Resenhas IME-USP 1999, Vol. 4, No. 2, 223 - 228.

General superalgebras of vector type and (γ, δ)-superalgebras

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Abstract: A general superalgebra of vector type is a superalgebra obtained by a certain double process from an associative and commutative algebra $A =$ with fixed derivation D and elements λ, μ, ν. We prove that any such a superalgebra is a superalgebra of (γ, δ) type. Conversely, any simple = finite dimensional nonassociative (γ, δ) superalgebra with $(\gamma, \delta) \neq$ (1,1) or (-1,0) is isomorphic to a certain general superalgebra of vector type.

Let A be an associative and commutative algebra over a ring of scalars Φ, with fixed nonzero derivation $D \in Der\,(A)$, and elements $\lambda, \mu, \nu \in A$. Denote by $\bar{A}$ an isomorphic copy of a Φ-module A, with the isomorphism mapping $a \mapsto \bar{a}$. Consider the direct sum of Φ-modules $B = 3DA + \bar{A}$ and define multiplication on it by the rules

$$\begin{aligned} a \cdot b \ &= 3D \quad ab, \\ a \cdot \bar{b} \ &= 3D \quad \bar{a} \cdot b \ = 3D\ \overline{ab}, \\ \bar{a} \cdot \bar{b} \ &= 3D \quad \lambda\, ab + \mu\, D(a)b + \nu\, aD(b), \end{aligned}$$

where $a, b \in A$ and ab is the product in A. Define a Z_2-grading on B by setting $B_0 = 3DA$, $B_1 = 3D\bar{A}$; then B becomes a superalgebra, which we will denote by $B(A, D, \lambda, \mu, \nu)$ and call = a *general superalgebra of vector type.*

Various partial cases of this construction have been considered before: the superalgebras $B(A, D, 0, 1, -1)$ are just the *Jordan superalgebras = of vector type* [4, 5, 7, 8]; the superalgebras $B(A, D, \lambda, 2, 1)$ in = case *char* $\Phi = 3D3$ are *alternative* [9], and in case of = arbitrary characteristic are *(-1,1) superalgebras* [9, 10].

Conversely, it was proved in [9] that any simple nontrivial nonassociative alternative superalgebra of dimension more than six is = isomorphic to a superalgebra $B(A, D, \lambda, 2, 1)$, with A being a D-simple algebra = of characteristic 3. Similarly, any simple nonassociative (-1,1) = superalgebra of positive characteristic $p > 3$ is isomorphic to a superalgebra = $B(A, D, \lambda, 2, 1)$ [10]. In particular, any simple finite dimensional nonassociative = (-1,1) superalgebra always has a positive characteristic and so is isomorphic = to $B(A, D, \lambda, 2, 1)$.

In this paper we give a similar characterization for a general superalgebra of vector type $B(A, D, \lambda, \mu, \nu)$ with $\mu \neq \pm\nu$. We first show = that any such a superalgebra is a so called (γ, δ) *superalgebra* (see = below), and then we prove that, under certain conditions, a simple = nonassociative (γ, δ) superalgebra is isomorphic to $B(A, D, \lambda, \mu, \nu)$.

Let us start with the definitions. Throughout the paper, if otherwise is not

[1] Supported by the RFFI grant 99-01-00499

stating, the word = "(super)algebra" means a (super)algebra over an associative and commutative ring of = scalars Φ with $1/6 \in \Phi$.

An algebra A is called a (γ, δ) *algebra* if it satisfies the identities:

$$\begin{aligned} (x,y,z) + \gamma(y,x,z) - \delta(z,x,y) &= 3D \quad 0, \\ (x,y,z) + (y,z,x) + (z,x,y) &= 3D \quad 0, \end{aligned}$$

where $(x,y,z) = 3D(xy)z - x(yz)$ denotes the *associator* of elements $= x, y, z$, and γ, δ are some elements from Φ, satisfying the equality $\gamma^2 - \delta^2 + \delta - 1 = 3D0$.

These algebras were introduced in 1949 by A.Albert [1] in the study = of *2-varieties* of algebras, that is, the varieties in which for any = ideal I its square I^2 is again an ideal. Together with alternative algebras, = the varieties of (γ, δ) algebras for different γ, δ give all the = possible examples of homogeneous 2-varieties of algebras that contain strictly = the class of associative algebras.

