}_{o}^{Q_i}$, $\mathbf{C}_{1}^{Q_i}$ and $\mathbf{C}_{\infty}^{Q_i}$ coincide. The Corollary follows by observing that $\tau_{Q_i}(M) = M$ if and only if $M \in \mathbf{C}_{o}^{Q_i}$. □

Lemma 3 . (For the proof, see [**CP**], Lemma 2.2.) *Let us assume that Λ is weakly triangular and that all $\tau_{\mathbf{Q}_i}(\Lambda)\,(i = 1, 2, \cdots, n)$ are in $\mathcal{P}^{<\infty}(\Lambda)$. Then, $M \in \mathcal{P}^{<\infty}(\Lambda)$ implies $\tau_{\mathbf{Q}_i}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ for all $i \leq n$.*

Lemma 4 . *Let us assume that Λ is weakly triangular and that $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ have finite projective dimension. Then $\Gamma_i = \mathrm{End}_{\Lambda}(\mathbf{Q}_i)^{\mathrm{op}}$ has the same properties for each $i \leq n$.*

Proof. We get that Γ_i is weakly triangular from the fact that $\Lambda(\mathbf{Q}_i, -)$ defines an equivalence of categories between $\mathrm{add}\mathbf{Q}_i$ and the category of projective Γ_i-modules.

We observe next that, if Q is a projective Λ-module and if $P, X \in \mathbf{C}_1^{Q}$, then

$$\tau_{\Lambda(Q,P)} \Lambda(Q, X) = \Lambda(Q, \tau_P X).$$

It follows then that

$$\mathbf{A}_{\Gamma_i, t} =: \Lambda(\mathbf{Q}_i, \mathbf{P}_t) / \tau_{\oplus_{j<t} \Lambda(\mathbf{Q}_i, \mathbf{P}_j)} \Lambda(\mathbf{Q}_i, \mathbf{P}_t) \cong \Lambda(\mathbf{Q}_i, \mathbf{A}_t).$$

On the other hand, our hypothesis and Corollary 1 imply that the Γ_i-projective dimension of $\Lambda(\mathbf{Q}_i, \mathbf{A}_t)$ is finite. □

We state for convenience the following result of [**CP**].

Lemma 5 . ([**CP**], Lemma 2.2. *Let us assume that Λ is weakly triangular and that all $\tau_{\mathbf{Q}_i}(\Lambda)\,(i = 1, 2, \cdots, n)$ are in $\mathcal{P}^{<\infty}(\Lambda)$. Then, $M \in \mathcal{P}^{<\infty}(\Lambda)$ implies $\tau_{\mathbf{Q}_i}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ for all $i \leq n$.*

Proof of Theorem 5. Obviously, (ii) implies (i). To prove the converse we assume that (i) holds, and observe that proving (ii) amounts to proving that $\tau_{\mathbf{Q}_i}(\mathbf{P}_j)$ is in $\mathcal{P}^{<\infty}(\Lambda)$ for all $i, j = 1, 2, \cdots, n$. We prove this by induction on j. We observe that the statement holds for $j = 1$ and assume that it is true for

$j \leq k-1$. We will prove that $\tau_{\mathbf{Q}_i}(\mathbf{P}_k) \in \mathcal{P}^{<\infty}(\Lambda)\, \forall i = 1, 2, \cdots, n$ by induction on n.

Since we are assuming that the projective dimension of $\mathbf{A}_k$ is finite it follows that $\tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k)$ has finite projective dimension. Thus, by Corollary 2, we get that

$$\mathrm{pd}_{\Gamma_{k-1}} \Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k)) < \infty.$$

