Resenhas IME-USP 1999, Vol. 4, No. 2, 229 - 248.

Anosov flows induced by partially hyperbolic Σ-geodesic flows

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Abstract: We construct Anosov flows related with partially hyperbolic flows on codimension 1 non-integrable orientable distributions of compact Riemannian manifolds. The distributions are constant umbilical and need a volume preserved. The manifolds are supposed to have sufficiently negative sectional curvatures on the planes contained in the distribution.

Key words: non-holonomic constraints Σ, non-integrable distributions Σ, Σ-geodesic flows, partially hyperbolic flows, Anosov flows.

1 Introduction

The flows, nowadays called Anosov flows, were extensively studied by D. V. Anosov in [A], 1967. The interest on Anosov flows follows from the fact that they constitute a class of non trivial dynamical systems which are structurally stable. In fact, an Anosov flow which is Hölder C^1 and has an invariant measure (generated by a volume's form) is ergodic .

The structural stability for Hölder C^1 Anosov flows, as well as the ergodicity when there is an invariant measure, were proved by Anosov in his book [A]. There it was also proved that the geodesic flow on the unitary tangent bundle of a compact Riemannian manifold, having strictly negative all their sectional curvatures, is an Anosov flow. The Anosov flows are a special case of the so called **partially hyperbolic flows**:

Definition 1 *Let Q be a compact C^∞ manifold . A non singular smooth flow $T^t : Q \to Q$ is partially hyperbolic if the derivative (variational) flow $DT^t : TQ \to TQ$ satisfies:*

i) for any $p \in Q$, $T_pQ = \mathcal{X}_p \oplus \mathcal{Y}_p \oplus \mathcal{Z}_p$, where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are invariant sub-bundles of TQ, $\dim \mathcal{X}_p = l \geq 1, \dim \mathcal{Y}_p = k \geq 1, \mathcal{Z}_p \supset [(T^tp)'_{t=0}]$;

ii) there exist a Riemannian metric on Q and numbers $a, c > 0$ such that for all $p \in Q$ we have:

$$|DT^t\xi| \leq a|\xi|e^{-ct}, \forall t \geq 0, \ (or \ \ |DT^t\xi| \geq a^{-1}|\xi|e^{-ct}, \forall t \leq 0), \ \forall \xi \in \mathcal{X}_p;$$

$$|DT^t\eta| \leq a|\eta|e^{ct}, \forall t \leq 0, \ (or \ \ |DT^t\eta| \geq a^{-1}|\eta|e^{ct}, \forall t \geq 0), \ \forall \eta \in \mathcal{Y}_p.$$

$\mathcal{X}$ and $\mathcal{Y}$ are said to be uniformly contracting and expanding , respectively;

iii) $\mathcal{Z}$ is neutral in the sense that it is neither uniformly contracting nor uniformly expanding .

[1]Partially supported by CNPQ-JNICT and Dpt. Matemática IST-UTL, Lisbon

[2]Partially supported by Fapesp,proc.97/05484-1 (Projeto Temático 95/0790-1)

If, in particular, $\mathcal{Z}_p = [(T^t p)'_{t=0}]$ *, iii) is satisfied and the flow is said to be hyperbolic or Anosov.*

The consideration of a smooth non-integrable distribution Σ on a Riemannian manifold (in a sense we could start with a sub-Riemannian structure - see Kupka [Ku], 1996), enabled us to obtain partially hyperbolic flows on the unitary vector bundle defined by Σ. To obtain the hyperbolic properties of the trajectories of these flows we need to assume, among other suitable hypotheses, some negativeness for the sectional curvatures on the 2-planes contained in Σ (and only in Σ).

Our motivation came from Mechanics when one considers motions with nonholonomic perfect constraints, that is, satisfying the d'Alembert principle (see, for instance, Cartan [C], 1928; Harle [Ha], 1977; Fusco and Oliva [FO], 1986).

In the present paper we construct a class of examples of partially hyperbolic Σ-geodesic flows, which, in some particular cases, will induce a related Anosov flow. Let M be a compact Riemannian C^∞ manifold, $\dim M = m$, ∇ be the associated Levi-Civita connection, and (TM, M, π) the tangent bundle. Consider on M a non-integrable distribution Σ, that is, it is given $\Sigma M \subset TM$ a C^∞ sub-vector bundle of TM , with fibers $\Sigma_q M$, $\dim \Sigma_q M = n$, $\forall q \in M$, $n < m$ and let $\Sigma^\perp M \subset TM$ be the vector bundle corresponding to the orthogonal distribution $\Sigma^\perp$. One can introduce the **total second fundamental form** of Σ (see [KO]) :

$$B : TM \times_M \Sigma M \to \Sigma^\perp M$$

defined by: for any $X \in T_q M$, $Y \in \Sigma_q M$, $Z \in \Sigma_q^\perp M$, let $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ be local extentions of X, Y, Z respectively. Then

$$< B(X,Y), Z > = < \nabla_{\tilde{X}} \tilde{Z}, \tilde{Y} > (q) = - < \tilde{Z}, \nabla_{\tilde{X}} \tilde{Y} > (q). \tag{1}$$

It is not dificult to see that the total second fundamental form is well defined, that is:

Lemma 1 *The value of B does not depend on the chosed extentions.*

The **Σ-geodesic flow** corresponds to the d'Alembert trajectories (see [C], [Ha] and [FO]) of a constrained particle free of external forces, and it is determined by the equation:

$$\nabla_{\dot{q}} \dot{q} + B(\dot{q}, \dot{q}) = 0. \tag{2}$$

Any solution of this equation with initial condition $v \in \Sigma M$ will be called a **trajectory** . We remark that , since M is compact, M is complete in the sense that all trajectories will be curves defined for all $t \in I\!R$. Note that for any trajectory, $\dot{q}(t) \in \Sigma_{q(t)}$, $\forall t \in I\!R$ and

$$\frac{d}{dt} \|\dot{q}\|^2 = 2 < \dot{q}, \nabla_{\dot{q}} \dot{q} > = 0.$$

Definition 2 *The compact $(m + n - 1)$-dimensional manifold $\Sigma_1 M$ is the submanifold of ΣM given by $\{(q, v) \in \Sigma M$ such that $< v, v > = 1\}$.*

Throughout this paper we will consider the Σ-geodesic flow restricted to $\Sigma_1 M$. The vector field $X_{\Sigma_1 M} \in \mathcal{X}(\Sigma_1 M)$ is the one defined by the flow (2) restricted to $\Sigma_1 M$.