According to the general definition of a superalgebra in a given = homogeneous variety of algebras (see [11]), a superalgebra $R = 3DR_0 + R_1$ is a (γ, δ) *superalgebra* if and only if it satisfies the = (super)identities:

$$(x,y,z) + = (-1)^{p(x)p(y)}\gamma(y,x,z) - (-1)^{(p(x)+p(y))p(z)}\delta(z,x,y) = 3D \quad 0 \quad (1)$$

$$(x,y,z) + = (-1)^{p(x)(p(y)+p(z))}(y,z,x) + (-1)^{(p(x)+p(y))p(z)}(z,x,y) = 3D \quad 0 \quad (2)$$

where $x, y, z \in R_0 \cup R_1$ and $p(r) \in \{0,1\}$ denotes a parity index of a homogeneous element $r : p(r) = 3Di$ if $r \in= R_i$.

In the sequel $B = 3DA + M$ will denote a (γ, δ) superalgebra with $= A = 3DB_0$, $M = 3DB_1$. Note that A is a (γ, δ) subalgebra of B, and M is a $(\gamma, \delta) =$ bimodule over A.

It was proved in [2] that any simple (γ, δ) algebra of characteristic $\neq 2, 3$, with $(\gamma, \delta) \neq$ (1,1),(-1,0), is = associative. We will see now that this statement is not true any more in the case of = (γ, δ) superalgebras.

Theorem 1 *Any general superalgebra of vector type $B(A, D, \lambda, \mu, \nu)$ with $\mu \neq \pm\nu$ is a = (γ, δ) superalgebra for $\gamma = 3D\frac{-\mu^2+\mu\nu-\nu^2}{\mu^2-\nu^2}$, $\delta = 3D\frac{2\mu\nu-\nu^2}{\mu^2-\nu^2}$. This superalgebra is simple if and = only if the algebra A is D-simple; and if $D(A)A^2 \neq 0$, then $B(A, D, \lambda, \mu, \nu)$ is not associative.*

Proof. Since $\bar{A}$ is an associative bimodule over A, it suffices to = consider only the associators that contain at least two elements from $\bar{A}$. = For any $a, b, c \in A$ we have

$$(a, \bar{b}, \bar{c}) \quad = 3D \quad \mu D(a)bc, \tag{3}$$

$$(\bar{a}, b, \bar{c}) \quad = 3D \quad (\mu - \nu)aD(b)c, \tag{4}$$

$$(\bar{a}, \bar{b}, c) \quad = 3D \quad -\nu abD(c), \tag{5}$$

$$(\bar{a}, \bar{b}, \bar{c}) \quad = 3D \quad \mu = \overline{D(a)bc} + (\nu - \mu)\overline{aD(b)c} - \nu\overline{abD(c)}. \tag{6}$$

It follows easily from (3)–(6) that the identity (2) = holds in $B(A,D,\lambda,\mu,\nu)$. Furthermore, let

$$\gamma = 3D\frac{-\mu^2+\mu\nu-\nu^2}{\mu^2-\nu^2}, \quad \delta = 3D\frac{2\mu\nu-\nu^2}{\mu^2-\nu^2},$$

then the equality $= \gamma^2-\delta^2+\delta-1 = 3D0$ is straightforward, and we have by (3)–(6)