It follows from Lemma 1 that the induction hypothesis applies to the algebra Γ_{k-1}. Then we can apply Lemma 2 to the Γ_{k-1}-module $\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))$ and conclude that

$$\Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_i}(\tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))) = \tau_{\Lambda(\mathbf{Q}_{k-1},\mathbf{Q}_i)} \Lambda(\mathbf{Q}_{k-1}, \tau_{\mathbf{Q}_{k-1}}(\mathbf{P}_k))$$

has finite projective dimension over Γ_{k-1}, for all $i \leq k-1$. But this finishes the proof, because $\tau_{\mathbf{Q}_i}(\tau_{\mathbf{Q}_{k-1}}(\mathrm{P}_k)) = \tau_{\mathbf{Q}_i}(\mathrm{P}_k)$ and, applying Corollary 2 once more, we obtain $\tau_{\mathbf{Q}_i}(\mathbf{P}_k) \in \mathcal{P}^{<\infty}(\Lambda)$, as desired. □

The results stated in the following theorem were proven in **[CP]** for weakly triangular algebras such that $\mathrm{pd}_\Lambda \tau_{\mathbf{Q}_i}(\Lambda)$ is finite for all $i = 1, 2, \cdots, n$. In view of Theorem 5, we replace the last condition by the assumption that Λ is a nice weakly triangular algebra.

Theorem 6 . **[CP]** *Let Λ be a nice weakly triangular artin ring such that all the $\mathbf{A}_i$ $(i = 1, 2, ..., n)$ have finite projective dimension and let $M \in \mathrm{mod}(\Lambda)$. Then*

1. *If M is in $\mathcal{P}^{<\infty}(\Lambda)$ then $\tau_{\mathbf{Q}_j}(M)$ is in $\mathcal{P}^{<\infty}(\Lambda)$, for all $j = 1, 2, ..., n$, and the factor $\tau_{\mathbf{Q}_j}(M)/\tau_{\mathbf{Q}_{j-1}}(M)$ is a free $\mathbf{A}_j$-module.*
2. *M is in $\mathcal{P}^{<\infty}(\Lambda)$ if and only if M admits a filtration with factors in $\{\mathbf{A}_i / i = 1, 2, ..., n\}$.*
3. *The finitistic projective dimension of Λ is the maximum of the projective dimensions of $\mathbf{A}_1, \cdots, \mathbf{A}_n$.*
4. *$\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite, closed under extensions and hence has Auslander-Reiten sequences.*

Proof. The proof will be omitted here because it will not contribute with any interesting new facts or arguments. See **[CP]**.

Remark 4 *It follows from the last example in* **[CP]** *that the hypothesis of the theorem do not imply that all idempotent ideals of Λ have finite projective dimension.*

Now, we are going to study the subcategory $\mathcal{P}^{<\infty}(\Lambda)$ of modules of finite projective dimension for these *nice* algebras. By Theorem 6, $\mathcal{P}^{<\infty}(\Lambda) = \mathcal{F}(\mathcal{A})$ and has Auslander-Reiten sequences. One of the questions posed in [**AS**] for subcategories having Auslander-Reiten sequences is to determine their ext-projective and their ext-injective objects.

Obviously, there is no difficulty in describing the $\mathcal{P}^{<\infty}(\Lambda)$ ext-projectives: they are just the Λ-projective modules, but the question of describing the ext-injective modules of $\mathcal{P}^{<\infty}(\Lambda)$, which was solved in [**MMP**], is not so trivial.

This will be our task in this section. We would like to point out that in [**MMP**] there is given also a proof that this $\mathcal{P}^{<\infty}(\Lambda)$ satisfies Auslander-Butler theorem, i. e. its Grothendieck group (modulo $\mathbf{K}_0(\mathcal{P}^{\infty}, 0)$) is generated by the relations defined by the Auslander-Reiten sequences of $\mathcal{P}^{<\infty}(\Lambda)$ if and only if $\mathcal{P}^{<\infty}(\Lambda)$ is of finite type.