The conservation of a volume by the geodesic flow represents an important feature in this theory, and furnishes a natural invariant measure. It is natural to ask under which hypoteses there is a natural volume preserved by the Σ-geodesic flow. Kupka and Oliva (see [KO]) gave an answer to this question. In the codimension one case, that is, $n = m - 1$, when there exists a globally defined unitary normal (to Σ) vector field N, that is, when Σ is orientable, conservation of a volume is equivalent to the condition

$$\nabla_N N = 0. \tag{3}$$

In section 2 we will show how the metric on M can induce a Riemannian metric on $\Sigma_1 M$.

In section 3 we will derive the system of variational equations to $X_{\Sigma_1 M}$ and decompose it in some natural components, that decouple the system. In section 4 we will prove our main results, which we will state briefly here.

Definition 3 *The codimension 1 distribution Σ is said to be totally umbilical if for any $X, Y \in \Sigma_q M$, $\forall q \in M$:*

$$< B^s(X,Y), N >= \frac{1}{2} < B(X,Y) + B(Y,X), N >= h(q) < X, Y > \tag{4}$$

where $h \in C^\infty$. If h is a constant function we say that the distribution is constant umbilical. When $h \equiv 0$ we say that the distribution is totally geodesic.

We are now able to state the following:

Theorem 1 *Let M be a compact Riemannian m-dimensional manifold and ΣM an orientable codimension 1 distribution, constant umbilical. Assume that $X_{\Sigma_1 M}$ conserves volume. Suppose also that the sectional curvature on Σ satisfies $-K(X, Y) > h^2 + 4 < B(X,Y), N >^2$ for any orthogonal and normalized vectors $X, Y \in \Sigma_q M$, $\forall q \in M$. In this case, the Σ-geodesic flow on $\Sigma_1 M$ is partially hyperbolic.*

Note that, in the case of an integrable distribution, the condition on the sectional curvature is equivalent to the negativeness of the sectional curvature on the integrable leaves.

In the special case where the distribution is totally geodesic we really obtain an Anosov flow. In section 3 , we will construct a vector field W on $\Sigma_1 M$ such that if $\mathcal{W} = [W]$ is the one-dimensional vector-bundle defined by W then:

Theorem 2 *Under the same hypoteses of Theorem 1, assume that $h = 0$. Then there exists a vector field W commuting with $X_{\Sigma_1 M}$ such that $\mathcal{W} = [W]$ is invariant, and, if $\Sigma_1 M/\mathcal{W}$ is a regular manifold, the Σ-geodesic flow induces an Anosov flow on this quotient, so ergodic and structurally stable.*

Note that if $\Sigma_1 M/\mathcal{W}$ is not a regular manifold, we will still have an Anosov flow on a manifold with singularities.

In section 5 we present an example of this class of Anosov flows with $\Sigma_1 M/\mathcal{W}$ regular.

2 Horizontal and Vertical Subspaces. The Induced Metric on $\Sigma_1 M$

We define, as usually, the vertical subspace at a point $v_q \in \Sigma_1 M$ by:

$$\mathcal{V}_{v_q} = d\pi_{v_q}^{-1}(0) \quad \subset \quad T_{v_q}(\Sigma_1 M). \tag{5}$$

Note that with this definition $dim\mathcal{V}_{v_q} = n-1$.

The **vertical lift** of $u_q \in \Sigma_q M \cap [v_q]^\perp$ at a point $v_q \in \Sigma_1 M$ is then defined by:

$$C_{v_q}(u_q) = \left(q, \frac{v_q + su_q}{\| v_q + su_q \|} \right)' \Bigg|_{s=0} . \tag{6}$$

It is not dificult to see that with these definitions one has the following properties:

Lemma 2 $d\pi_{v_q} \circ C_{v_q} = 0$ *and* $C_{v_q} : \Sigma_q M \cap [v_q]^\perp \to \mathcal{V}_{v_q}$ *is an isomorphism.*

Let us now define a map $K_{v_q} : \mathcal{V}_{v_q} \to T_q M$. If $A_{v_q} \in \mathcal{V}_{v_q}$, let Z be a curve on $\Sigma_1 M$ such that $Z(0) = v_q$ and $Z'(0) = A_{v_q}$. Then, if $Z(t) = (Z_1(t), Z_2(t))$, with $Z_1(t) \equiv q \in M$ and $Z_2(t) \in T_{Z_1(t)} M$, we finally set $K_{v_q}(A_{v_q}) = Z_2'(0)$. It is not dificult to prove that:

Lemma 3 K_{v_q} *is a well defined linear map ,its image is given by* $Im(K_{v_q}) = \Sigma_q M \cap [v_q]^\perp$ *and* K_{v_q} *is the inverse of the vertical lift.*

Let us now define the **horizontal lift** H_{v_q} of a vector $w_q \in T_q M$, at a point $v_q \in \Sigma_1 M$. For this, let $\gamma(t)$ be a curve in M such that $\gamma(0) = q$, $\gamma'(0) = w_q$ and let $V(t)$ be the parallel transport of v_q along $\gamma(t)$. Then, if $PV(t) \in \Sigma_{\gamma(t)}$ denotes the orthogonal projection of $V(t)$, one defines:

$$H_{v_q}(w_q) = \left(\gamma(t), \frac{PV(t)}{\| PV(t) \|} \right)' \Bigg|_{t=0} \in T_{v_q} \Sigma_1 M \tag{7}$$

It is easy to prove that:

Lemma 4 *The horizontal lift* $H_{v_q} : T_q M \to T_{v_q} \Sigma_1 M$ *is a linear injective map and* $d\pi_{v_q} \circ H_{v_q} = id_{|T_q M}$.

The horizontal subspace at $v_q \in \Sigma_1 M$ is then defined as the image $\mathcal{H}_{v_q} = Im(H_{v_q}) \subset T_{v_q}(\Sigma_1 M)$. Note that $dim \mathcal{H}_{v_q} = m$.