$$\begin{aligned}
(a,\bar b,\bar c)+\gamma(\bar b,a,\bar c)-\delta(\bar c,a,\bar b) &= 3D\ \mu D(a)bc+\gamma(\mu-\nu)bD(a)c\\
&\quad -\delta(\mu-\nu)cD(a)b\\
&= 3D(\mu+(\gamma-\delta)(\mu-\nu))D(a)bc = 3D0,\\
(\bar a,b,\bar c)+\gamma(b,\bar a,\bar c)-\delta(\bar c,\bar a,b) &= 3D(\mu-\nu)aD(b)c+\gamma\mu D(b)ac+\delta\nu caD(b)\\
&= 3D\ (\mu-\nu+\gamma\mu+\delta\nu)aD(b)c = 3D0,\\
(\bar a,\bar b,c)-\gamma(\bar b,\bar a,c)+\delta(c,\bar a,\bar b) &= 3D\ --\nu abD(c)+\gamma\nu baD(c)+\delta\mu D(c)ab\\
&= 3D\ (-\nu+\gamma\nu+\delta\mu)abD(c) = 3D0,\\
(\bar a,\bar b,\bar c)-\gamma(\bar b,\bar a,\bar c)+\delta(\bar c,\bar a,=\bar b) &= 3D\ \mu\overline{D(a)bc}+(\nu-\mu)\overline{aD(b)c}-\nu\overline{abD(c)}\\
&\quad -\gamma(\mu\overline{D(b)ac}+(\nu-\mu)\overline{bD(a)c}-\nu\overline{baD(c)})\\
&\quad +\delta=(\mu\overline{D(c)ab}+(\nu-\mu)\overline{cD(a)b}\\
&\quad -\nu\overline{caD(b)}) = 3D0.
\end{aligned}$$

Therefore, (1) holds in $B(A,D,\lambda,\mu,\nu)$ too, and $B(A,D,\lambda,\mu,\nu)$ is a (γ,δ) = superalgebra.

It is clear that for any D-ideal I of A the set $I+\bar I$ is an = ideal of $B(A,D,\lambda,\mu,\nu)$, so the D-simplicity of A is a necessary condition for the = simplicity of $B(A,D,\lambda,\mu,\nu)$. On the other hand, if A is D-simple, then the Jordan superalgebra of vector type $B(A,D,0,\alpha,-\alpha)$ is simple for any $0\neq\alpha\in\Phi$ (see [4, 8]). Therefore, the = supersymmetrized superalgebra $B(A,D,\lambda,\mu,\nu)^+ \cong B(A,D,0,\mu-\nu,\nu-\mu)$ is simple, which yields immediately the simplicity of $B(A,D,\lambda,\mu,\nu)$. □

Let now $B = 3DA+M$ be a (γ,δ) superalgebra with $(\gamma,\delta)\neq$ (1,1),(-1,0). (Note that any (1,1) superalgebra is = antiisomorphic to a (-1,0) superalgebra.)

Lemma 1 *If B is simple and not associative, then it satisfies the = superidentity*

$$\langle\langle x,y\rangle,z\rangle = 3D0, \tag{7}$$

where x,y,z are homogeneous and $\langle x,y\rangle = 3Dxy-(-1)^{p(x)p(y)}yx$.

Proof. Since B is simple and not associative, it coincides with its = associator ideal $D(B)$. Therefore, it suffices to prove that the associator ideal of any (γ,δ) superalgebra R satisfies (7). Let $G = 3DG_0+G_1$ = be a Grassmann algebra, consider the *Grassmann envelope* = $G(R) = 3DG_0\otimes R_0+G_1\otimes R_1$ of the superalgebra

R. The algebra $G(R)$ is an = ordinary (γ, δ) algebra, with $\gamma - 2\delta + 1 \neq 0$, so by [3] its = associator ideal $D(G(R))$ satisfies the identity $[[x, y], z] = 3D0$. From here, by standard arguments on Grassmann envelope, we conclude that $D(R)$ satisfies (7). □

The following lemma shows that, in the presence of identity = (7), the study of (γ, δ) (super)algebras is reduced to (-1,1) = (super)algebras. This fact, in the algebra case, was observed by the author in the beginning of seventies (see [6, Proposition 4]); we used the = modification of this fact given in [3, lemma 6].

Lemma 2 *Let B be a (γ, δ) superalgebra that satisfies identity = (7). For any $\alpha \in \Phi$ denote by $B(\alpha)$ the superalgebra, obtained = from B by introducing the new multiplication*

$$x \cdot_\alpha y = 3D\alpha xy + (1 - \alpha)(-1)^{p(x)p(y)} yx.$$

Then, the superalgebra $B' = 3DB(1 - \gamma - \delta)$ is a (-1,1) superalgebra, and $B = 3DB'(\beta)$ for $\beta = 3D\frac{1-\gamma+\delta}{3}$.