We will refer to notions relative to $\mathcal{P}^{<\infty}(\Lambda)$, such as $\mathcal{P}^{<\infty}(\Lambda)$-injective, $\mathcal{P}^{<\infty}(\Lambda)$-projective, $\mathcal{P}^{<\infty}(\Lambda)$-simple, etc., and we will show also that one can define the $\mathcal{P}^{<\infty}(\Lambda)$-socle of a module in $\mathcal{P}^{<\infty}(\Lambda)$. As we already mentioned, it is clear that the $\mathcal{P}^{<\infty}(\Lambda)$-projective objects are the projective Λ-modules. On the other hand, using the fact that $\mathcal{P}^{<\infty}(\Lambda)$ is closed under cokernels of monomorphisms, it follows that the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules coincide with the ext-injective modules in $\mathcal{P}^{<\infty}(\Lambda)$, that is, with the modules I such that $\operatorname{Ext}^1_\Lambda(X, I) = 0$ for all X in $\mathcal{P}^{<\infty}(\Lambda)$. Or, equivalently, with the modules I such that any exact sequence in $\mathcal{P}^{<\infty}(\Lambda)$ starting at I splits.

On the other hand, using that $\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite, we can give a different description of the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules in terms of the right $\mathcal{P}^{<\infty}(\Lambda)$-approximations of the indecomposable injective Λ-modules. Although these approximations may decompose, we will prove that they have only one indecomposable summand, up to isomorphism. The multiplicity of such summand will also be determined.

Definition 7 . *We say that the $\mathcal{P}^{<\infty}(\Lambda)$-socle of $M \in \mathcal{P}^{<\infty}(\Lambda)$ is defined and is equal to V when $M \in \mathcal{P}^{<\infty}(\Lambda)$ has a unique maximal $\mathcal{P}^{<\infty}(\Lambda)$-semisimple submodule, V.*

Let us show that the $\mathcal{P}^{<\infty}(\Lambda)$-socle is defined for all $M \in \mathcal{P}^{<\infty}(\Lambda)$.

Proposition 5 *Let Λ be a nice weakly triangular algebra. Then, for $M \in \mathcal{P}^{<\infty}(\Lambda)$, the family of $\mathcal{P}^{<\infty}(\Lambda)$-semisimple submodules of M has a maximum element.*

Proof. Let $M \in \mathcal{P}^{<\infty}(\Lambda)$. We know by Theorem 1 that $\tau_{Q_j}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ and

$$\tau_{Q_j}(M)/\tau_{Q_{j-1}}(M) \simeq \mathbf{A}_j^{n_j},$$

for some $n_j \geq 0$ and for all $j = 1, 2, ..., n$.

We will prove the proposition by induction on the minimal number k such that $\tau_{Q_k}(M) = M$. If $k = 1$ then $M \simeq \mathbf{A}_1^{n_1}$ and the result is true. Let k be greater than 1. We suppose that the proposition holds for modules N such that $\tau_{Q_{k-1}}(N) = N$ and let $M \in \mathcal{P}^{<\infty}(\Lambda)$ be such that $\tau_{Q_k}(M) = M$. We may assume that M is indecomposable. Since $\tau_{Q_{k-1}}(M) \in \mathcal{P}^{<\infty}(\Lambda)$ and $\tau_{Q_{k-1}}(\tau_{Q_{k-1}}(M)) = \tau_{Q_{k-1}}(M)$ we can apply the induction hypothesis and conclude that the socle of $\tau_{Q_{k-1}}(M)$ is defined.

We will prove that either $\mathbf{A}_k \simeq M$ or $\mathrm{soc}_{\mathcal{P}^{<\infty}(\Lambda)}(M)$ is defined and coincides with $\mathrm{soc}_{\mathcal{P}^{<\infty}(\Lambda)}(\tau_{Q_{k-1}}(M))$. To do so we consider a $\mathcal{P}^{<\infty}(\Lambda)$-simple submodule X of M and prove that either $M \simeq \mathbf{A}_k$ or $X \subseteq \tau_{Q_{k-1}}(M)$.