With the constructions above we obtain a generalization of the classical decomposition in horizontal and vertical spaces and lifts relatively to the Σ-geodesic flow. As in the geodesic case, we have:

Lemma 5 *The tangent space of $\Sigma_1 M$ is the direct sum of its horizontal and vertical spaces, that is , $T_{v_q}(\Sigma_1 M) = \mathcal{V}_{v_q} \oplus \mathcal{H}_{v_q}$.*

As a matter of fact, the map K_{v_q} acts on any element $A_{v_q} \in T_{v_q}(\Sigma_1 M)$; in fact ,if $Z(t) = (Z_1(t), Z_2(t))$ is a curve on $\Sigma_1 M$ such that $Z(0) = v_q$ and $Z'(0) = A_{v_q}$, it is enough to define $K_{v_q}(A_{v_q})$ as the covariant derivative at $t = 0$ of $Z_2(t)$ with respect to $Z_1'(t)$. It is clear that this definition extends the one introduced above when $A_{v_q} \in \mathcal{V}_{v_q}$. Note that if $U_{v_q} \in \mathcal{H}_{v_q}$ then there exists $w_q \in T_q M$ such that $U_{v_q} = H_{v_q}(w_q)$. Using the same notation as in the definition of horizontal lift, we have :

$$K_{v_q}(U_{v_q}) = \nabla w_q \left. \frac{PV(t)}{\| PV(t) \|} \right|_{t=0} . \tag{8}$$

After some computation we obtain that:

Lemma 6 $K_{v_q}(U_{v_q}) = -B(d\pi(U_q), v_q)$.

We are now able to define the map that will allow us to turn $\Sigma_1 M$ into a Riemannian manifold:

Definition 4 *$i_{v_q} : T_{v_q}(\Sigma_1 M) \to T_q M \times T_q M$ is the linear injective map constructed as follows: if $A_{v_q} \in T_{v_q}(\Sigma_1 M)$ and $A_{v_q} = A^v_{v_q} + A^h_{v_q}$, let*

$$i_{v_q}(A_{v_q}) = \left(d\pi_{v_q}(A_{v_q}), -B(d\pi_{v_q}(A_{v_q}), v_q) + K_{v_q}(A^v_{v_q}) \right), \tag{9}$$

with inverse (on the image) given by:

$$i^{-1}_{v_q}(w_q, -B(w_q, v_q) + u_q) = H_{v_q}(w_q) + C_{v_q}(u_q) , \tag{10}$$

for all $w_q \in T_q M$, $u_q \in \Sigma_q M \cap [v_q]^{\perp}$.

Definition 5 *The metric on $\Sigma_1 M$ is defined by:*

$$< A_1, A_2 >_{v_q} = < i_{v_q}(A_1), i_{v_q}(A_2) >_{T_q M \times T_q M} =$$

$$= < d\pi_{v_q}(A_1), d\pi_{v_q}(A_2) >_q + < K_{v_q}(A_1), K_{v_q}(A_2) >_q . \tag{11}$$

3 The Variational Equation

Let T^t be the one parameter group of diffeomorfisms generated by the vector field $X_{\Sigma_1 M}$, that is, the flow defined by equation (2) on $\Sigma_1 M$. The variational equation for (2) is the equation that determines the time evolution of a vector $A \in T_{(q_0,\dot{q}_0)}(\Sigma_1 M)$ under the derivative $D_{(q_0,\dot{q}_0)}T^t$ that is, the equation that gives

$$A(t) = D_{(q_0,\dot{q}_0)}T^t(A) \in T_{(q(t),\dot{q}(t))}(\Sigma_1 M),$$

$q(t)$ being the trajectory defined by $(q_0, \dot{q}_0)$.

Consider a local coordinate system at $(q_0, \dot{q}_0) \in \Sigma_1 M$ of the type $(\varphi = (q_1, ..., q_m), \psi)$ and let $(\gamma(s), \Gamma(s))$, $s \in (-\epsilon, +\epsilon)$, be a curve in $\Sigma_1 M$ such that $(\gamma(0), \Gamma(0)) = (q_0, \dot{q}_0)$ and

$$\frac{d}{ds}(\gamma(s), \Gamma(s))\bigg|_{s=0} = (J_0, \mathcal{J}_0) = A.$$

Take, for each $s \in (-\epsilon, +\epsilon)$, the trajectory $(q(t,s), \dot{q}(t,s)) = T^t(\gamma(s), \Gamma(s))$. Note that it is defined a 2-dimensional submanifold of $\Sigma_1 M$. Then,

$$A(t,s) = (J(t,s), \mathcal{J}(t,s)) = D_{(\gamma(s),\Gamma(s))}T^t\left(\frac{d}{ds}(\gamma(s), \Gamma(s))\right) =$$

$$= \frac{d}{ds}T^t(\gamma(s), \Gamma(s)) = \left(\frac{d}{ds}q(t,s), \frac{d}{ds}\left(\frac{d}{dt}q(t,s)\right)\right)$$

defines, for each $s \in (-\epsilon, +\epsilon)$, a solution of the variational equation, which, from now on, will be called a **Jacobi field**. By construction, $\{q(t,s),\ s \in (-\epsilon, +\epsilon),\ t \in \mathbb{R}\}$ defines also a 2-dimensional submanifold of M , and so $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$. Since $\frac{\partial}{\partial s} = J$ and $\frac{\partial}{\partial t} = \dot{q}$, then, as elements of $T_{q(t,s)}M$, $[J, \dot{q}]$ is well defined and vanishes. Note that, as elements of $T_{(q,\dot{q})(t,s)}\Sigma_1 M$, one has $\frac{\partial}{\partial s} = (J, \mathcal{J})$ and $\frac{\partial}{\partial t} = X_{\Sigma_1 M}$, which implies that $[A(.,.), X_{\Sigma_1 M}] = 0$. For $s = 0$ the above construction provides the Jacobi field over the trajectory $(q(t), \dot{q}(t))$. We remark that a Jacobi field over a given trajectory does not depend on the curve chosen as a representant of A. It will be usefull to consider the image of a Jacobi field by the map $i_{(q,\dot{q})}$, that is, as a pair of vectors in $T_q M$.

Lemma 7 *If $A(t)$ is a Jacobi field over a trajectory $(q(t), \dot{q}(t)) \in \Sigma_1 M$, then*

$$i_{(q(t),\dot{q}(t))}(A(t)) = (J(t), \nabla_{\dot{q}(t)} J(t)) \tag{12}$$

where the second component is the covariant derivative of $J(t)$ with respect to $\dot{q}(t)$.

Since the second component of the image of a Jacobi field depends only of its first component, J will also be called a Jacobi field along $q = q(t)$. It is simple, now, to derive the variational equation (see also [Ha]),

Proposition 1 *The variational equation for the Jacobi fields is given by:*

$$\nabla_{\dot{q}} \nabla_{\dot{q}} J = R(\dot{q}, J)\dot{q} - \nabla_J B(\dot{q}, \dot{q}) \tag{13}$$

where R is the curvature tensor of M. Given an initial condition

$$(J(0), \nabla_{\dot{q}} J(0)) \in i_{(q(0),\dot{q}(0))}(T_{(q(0),\dot{q}(0))}\Sigma_1 M),$$

(13) has a unique solution, with this initial condition.