Proof. Consider the Grassmann envelope $G(B)$, which is an ordinary (γ, δ) = algebra. It is easy to check that $G(B)(\alpha) = 3DG(B(\alpha))$ for any = $\alpha \in \Phi$. Therefore, by [3, lemma 6], the algebra $G(B') = 3DG(B)(1 - \gamma - \delta)$ = is a (-1,1) algebra,which proves that B' is a (-1,1) superalgebra. Moreover, by = the same lemma we have the equality $(G(B)(1 - \gamma - \delta))(\beta) = 3DG(B)$ for $\beta = 3D\frac{1-\gamma+\delta}{3}$, which proves that $B'(\beta) = 3DB$. □

We can give now the description of simple (γ, δ) superalgebras.

Theorem 2 *Let $B = 3DA + M$ be a simple nonassociative (γ, δ) superalgebra of characteristic $\neq 2, 3$, with $(\gamma, \delta) \neq$ (1,1),(-1,0). Then $(B, A, A) = 3D(A, B, A) = 3D[A, B] = 3D0$, and there exist $x_1, \ldots, x_n \in M$ = such that $M = 3DAx_1 + \ldots + Ax_n$ and the product in M is defined by*

$$ax_i \cdot bx_j = 3D\lambda_{ij} \cdot = ab + (-\gamma + \delta)D_{ij}(a)b + (-1 - \gamma + \delta)D_{ij}(b)a, \ i, j = 3D1, \ldots, n,$$

where $\lambda_{ij} \in A, D_{ij} = 3DD_{ji} \in Der\, A$. In particular, if = $n = 3D1$ then B is isomorphic to a superalgebra $B(A, D, \lambda, -\gamma + \delta, -1 - \gamma + \delta)$, where A is a (unital) commutative and associative D-simple algebra = with $0 \neq D \in Der\, A$, $\lambda \in A$.

Proof. Let $\alpha = 3D1 - \gamma - \delta$, $\beta = 3D\frac{1-\gamma+\delta}{3}$, then by lemmas 1 and 2 we = have that $B' = 3DB(\alpha)$ is a (-1,1) superalgebra and $B = 3DB'(\beta)$. It is = obvious that the two-sided ideals of B and B' are the same; hence B' is simple. Furthermore, since B is not associative, neither is B'. Therefore, = by [10], B' has the following properties:

(i) A is a commutative and associative subalgebra of B', and B' = is an associative and commutative A-bimodule;

(ii) there exist $x_1, \ldots, x_n \in M$ such that $M = 3DAx_1 + \ldots + Ax_n$ and the product of odd elements in B' is = defined by

$$ax_i \cdot bx_j = 3D\lambda_{ij} \cdot ab + 2D_{ij}(a)b + D_{ij}(b)a, \ i,j = 3D1, \ldots, n,$$

where $\lambda_{ij} \in A, D_{ij} = 3DD_{ji} \in Der A$.

It follows immediately that B also satisfies (i) and the first part of (ii). As for the product of the elements of M in B is concerned, it is given by

$$ax_i \cdot \dot{b}x_j = 3D(2\beta-1)\lambda_{ij} \cdot = ab + (3\beta-1)D_{ij}(a)b + (3\beta-2)D_{ij}(b)a, \ i,j = 3D1, \ldots, n.$$

The theorem now is obvious. □

Corollary 1 *Let $B = 3DA + M$ be a simple nonassociative (γ, δ) superalgebra of characteristic $\neq 2, 3$, with $(\gamma, \delta) \neq$ (1,1),(-1,0). Assume = that one of the following conditions is satisfied:*

(i) B is of positive characteristic;

(ii) B is finite dimensional;

(iii) A is a polynomial algebra on a finite number of variables;

(iv) A is a local algebra.

Then B is isomorphic to $B(A, D, \lambda, -\gamma + \delta, -1 - \gamma + \delta)$.

The proof follows easily from [10] in view of the fact that the = condition $n = 3D1$ in the theorem is satisfied by B if and only if it is = satisfied by the (-1,1) superalgebra B'. □

As in the case of (-1,1) superalgebras [10], we could not find any example of a simple nonassociative (γ, δ) superalgebra which would = not be isomorphic to a superalgebra of the type $B(A, D, \lambda, \mu, \nu)$. So it is still an open = question whether such superalgebras exist. Notice that in case a new simple = (γ, δ) superalgebra B exists, its attached superalgebra B^+ would give a new example of a simple Jordan superalgebra.

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