If $X \simeq \mathbf{A}_i$ with $i < k$ then the second assertion holds. So we assume $X \simeq \mathbf{A}_k$, and let $j : X \to M$ be the inclusion and $\pi : M \to \tau_{Q_k}(M)/\tau_{Q_{k-1}}(M)$ the canonical map. Since the only composition factor of X is $\mathbf{S}_k$, which is not a composition factor of $\tau_{Q_{k-1}}(M)$, it follows that $X \cap \tau_{Q_{k-1}}(M) = 0$, so πj is a monomorphism.

Since $\mathbf{A}_k$ is a local artinian ring then any monomorphism $f : \mathbf{A}_k \to \mathbf{A}_k^{n_k}$ splits. This follows from the fact that $0 \to \mathbf{A}_k \to \mathbf{A}_k^{n_k} \to \mathrm{Coker}(f) \to 0$ is a projective resolution of Coker(f) and is not minimal.

Since $X \simeq \mathbf{A}_k$ and $\tau_{Q_k}(M)/\tau_{Q_{k-1}}(M) \simeq \mathbf{A}_k^{n_k}$, for some $n_k \geq 0$, it follows that the monomorphism πj splits. Thus $j : X \to M$ splits. Since M is indecomposable this proves that $M \simeq X \simeq \mathbf{A}_k$, ending the proof of the proposition. $\square$

We go on now to study the $\mathcal{P}^{<\infty}(\Lambda)$-injective modules. The following proposition proves in particular that the $\mathcal{P}^{<\infty}(\Lambda)$-injective indecomposable modules have $\mathcal{P}^{<\infty}(\Lambda)$-simple socle and are the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelopes (in the sense that the corresponding inclusions are minimal morphisms) of $\mathbf{A}_1, \cdots, \mathbf{A}_n$.

Proposition 6 . *Assume Λ is weakly triangular and the modules $\mathbf{A}_1, \cdots, \mathbf{A}_n$ have finite projective dimension. Let*

$$\widetilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$$

be the minimal right $\mathcal{P}^{<\infty}(\Lambda)$-approximation of the indecomposable injective Λ-module $\mathbf{I}_i$, and let $\widetilde{\mathbf{I}}_i = \coprod_j \widetilde{\mathbf{I}}_{ij}$, with $\widetilde{\mathbf{I}}_{ij}$ indecomposable. Then

1. *$\{\widetilde{\mathbf{I}}_{ij}\}_{i,j}$ is the set of indecomposable $\mathcal{P}^{<\infty}(\Lambda)$-injective objects, up to isomorphism. Thus the category $\mathcal{P}^{<\infty}(\Lambda)$ has enough injectives.*

2. *$\mathbf{A}_i$ is not a $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\widetilde{\mathbf{I}}_j$, for $i < j$.*

3. *$\widetilde{\mathbf{I}}_{nj} \cong \mathbf{A}_n$, for all j.*

4. *$\widetilde{\mathbf{I}}_{ij} \simeq \widetilde{\mathbf{I}}_{i1}$, for all i, j.*

5. *$soc_{\mathcal{P}^{<\infty}(\Lambda)}(\widetilde{\mathbf{I}}_{i1}) = \mathbf{A}_i$.*

6. $\widetilde{\mathbf{I}}_i = (\widetilde{\mathbf{I}}_{i1})^{n_i}$, *where* n_i *is the smallest number of copies of* $\mathbf{A}_i$ *necessary to cover the* $\mathbf{A}_i$*-injective envelope* $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$ *of* $\mathbf{S}_i$. *That is,* n_i *is the length of* $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)/rad(\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i))$.

PROOF. 1) Since $\mathcal{P}^{<\infty}(\Lambda)$ contains the projective Λ-modules, all the approximations $\widetilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ are epimorphisms. We start by proving that all $\widetilde{\mathbf{I}}_i$ are $\mathcal{P}^{<\infty}(\Lambda)$-injective. Let

$$0 \to \widetilde{\mathbf{I}}_i \xrightarrow{f} X \xrightarrow{g} Y \to 0$$

be an exact sequence in $\mathcal{P}^{<\infty}(\Lambda)$. According to the remark at the beginning of this section we only need to prove that this sequence splits. Since f is a monomorphism and $\mathbf{I}_i$ is injective there is a morphism $t : X \to \mathbf{I}_i$ such that $\phi_i = tf$. Since X is in $\mathcal{P}^{<\infty}(\Lambda)$ then $t : X \to \mathbf{I}_i$ factors through the $\mathcal{P}^{<\infty}(\Lambda)$-approximation $\widetilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ That is, there is $h : X \to \widetilde{\mathbf{I}}_i$ such that $t = \phi_i h$. So $\phi_i = \phi_i hf$. The minimality of ϕ_i implies that hf is the identity map, so f splits, as required.