Proof: We consider $q(t,s)$ and $J(t,s)$ as above. Then, the left side of (13) is equal to $\nabla_{\dot{q}} \nabla_J \dot{q}$, and it will be enough to use the definiton of the curvature tensor, and the trajectory equation (2) ∎.

We will consider, from now on, a codimension 1 orientable distribution, that conserves volume, that is, $\Sigma^{\perp} M = [N]$, where N is a normalized vector field satisfying (3). In this case, there exists a C^{∞} function

$$k : M \times_M TM \times_M \Sigma M \to R,$$

linear on the second and third variables, such that :

$$B(X,Y)(q) = k(q,X,Y)N(q) \ , \quad \forall X \in T_q M, \ Y \in \Sigma_q M, \forall q \in M. \tag{14}$$

Whenever the meaning is clear, we will ommit the dependence of k on the first variable.

Given a trajectory $(q,\dot{q})$, let $\bar{P}_{(q,\dot{q})}$ be the orthogonal projection of $T_q M$ onto $\Sigma_q M \cap [\dot{q}]^{\perp}$. It will be usefull to decompose a Jacobi field J over $(q,\dot{q})$ as:

$$J(t) = a(t)\dot{q}(t) + Z(t) + b(t)N(q(t)), \tag{15}$$

where $Z(t) = \bar{P}_{(q(t),\dot{q}(t))} J(t)$. We will ommit the dependence on the trajectory and time whenever it does not cause confusion.

Let us study the first derivative of J . From (15) one obtains,

$$\nabla_{\dot{q}} J = \dot{a}\dot{q} + a\nabla_{\dot{q}}\dot{q} + \nabla_{\dot{q}} Z + \dot{b}N + b\nabla_{\dot{q}} N. \tag{16}$$

But

$$\nabla_{\dot{q}} Z = \bar{P}\nabla_{\dot{q}} Z + < \nabla_{\dot{q}} Z, N > N$$

and

$$\nabla_{\dot{q}} N = \bar{P}\nabla_{\dot{q}} N + k(\dot{q},\dot{q})\dot{q}.$$

With these relations and (2), (16) gives:

$$\nabla_{\dot{q}} J = (\dot{a} + bk(\dot{q},\dot{q}))\dot{q} + \bar{P}\nabla_{\dot{q}} Z + b\bar{P}\nabla_{\dot{q}} N +$$

$$+(-ak(\dot{q},\dot{q})- < Z, \nabla_{\dot{q}} N > +\dot{b})N. \tag{17}$$

But
$$< \nabla_{\dot{q}} J, \dot{q} >= \frac{1}{2} J < \dot{q}, \dot{q} >= 0,$$
where the derivative in the second term is taken on the 2-dimensional submanifold defined above. From now on, derivatives of functions of $\dot{q}$ or covariant derivatives of vector fields depending on $\dot{q}$, with respect to J, will mean derivatives(or covariant derivatives) on that 2-dimensional manifold.

Then, for the first component of the Jacobi field we get the equation:

$$\dot{a} = -bk(\dot{q},\dot{q}) \ . \tag{18}$$

On the other hand, we have that:

$$J < \dot{q}, N >=< \nabla_J \dot{q}, N > + < \dot{q}, \nabla_J N >= 0,$$

which implies that the normal component of the Jacobi field satisfies :

$$\dot{b} =< B(\dot{q}, Z) - B(Z, \dot{q}), N >= 2 < B^a(\dot{q}, Z), N >\equiv 2k^a(\dot{q}, Z), \tag{19}$$

where B^a is the skewsymmetric component of B on $\Sigma M \times \Sigma M$.

To study the conditions for the variational flow on $T(\Sigma M)$ to be Anosov, we need to analyse the norms of Jacobi fields. We recall that $\|A\|^2 = \|J\|^2 + \|\nabla_{\dot{q}} J\|^2$. To know the exponential decay of a particular $\|A\|$ we need to check the exponential decay of each component. To do so, we will first show that, under suitable hypotheses, $\|Z(t)\|^2$ is a convex function . First we note that:

$$\frac{1}{2}\frac{d^2}{dt^2}\|J\|^2 =< \nabla_{\dot{q}} \nabla_{\dot{q}} J, J > +\|\nabla_{\dot{q}} J\|^2. \tag{20}$$

In the sequel we will assume that the distribution is constant umbilical. Note that, in this case, $B^a(X,Y) = B(X,Y)$ whenever X, Y are orthogonal. From (13), (15) and (2) we have that:

$$< \nabla_{\dot{q}} \nabla_{\dot{q}} J, J >=< R(\dot{q}, Z)\dot{q}, Z > +b^2 < R(\dot{q}, N)\dot{q}, N > +$$

$$+2b < R(\dot{q}, Z)\dot{q}, N > -h < \nabla_J N, Z + a\dot{q} >= -\|Z\|^2 K(\dot{q}, \hat{Z})+$$

$$+b^2 < R(\dot{q}, N)\dot{q}, N > +2b < R(\dot{q}, Z)\dot{q}, N > -h^2 a^2 - h^2\|Z\|^2, \tag{21}$$

where $\hat{Z} = \frac{Z}{\|Z\|}$. Note that the last formula also holds for $Z = 0$.

Let $\tilde{Q}$ be a local extension of $\dot{q}$, such that $< \tilde{Q}, N >= 0$ and $< \tilde{Q}, \tilde{Q} >= 1$, and let $Y_1, ..., Y_{m-2}$ be a local orthonormal basis of $\Sigma \cap [\tilde{Q}]^\perp$. Then, it is not difficult to show that :

$$\nabla_{\tilde{Q}}N = h\tilde{Q} + \sum_i < \nabla_{\tilde{Q}}N, Y_i > Y_i = h\tilde{Q} + \sum_i k(\tilde{Q}, Y_i)Y_i$$

and

$$[\tilde{Q}, N] = h\tilde{Q} + \sum_i \left(k(\tilde{Q}, Y_i) - < \nabla_N \tilde{Q}, Y_i >\right) Y_i$$

which implies that:

$$< R(\dot{q}, N)\dot{q}, N > = < \nabla_N \nabla_{\tilde{Q}} N, \tilde{Q} > - < \nabla_{\tilde{Q}} \nabla_N N, \tilde{Q} > -$$

$$- < \nabla_{[N, \tilde{Q}]} N, \tilde{Q} > = h < \nabla_N \tilde{Q}, \tilde{Q} > - < \sum_i k(\tilde{Q}, Y_i) Y_i, \nabla_N \tilde{Q} > +$$