To prove the converse, let J be an indecomposable $\mathcal{P}^{<\infty}(\Lambda)$-injective module in $\mathcal{P}^{<\infty}(\Lambda)$, and let $j : J \to I(J)$ be its injective envelope in mod Λ. Since J is in $\mathcal{P}^{<\infty}(\Lambda)$ the map j factors through the $\mathcal{P}^{<\infty}(\Lambda)$-approximation

$$\widetilde{\mathbf{I}}(J) \xrightarrow{\phi} \mathbf{I}(J)$$

Thus there is $h : J \to \widetilde{\mathbf{I}}(J)$ such that $j = \phi h$. Since j is a monomorphism h is a monomorphism too and therefore it splits, because J is $\mathcal{P}^{<\infty}(\Lambda)$-injective. This ends the proof of 1).

2) We observe first that $\Lambda(\mathbf{P}_i, \mathbf{I}_j) = 0$ for $i < j$, because in this case $\Lambda(\mathbf{P}_j, \mathbf{P}_i) = 0$. So the minimal right $\mathcal{P}^{<\infty}(\Lambda)$ -approximation

$$\widetilde{\mathbf{I}}_j \xrightarrow{\phi_j} \mathbf{I}_j$$

vanishes on the trace of $\mathbf{Q}_i$ in $\widetilde{\mathbf{I}}_j$, for $i < j$.

Hence, ϕ_j factors through $\tilde{\mathbf{I}}_j/\tau_{\mathbf{Q}_i}(\widetilde{\mathbf{I}}_j)$. Since this quotient is in $\mathcal{P}^{<\infty}(\Lambda)$ (see Theorem 6) we deduce from the minimality of ϕ_j that the trace of $\mathbf{Q}_i$ in $\widetilde{\mathbf{I}}_j$ is 0, proving 2).

3) We know by 2) that the only $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\widetilde{\mathbf{I}}_n$ is $\mathbf{A}_n$. So, for each summand $\widetilde{\mathbf{I}}_{nj}$ of $\mathbf{I}_n$ there is an epimorphism $f : \widetilde{\mathbf{I}}_{nj} \to \mathbf{A}_n$ in mod $\mathbf{A}_n$. Since $\widetilde{\mathbf{I}}_{nj}$ is indecomposable it follows that f is an isomorphism, proving 3).

4) Since the approximation $\widetilde{\mathbf{I}}_i \xrightarrow{\phi_i} \mathbf{I}_i$ is minimal it follows that $\mathbf{S}_i$ is a composition factor of all summands $\widetilde{\mathbf{I}}_{ij}$ of $\widetilde{\mathbf{I}}_i$. Thus $\tau_{P_i}(\widetilde{\mathbf{I}}_{ij}) \neq 0$ for all j. Since we know by 2) that $\mathrm{Hom}_\Lambda(\mathbf{P}_k, \widetilde{\mathbf{I}}_i) = 0$ for all $k < i$, it follows that $\tau_{\mathbf{P}_i}(\widetilde{\mathbf{I}}_{ij}) = \tau_{\mathbf{Q}_i}(\widetilde{\mathbf{I}}_{ij})/\tau_{\mathbf{Q}_{i-1}}(\widetilde{\mathbf{I}}_{ij})$. We know then by Theorem 6 that $\tau_{\mathbf{P}_i}(\widetilde{\mathbf{I}}_{ij})$ is a free $\mathbf{A}_i$-module. That is, $\tau_{\mathbf{P}_i}(\widetilde{\mathbf{I}}_{ij}) = \mathbf{A}_i^{k_i}$, for some $k_i > 0$.