$$+ h^2 + \sum_i k(\dot{q}, Y_i) k(Y_i, \dot{q}) - \sum_i < \nabla_N \tilde{Q}, Y_i > < \nabla_{Y_i} N, \tilde{Q} > =$$

$$= h^2 - \sum_i k(\dot{q}, Y_i)^2. \tag{22}$$

In order to calculate the the third curvature term in (21) we first establish that:

$$< \nabla_{\tilde{Q}} \nabla_Z \tilde{Q}, N > = < \nabla_{\tilde{Q}} \left(\bar{P} \nabla_Z \tilde{Q} - k(Z, \tilde{Q}) N\right), N > =$$

$$= - < \bar{P} \nabla_Z \tilde{Q}, \nabla_{\tilde{Q}} N > - \frac{d}{dt}\left(k(Z, \tilde{Q})\right) =$$

$$= - \sum_i k(\dot{q}, Y_i) < \nabla_Z \tilde{Q}, Y_i > - \frac{d}{dt}\left(k(Z, \tilde{Q})\right) \;,$$

$$< \nabla_Z \nabla_{\tilde{Q}} \tilde{Q}, N > = < \nabla_Z \left(-hN + \bar{P} \nabla_{\tilde{Q}} \tilde{Q}\right), N > =$$

$$= - \sum_i < \nabla_{\dot{q}} \dot{q}, Y_i > < Y_i, \nabla_Z N > = 0 \;,$$

$$[\tilde{Q}, Z] = < \nabla_{\tilde{Q}} Z, \tilde{Q} > \tilde{Q} + \sum_i \left(< \nabla_{\tilde{Q}} Z, Y_i > - < \nabla_Z \tilde{Q}, Y_i >\right) Y_i -$$

$$-2k(\dot{q}, Z)N$$

and

$$< \nabla_{[\tilde{Q}, Z]} \tilde{Q}, N > = \sum_i k(Y_i, \dot{q}) \left(< \nabla_Z \tilde{Q}, Y_i > - < \nabla_{\tilde{Q}} Z, Y_i >\right) \;.$$

The computations above imply:

$$< R(\dot{q}, Z)\dot{q}, N >= \frac{d}{dt}\left(k(\dot{q}, Z)\right) - \sum_i k(\dot{q}, Y_i) < \nabla_{\dot{q}} Z, Y_i > . \tag{23}$$

With the substitution of equations (22) and (23) into equation (21) and, then, using equations (20) and (15), it is possible to prove that:

Proposition 2 *Let Σ be an orientable codimension 1 constant umbilical distribution, with conservation of volume, and let $Z = \bar{P}_{(q,\dot{q})}J$ be the projection onto $\Sigma_q M \cap [\dot{q}]^\perp$ of a Jacobi field J along a trajectory $(q, \dot{q})$. Then one has*

$$\frac{1}{2}\frac{d^2}{dt^2}\|Z\|^2 = \|\nabla_{\dot{q}} Z\|^2 + \|Z\|^2(-K(\dot{q}, \hat{Z}) - h^2 - 4k^2(\dot{q}, \hat{Z})), \tag{24}$$

where $\hat{Z} = \frac{Z}{\|Z\|}$.

4 Main Results

The proof of Theorem 1 has two distinct parts. The first one is to show the existence of $\mathcal{Z}_{(q,\dot{q})}$ and its properties. The second will be the construction of the subspaces $\mathcal{X}_{(q,\dot{q})}$ and $\mathcal{Y}_{(q,\dot{q})}$ as in the Anosov's proof for the geodesic flow on a Riemannian compact manifold of strictly negative sectional curvature. In the geodesic flow case it is well known that $J(t) = a\dot{q}(t)$ is a particular Jacobi field ; but, for a Σ-geodesic flow, there will be the following particular Jacobi fields:

Proposition 3 *Under the same hypotheses of Proposition 2, for any given trajectory there are Jacobi fields such that $Z \equiv 0$. They are linear combinations of:*

$$(J_1, \nabla_{\dot{q}} J_1) = (\dot{q}, -hN), \tag{25}$$

$$(J_2, \nabla_{\dot{q}} J_2) = (-ht\dot{q} + N, h^2 tN + \bar{P}\nabla_{\dot{q}} N). \tag{26}$$

Moreover, if it is given an initial condition of the form

$$a(\dot{q}(0), -hN(q(0))) + b(N(q(0)), \bar{P}(\nabla_{\dot{q}} N)(q(0))), \tag{27}$$

the corresponding Jacobi field will be given by $aJ_1 + bJ_2$.

Proof: We have to show that the vector field $J = a\dot{q} + bN$, with $\dot{b} = 2k^a(\dot{q}, Z) = 0$ and $\dot{a} = -bh = constant$, satisfies (13). Note first that:

$$\nabla_{\dot{q}} J = -ahN + b\sum_i k(\dot{q}, Y_i)Y_i$$

It is not difficult to see that the left side of equation (13) is given by:

$$\nabla_{\dot{q}}\nabla_{\dot{q}}J = -ah^2\dot{q}+$$

$$+\sum_i \left(-ahk(\dot{q},Y_i) + b\frac{d}{dt}k(\dot{q},Y_i) + b\sum_j \frac{d}{dt}k(\dot{q},Y_j) < \nabla_{\dot{q}}Y_j, Y_i >\right) Y_i +$$

$$+\left(bh^2 - b\sum_i k^2(\dot{Q},Y_i)\right) N = (I) \ . \tag{28}$$

On the other hand, from equations (22) and (23) one obtains:

$$R(\dot{q}, a\dot{q}+bN)\dot{q} = b\left(\left(h^2 - \sum_i k^2(\dot{q},Y_i)\right) N + \right.$$

$$\left. +\sum_i \left(\frac{d}{dt}k(\dot{q},Y_i) - \sum_j k(\dot{q},Y_j) < \nabla_{\dot{q}}Y_i, Y_j >\right) Y_i\right) = (II) \ . \tag{29}$$

To conclude the proof, we just observe that:

$$h\nabla_J N = ha\left(h\dot{q} + \sum_i k(\dot{q},Y_i)Y_i\right) = (III) \ . \tag{30}$$

These particular solutions will be called **special** Jacobi fields.

If we consider the subspace $\mathcal{W}_{(q,\dot{q})} = i^{-1}_{(q,\dot{q})}[(N, \bar{P}\nabla_{\dot{q}}N)]$, by Proposition 3 it will be a subspace of $\mathcal{Z}_{(q,\dot{q})}$.

A remarkable fact that occours under our hypotheses is that:

Proposition 4 *The equation for the Z component of a Jacobi field is given by:*

$$\nabla_{\dot{q}}\nabla_{\dot{q}}Z = R(\dot{q},Z)\dot{q} + h\nabla_Z N -$$

$$-4k^a(\dot{q},Z)\left(\sum_i k(\dot{q},Y_i)Y_i + \frac{h}{2}\dot{q}\right) - 2\frac{d}{dt}\left(k^a(\dot{q},Z)\right)N \ , \tag{31}$$

that is, the equation decouples from the components in N and $\dot{q}$.

Proof: Using the notation defined in equations (28) , (29) and (30), we get:

$$\nabla_{\dot{q}}\nabla_{\dot{q}}J = (I) + 4k^a(\dot{q},Z)\left(\sum_i k(\dot{q},Y_i)Y_i + \frac{h}{2}\dot{q}\right) +$$

$$+2\frac{d}{dt}\left(k^a(\dot{q},Z)\right)N+\nabla_{\dot{q}}\nabla_{\dot{q}}Z\ ,$$
$$R(\dot{q},J)\dot{q}=(II)+R(\dot{q},Z)\dot{q},$$

and

$$h\nabla_J N=(III)+h\nabla_Z N\ ;$$

taking into account equation (13), the proof is complete.

As a corollary of proposition 4, we see that equation (31) defines a linear flow F^t , $t\in \mathbb{R}$, on a vector bundle over $\Sigma_1 M$ with $(2m-4)$-dimensional fibers $V_{(q,\dot{q})}$ given by:

$$V_{(q,\dot{q})}=\{(\xi,\eta)\ s.t.\ \xi\in\Sigma_{(q,\dot{q})}\cap[\dot{q}]^{\perp},\eta\in[\dot{q}]^{\perp},<\eta,N>=k(\dot{q},\xi)\}$$
$$\subset T_qM\times T_qM$$

and

$$F^t_{(q,\dot{q})(0)}(\xi,\eta)=\Big(Z(t),\Big(\nabla_{\dot{q}}Z+2k^a(\dot{q},Z)N\Big)(t)\Big)\equiv(Z(t),\delta Z(t))\ ,$$

where $(\xi,\eta)\in V_{(q,\dot{q})(0)}$ and $Z(t)$ is the solution of equation (31), with initial condition

$$\Big(Z(0),\Big(\nabla_{\dot{q}}Z+2k^a(\dot{q},Z)N\Big)(0)\Big)=(Z(0),\delta Z(0))=(\xi,\eta).$$

Now we will restrict ourselves to a distribution Σ such that all the sectional curvatures on planes of Σ are sufficiently negative, that is, such that:

$$-K(X,Y)>h^2+4k^2(X,Y), \tag{32}$$

for any orthonormal vectors X,Y on $\Sigma_{1q}M$ and $q\in M$. Note that, since $\Sigma_1 M$ is a compact manifold, then there exists a positive constant ν such that:

$$-K(\dot{q},\hat{Z})-h^2-4k^2(\dot{q},\hat{Z})>\nu^2. \tag{33}$$

Suppose that for some time t_0 we have that the second member of equation (24) vanishes. This implies that $\|\nabla_{\dot{q}}Z(t_0)\|^2=0$ and that $Z(t_0)=0$ .Then, equations (15) and (17) imply that for $t=t_0$ we have an initial condition of the form (27), and then $Z\equiv 0$, that is, J is a special Jacobi field. In this case, considering (24) we have proved that:

Lemma 8 *Under the above condition (32) and the hypoteses of Proposition 2, if J is not a special Jacobi field, then $\|Z\|^2$ is a convex function, that is:*

$$\frac{d^2}{dt^2}\|Z\|^2>0. \tag{34}$$

Definition 6 *Let*

$$\mathcal{X}_{(q,\dot{q})} = \{A \in T_{(q,\dot{q})}(\Sigma_1 M) \ s.\ t.\ \|DT^t A\| \to 0,\ for\ t \to \infty\}, \tag{35}$$

$$\mathcal{Y}_{(q,\dot{q})} = \{A \in T_{(q,\dot{q})}(\Sigma_1 M) \ s.\ t.\ \|DT^t A\| \to 0,\ for\ t \to -\infty\}. \tag{36}$$

Note that , by definition, we have :

$$(DT^t)\mathcal{X}_{(q,\dot{q})} = \mathcal{X}_{T^t(q,\dot{q})} \ \text{ and } \ (DT^t)\mathcal{Y}_{(q,\dot{q})} = \mathcal{Y}_{T^t(q,\dot{q})}, \tag{37}$$

for any $t \in R$ and $(q, \dot{q}) \in \Sigma_1 M$.

Moreover, the special Jacobi fields do not belong to these subspaces. Since $A \in \mathcal{X}_{(q,\dot{q})}$ implies that $\|Z\| \to 0$ as $t \to \infty$ and $A \in \mathcal{Y}_{(q,\dot{q})}$ implies that $\|Z\| \to 0$ as $t \to -\infty$,the convexity of $\|Z\|^2$ assures that $\mathcal{X}_{(q,\dot{q})} \cap \mathcal{Y}_{(q,\dot{q})} = \{0\}$.

We are now able to prove the basic facts for the proof of Theorem 1, namely:

Lemma 9 *Given a vector $\xi \in \Sigma_{1q} M \cap [\dot{q}]^\perp$ and a fixed time s ,there exists a Jacobi field such that $Z(0) = \xi$ and $Z(s) = 0$. Moreover, two such Jacobi fields differ by a special Jacobi field.*

Proof: The subspace of vectors $(0, \eta) \in V_{(q.\dot{q})(0)}$ has dimension $m - 2$. The component $Z(t) \equiv P_1 F^t_{(q,\dot{q})(0)}(0, \eta)$ corresponds to a linear transformation (for any fixed t), so

$$m - 2 = dim\ Ker\left(P_1 F^t(q, \dot{q})(0)\right) + dim\ Im\left(P_1 F^t(q, \dot{q})(0)\right).$$

But, from lemma 8, $dim\ Ker\left(P_1 F^t(q, \dot{q})(0)\right) = 0$, so the map is onto $\Sigma_{q(t)} M \cap [\dot{q}(t)]^\perp$. Reversing time and taking into account translation invariance, the lemma is proved.