We can therefore consider monomorphisms $\mathbf{A}_i \to \widetilde{\mathbf{I}}_{ij}$, $\mathbf{A}_i \to \widetilde{\mathbf{I}}_{i1}$. Since both $\widetilde{\mathbf{I}}_{ij}$ and $\widetilde{\mathbf{I}}_{i1}$ are $\mathcal{P}^{<\infty}(\Lambda)$-injective and indecomposable, we get that they are isomorphic. This proves 4).

5) Using that $\Lambda(\mathbf{A}_j, \mathbf{I}_i) = 0$ and the minimality of the approximation ϕ_i it follows that $\mathbf{A}_j$ is not contained in $\tilde{\mathbf{I}}_i$, for $j \neq i$. Thus, to prove 5) we only need to prove that the integer k_i above considered is 1. We start by observing the following consequence of 2). If $M \in \mathcal{P}^{<\infty}(\Lambda)$ has $\mathcal{P}^{<\infty}(\Lambda)$-composition factors in $\{\mathbf{A}_i, \cdots, \mathbf{A}_n\}$, then all the modules that occur in a minimal injective $\mathcal{P}^{<\infty}(\Lambda)$-coresolution of M have the same property. This fact follows by induction on the length of the coresolution, and using 2). We know that the length of the coresolution is finite because the projective dimension of the modules in $\mathcal{P}^{<\infty}(\Lambda)$ are bounded.

To prove 5) we proceed by decreasing induction on i, using that the statement is true for $i = n$, by 3). Let

$$0 \to \mathbf{A}_i \to E_1 \to E_2 \to \cdots \to E_t \to 0$$

be a minimal $\mathcal{P}^{<\infty}(\Lambda)$-injective coresolution of $\mathbf{A}_i$.

Let m_k denote the number of times that $\widetilde{\mathbf{I}}_{i1}$ occurs as a summand of E_k, and let n_i be the multiplicity of $\mathbf{A}_i$ as $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\widetilde{\mathbf{I}}_{i1}$. We want to prove that $n_i = 1$. From the above observation we know that the E_i's have $\mathcal{P}^{<\infty}(\Lambda)$-composition factors in $\{\mathbf{A}_1, \cdots, \mathbf{A}_n\}$. So, the only possible summands of E_i are $\widetilde{\mathbf{I}}_{i1}, \cdots, \widetilde{\mathbf{I}}_{n1}$. From 2) we know that $\mathbf{A}_i$ is not a $\mathcal{P}^{<\infty}(\Lambda)$-composition factor of $\widetilde{\mathbf{I}}_{j1}$ for $j > i$. Thus the multiplicity of $\mathbf{A}_i$ in E_k is $m_k n_i$.Then, from the above coresolution, we obtain

$$1 + \sum_{k=1}^{t} (-1)^k m_k n_i = 0$$

from which it follows that n_i divides 1. So $n_i = 1$, as required.

6) Since $\mathbf{A}_i = \Lambda/\tau_{\hat{\mathbf{P}}_i}(\Lambda)$, then the $\mathbf{A}_i$-injective envelope of $\mathbf{S}_i$ is $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) = \{x \in \mathbf{I}_i : \tau_{\hat{\mathbf{P}}_i}\Lambda.x = 0\}$.