Definition 7 *Let us introduce the subspaces :*

$$\bar{X}(q, \dot{q}) = \{(\xi, \eta) \in V_{(q,\dot{q})} \ s.t. \ \| F^t(\xi, \eta) \| \to 0 \ s.t. \ \to \infty\},$$

$$\bar{Y}(q, \dot{q}) = \{(\xi, \eta) \in V_{(q,\dot{q})} \ s.t. \ \| F^t(\xi, \eta) \| \to 0 \ s.t. \ \to -\infty\}.$$

We will show , in the sequel , that $\bar{X}(q, \dot{q})$ and $\bar{Y}(q, \dot{q})$ are indeed $(m - 2)$-dimensional subspaces of $V_{(q,\dot{q})}$.

Lemma 10 *There exists a $(m - 2)$-dimensional subspace of $V_{(q,\dot{q})(0)}$ such that $Z(t) = P_1 F^t_{(q,\dot{q})(0)}(\xi, \eta) \to 0$ as $t \to \infty$, for all (ξ, η) in that subspace.*

Proof: For a fixed $\xi \in \Sigma_{q(0)}M \cap [\dot{q}(0)]^{\perp}$, consider the solution of (31) given by lemma 9, satisfying $Z_n(0) = \xi$ and $Z_n(n) = 0$. Let η_n be the vector

$$\eta_n = \delta Z_n(0) = \left(\nabla_{\dot{q}} Z_n + 2k^a(\dot{q}, Z_n)\right)(0).$$

Suppose, by contradiction, that there exists a subsequence $n_i \to \infty$ such that $\| \eta_{n_i} \| \to \infty$. Let us consider $Y_i = \frac{Z_{n_i}}{\|\eta_{n_i}\|}$. By convexity (lemma 8) and the definitions of Z_n and Y_i , $\| Y_i \|^2$ is decreasing for $t \in [0, n_i]$, with $Y_i(0) = \frac{\xi}{\eta_{n_i}}$ and $\delta Y_i(0) = \frac{\eta_{n_i}}{\|\eta_{n_i}\|}$. Since $\| (Y_i(0), \delta Y_i(0)) \|$ is bounded, there is a subsequence $i_l \to \infty$, such that $(Y_{i_l}(0), \delta Y_{i_l}(0)) \to (0, \eta)$ as $l \to \infty$, with $\| \eta \| = 1$. We remark that, in fact, $(0, \eta) \in V_{(q,\dot{q})(0)}$.

On the other hand, the limit solution $Y(t)$ (that is, the solution of(31) with initial conditions $(0, \eta)$) is , in any compact time interval $[0, T]$, a uniform limit of the functions Y_{i_l} such that $\| Y_{i_l} \|$ are decreasing functions (we consider only the i_l such that $n_{i_l} > T$), so $\| Y(t) \|$ is also a decreasing function. But $\| Y(0) \| = 0$ and $\| \delta Y(0) \| \neq 0$ that is $Y(t)$ is not identically zero , which is a contradiction.

Thus, there exist a constant A such that $\| \eta_n \| < A$, $\forall n \in \mathbb{N}$, and a subsequence n_i such that $(\xi, \eta_{n_i}) \to (\xi, \eta) \in V_{(q,\dot{q})(0)}$ as $n_i \to \infty$. The corresponding function $Z(t)$ is such that its norm is decreasing on any compact interval $[0, T]$, so will be decreasing for all positive t . If $\lim_{t\to\infty} \| Z(t) \|^2 = a^2 > 0$, then, by equations (24) and (33) we have $\frac{d^2\|Z(t)\|^2}{dt^2} \geq \nu a^2$, $\forall t \geq 0$, which is a contradiction , because this would imply $\| Z(t) \|^2$ unbounded. Since the construction above is valid for arbitrary $\xi \in \Sigma_{q(0)}M \cap [\dot{q}(0)]^{\perp}$, the lemma is proved.

Lemma 11 $dim\bar{X}_{(q,\dot{q})} = m - 2$.

Proof: Since $\delta Z(t) = \left(\nabla_{\dot{q}} Z + 2k^a(\dot{q}, Z)N\right)(t)$,it remains only to prove that a solution of equation (31) satisfying $\| Z(t) \| \to 0$ as $t \to \infty$, verifies also $\| \nabla_{\dot{q}} Z(t) \| \to 0$ as $t \to \infty$. But, by equation (24), it will be enough to show that $\frac{d^2\|Z(t)\|^2}{dt^2} \to 0$ as $t \to \infty$. By construction, $\frac{d\|Z(t)\|^2}{dt}$ is a strictly negative increasing function , so the proof is complete.

Note that $\bar{X}_{(q,\dot{q})}$ is invariant under the flow, that is $F^t\bar{X}_{(q,\dot{q})(0)} = \bar{X}_{(q,\dot{q})(t)}$. Analogously, $dim\bar{Y}_{(q,\dot{q})} = m - 2$ and $\bar{Y}_{(q,\dot{q})}$ is also invariant under tha flow. To obtain the contracting and expanding properties we will need the next elementary lemma :

Lemma 12 *Let f be a positive smooth function defined for $t \leq 0$ satisfying*

$$\ddot{f}(t) \geq \alpha^2 f(t) \ , \ \ \forall t \leq 0, \tag{38}$$

with $f(0) = f_0 \neq 0$ and $\dot{f}(0) < 0$. Then

$$f(t) \geq \frac{f_0}{2} e^{-\alpha t} \ \ , \ \ \forall t \leq 0. \tag{39}$$

Proof: See [A].

We are now able to state that:

Lemma 13 *For any $(\xi,\eta) \in \bar{X}_{(q,\dot{q})}$, we have that $\| Z(t) \| \leq c \| (\xi,\eta) \| e^{-\nu t}$, where $\nu > 0$ is the constant in equation (33) and $c > 0$ is a constant that does not depend neither on the points in $\Sigma_1 M$ nor on the initial conditions $(\xi,\eta) \in \bar{X}_{(q,\dot{q})(0)}$.*

Proof: Taking into account equations (24) and (33) , we get, by lemma 13 with $f(t) = \| Z(t) \|^2$ and $\alpha = \nu$, that $\| F^t_{(q,\dot{q})(0)}(\xi,\eta) \| \geq \frac{\|\xi\|}{2} e^{-\nu t}$, $\forall t \leq 0$. Then, since $\bar{X}_{(q,\dot{q})}$ is invariant under the flow, one obtains ,for any $t \geq 0$, that $\| F^{-t}_{(q,\dot{q})(t)} F^t_{(q,\dot{q})(0)}(\xi,\eta) \| \geq \frac{\|Z(t)\|}{2} e^{\nu t}$, which completes the proof.