Let $\pi : \mathbf{A}_i^{k_i} \to \mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$ be the $\mathbf{A}_i$-projective cover of $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i)$, and let $j : \mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) \to \mathbf{I}_i$ be the inclusion map. Then $j\pi$ factors through the minimal approximation $\phi_i : \tilde{\mathbf{I}}_i \to \mathbf{I}_i$. Since the domain of π is $\mathcal{P}^{<\infty}(\Lambda)$-semisimple then it factors also through the $\mathcal{P}^{<\infty}(\Lambda)$-socle $\mathbf{A}_i^{n_i}$ of $\tilde{\mathbf{I}}_i$. But this says that $\mathbf{A}_i^{n_i}$ also covers $\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i) = j(\mathbf{I}_{\mathbf{A}_i}(\mathbf{S}_i))$, implying that $\phi_i\ |_{\mathbf{A}_i^{n_i}}$ factors as $\pi\pi'$, where π' is a split epimorphism. Let $\mathbf{A}_i^{n_i} \cong \mathbf{A}_i^{k_i} \oplus \mathbf{A}_i^{n_i - k_i}$ be the induced factorization. Since $\tilde{\mathbf{I}}_{i1}$ is the $\mathcal{P}^{<\infty}(\Lambda)$-injective envelope of $\mathbf{A}_i$ we obtain a decomposition $\tilde{\mathbf{I}}_i \simeq \tilde{\mathbf{I}}_{i1}^{k_i} \oplus \tilde{\mathbf{I}}_{i1}^{n_i - k_i}$ so that ϕ_i vanishes on the second summand $\tilde{\mathbf{I}}_{i1}^{n_i - k_i}$. Since the approximation ϕ_i is minimal, this implies that $n_i - k_i = 0$, as desired. □

Proposition 5 shows that though the approximation of an indecomposable injective module has only one indecomposable summand up to isomorphism, it is not indecomposable. In fact the multiplicity of such summand can be arbitrarily large, as shown in the following example.

Example 3 . *Let* Λ *be a basic radical square zero local artin algebra of length* n. *Let* $\widetilde{\mathbf{I}}$ *be the minimal* $\mathcal{P}^{<\infty}(\Lambda)$*-approximation of the indecomposable injective* Λ*-module* $\mathbf{I}$. *Then the multiplicity of the indecomposable* $\mathcal{P}^{<\infty}(\Lambda)$*-injective module as a summand of* $\widetilde{\mathbf{I}}$ *is* $n-1$.

4 Appendix: Representations of algebras at IME-USP.

In 1979, after attending the second ICRA (= International Conference on Representations of Algebras) at Carleton University, Ottawa, Canada, we finally made our decision of shifting our research to this area, following the initial motivations received from the courses taught here in São Paulo by professors

V. Dlab (Carleton University, Ottawa) (1976) and

D. Simson (Toruń, Poland) (1978).

At this moment, our group was all but non-existent, since we counted only with some support from Prof. A. Jones and with a couple of students, E. N. Marcos and M. I. Martins. Shortly after, we participated in the ICRA III, in Puebla, Mexico (1980).

These first years demanded much work but gave a return of only small results. We received the scientific visits of

R. Bautista (Mexico) and

G. Michler (Germany),

our students

E. N. Marcos and

H. Panzarelli

completed their masters and

F. U. Coelho

joined the group. The first book and class-notes were published locally (*see* [**JM**, **M 2**]). An introductory course was given (*see* [**M 3**]) in a meeting at the Catholic University of Valparaíso, Chile, and at the V Algebra School, at UNICAMP, Campinas, both in 1982. We sent a student abroad for the first time to work for a Ph. D. in the area (E. N. Marcos, under M. Auslander at Brandeis).

Our first presentations of papers in the area took place at a meeting at the Ohio State University, in 1982 (*see* [**M 4**], and at the ICRA IV, again at Carleton University (*see* [**M 5**]).

Our group continued growing. Some master students, like

M. R. Pinto and

A. G. Chalom,

came into the group, and we continued receiving distinguished visitors, like

J. Carlson (Georgia, USA) (1984) and

P. Gabriel (Zürich) (1986).

We also persevered in our divulgation of our subject in local or regional events: VI Algebra School, Brasília, 1984, algebra meetings in Santiago de Chile (1985) and Córdoba, Argentina (1986) and VII Algebra School, at IMPA, Rio de Janeiro, 1986. F. U. Coelho finished his master with us and went to Liverpool for a Ph. D. under the supervision of M. C. R. Butler.

In 1988 we had our first scientific long term visit, at the University of Zürich, and were able to participate in a meeting at the Banach Center (Warsow) and to visit the German centers of Bielefeld and Paderborn.

In the following year, our small but enthusiastic group organized our first international meeting at USP in São Paulo, with the title

Workshop on Singularities and Representation Theory of Algebras (*see* [**Pr**]),

and received the scientific visits of

M. Auslander (Brandeis) and

I. Reiten (Trondheim).

In 1990, we participated in the ICRA V, Tsukuba, Japan, and in the International Congress of Mathematicians in Kyoto.

The representation theory group of IME-USP began to consolidate more and more, having incorporated two new researchers: E. N. Marcos (Ph. D. 1987) and F. U. Coelho (Ph. D. 1990) who began to publish, to give courses and to participate in our seminars. Their doctoral theses were published in 1992, 95 and 1990, 93 (respectively) (*see* [**E 1**, **E 2**] and [**C 1**, **C 2**]). Also by 1990 our permanent seminar in representations of algebras was born.

The three of us began to pay short visits to regional centers (Montevideo and Bahia Blanca) and the group received the visits of other mathematicians as

D. Happel (Chemnitz, Germany) and

L. Unger (Paderborn, Germany) (1992),

E. L. Green (Virginia Tech) (1992),

A. Skowroński (Toruń) (1993),
H. Meltzer (Chemnitz) (1995),
M-P. Malliavin (Paris VI) (1996)
Liu, Shao-Xue (Beijing) (1996)
María Inés Platzeck (Bahia Blanca, Argentina),
J. A. de la Peña (México),
B. Keller (Paris VII) (1997).
R. Martínez-Villa (México) (1997),
Martha Takane (México) (1998),
V. Dlab (Ottawa) (1998) and
O. Kerner (Düsseldorf, Germany) (1998).

Joint research projects were started with the groups of E. L. Green (Virginia Tech, USA) and of M. I. Platzeck (Universidad del Sur, Argentina) and a cooperation agreement in the frame of CAPES-COFECUB was arranged with the group of Marie-Paule Malliavin in Paris VI-VII. In 1993 we launched a project in cooperation with the algebraists of UNICAMP, Campinas, consisting in organizing regional meetings with special attention to the work of our students. They are called "Encontros em Álgebra", or ENAL's, and we have had already six of them, two in each odd year (1993, 1995, 1997), one in each center (São Paulo and Campinas).

Thus, slowly but firmly and steadily, we arrived to the present situation. We are now in the number of five professors, since M. I. Martins and A. G. Chalom defended their theses under our supervision in 1996 and 1998, respectively (*see* [**BEL**] and [**G**]). Another (Argentinean) student finished her doctorate with F. U. Coelho and we have presently another two (Brazilian) students of E. N. Marcos scheduled to finish in November, 1998 and March, 1999, respectively.

The students in the group have now grown to the number of 11 (8 of them working for their doctor degree) and they are coming from Brasil and from other Latin American countries, like Colombia and Uruguay.

Also, the joint works with other researchers are growing steadily. Each year the professors members of the group visit other centers (Virginia Tech, Mexico, Bielefeld, Münich in 1998), participate in congresses (ICM-98, Berlin, ICRA 8,5, Bielefeld and XV School of Algebra, Canela, Brasil in 1998), receive visitors and engage in cooperation agreements (the one within CAPES-COFECUB is ending this year and one with the Universidad Autónoma de México and another with the Universidad del Sur, Bahia Blanca, are expected to get approval to start in 1999).

Finally, we are proud of being able to mention that our *Grupo de Representações de Álgebras do Instituto de Matemática e Estatística da Universidade de São Paulo* is organizing, under the coordination of F. U. Coelho, the *Conference in Representations of Algebras - São Paulo*, scheduled for July of 1999, which is expected to have the participation of one hundred or more specialists of all over the world. It will be indeed a very nice way of celebrating the twentieth anniversary of the group.

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