It remains to show that $\| \nabla_{\dot{q}} Z(t) \|$ is also exponentialy contracting. In order to do so we first proof that :

Lemma 14 *$\bar{X}_{(q,\dot{q})}$ depends continuously on $(q,\dot{q})$.*

Proof: Let $(q_n,\dot{q}_n) \to (q_0,\dot{q}_0)$ as $n \to \infty$ and let $x_n = (\xi_n,\eta_n) \in \bar{X}_{(q_n,\dot{q}_n)}$ be such that $x_n \to x_0 = (\xi_0,\eta_0) \in V_{(q_0,\dot{q}_0)}$ as $n \to \infty$, as elements of $TM \times_M TM$. Then, by continuity of $P_1 F^t$ and lemma 13 we get:

$$\| Z(t) \| = \| P_1 F^t_{(q_0,\dot{q}_0)}(x_0) \| = \lim_{n\to\infty} \| P_1 F^t_{(q_n,\dot{q}_n)}(x_n) \| \leq$$

$$\leq \lim_{n\to\infty} c \| x_n \| e^{-\nu t} = c \| x_0 \| e^{-\nu t} .$$

By the proof of lemma 11, $x_0 \in \bar{X}_{(q_0,\dot{q}_0)}$. We showed that $\lim \bar{X}_{(q_n,\dot{q}_n)} \subset \bar{X}_{(q_0,\dot{q}_0)}$. Since $dim \bar{X}_{(q,\dot{q})} = m-2$ for any $(q,\dot{q})$, this concludes the proof.

We are now able to prove that:

Lemma 15 *There exists a constant $L > 0$, that does not depend neither on the point $(q_0,\dot{q}_0)$ nor on the initial condition in $\bar{X}_{(q_0,\dot{q}_0)}$, such that $\| \delta Z(t) \| \leq L \| Z(t) \|$, $\forall t \geq 0$, $\forall (q_0,\dot{q}_0) \in \Sigma_1 M$ and $\forall (Z(0),\delta Z(0)) \in \bar{X}_{(q_0,\dot{q}_0)}$.*

Proof: For a fixed point $(q_0,\dot{q}_0)$, we consider a basis of $\bar{X}_{(q_0,\dot{q}_0)}$ of the form (ξ_i,η_i) , $i = 1,...,m-2$, where $\xi_i \in T_{q_0}M$, $i = 1,...,m-2$, is an orthonormal basis of $\Sigma_{q_0} M \cap [\dot{q}_0]^\perp$. Since the dimension is finite , there exists a constant $L_{(q_0,\dot{q}_0)} > 0$ such that $\| \eta_i \|^2 < L^2_{(q_0,\dot{q}_0)} \| \xi_i \|^2$, $i = 1,...,m-2$; and, for an arbitrary vector $(\xi,\eta) = \sum_i a_i(\xi_i,\eta_i) \in \bar{X}_{(q_0,\dot{q}_0)}$, we also get $\| \eta \|^2 < \sum_i a_i^2 L^2_{(q_0,\dot{q}_0)} \| \xi_i \|^2 = L^2_{(q_0,\dot{q}_0)} \| \xi \|^2$, so the inequality in the lemma holds for any vector in $\bar{X}_{(q_0,\dot{q}_0)}$. By lemma 14 , it also holds in a neighbourhood of $(q_0,\dot{q}_0)$. Taking into account the compacity of $\Sigma_1 M$, the lemma is proved.

Lemma 16 *The flow F^t on $\bar{X}_{(q,\dot{q})}$ is uniformly contracting, that is ,*

$$\| F^t_{(q_0,\dot{q}_0)}(\xi,\eta) \| \leq C \| (\xi,\eta) \| e^{-\nu t} , \ \forall t \geq 0,$$

$$\| F^t_{(q_0,\dot{q}_0)}(\xi,\eta) \| \geq C' \| (\xi,\eta) \| e^{-\nu t} , \ \forall t \leq 0,$$

where the constants C and C' are positive and do not depend neither on the point $(q_0,\dot{q}_0)$ nor on the initial condition $(\xi,\eta) \in \bar{X}_{(q_0,\dot{q}_0)}$.

Proof: It is enough to observe that by lemma 15 and the proof of lemma 13, we have:

$$\| F^t_{(q_0,\dot{q}_0)}(\xi,\eta) \|^2 \geq c^2 \| \xi \|^2 e^{-2\nu t} \geq c^2 \left(\frac{\| \xi \|^2}{2} + \frac{\| \eta \|^2}{2L^2} \right) e^{-2\nu t} \geq$$

$$\geq C'^2 \| (\xi,\eta) \|^2 e^{-2\nu t} ,$$

where $C'^2 = c^2 min \left(\frac{1}{2}, \frac{1}{2L} \right)$. Now apply the same argument used in the proof of lemma 15.

Lemma 17 *The flow F^t on $\bar{Y}_{(q,\dot{q})}$ is uniformly expanding, that is ,*

$$\| F^t_{(q_0,\dot{q}_0)}(\xi,\eta) \| \geq D \| (\xi,\eta) \| e^{\nu t} , \ \forall t \geq 0,$$

$$\| F^t_{(q_0,\dot{q}_0)}(\xi,\eta) \| \leq D' \| (\xi,\eta) \| e^{\nu t} , \ \forall t \leq 0,$$

where the constants D and D' are positive and ddo not depend neither on the point $(q_0,\dot{q}_0)$ nor on the initial condition $(\xi,\eta) \in \bar{Y}_{(q_0,\dot{q}_0)}$.

Proof: Analogous to the proof of lemma 16.

To complete the proof of Theorem 1, we will show that for each element of $\bar{X}_{(q,\dot{q})}$ there is an element in $\mathcal{X}_{(q,\dot{q})}$, that takes a basis of $\bar{X}_{(q,\dot{q})}$ into a basis of $\mathcal{X}_{(q,\dot{q})}$, so that $dim\mathcal{X}_{(q,\dot{q})} = m - 2$. In order to do so , note that , by equation (19) and lemma 17:

$$|\dot{b}(t)| \leq \| 2k^a(\dot{q}, Z) \| \leq C_1 \| (\xi,\eta) \| e^{-\nu t} ,$$

for $\forall t \geq 0$ and initial condition

$$\tau = \Big(a_0\dot{q}(0) + \xi + b_0 N(q(0)), -a_0 h N(q(0)) + b_0 \bar{P} \nabla_{\dot{q}} N(0) + \eta \Big) ,$$

where $(\xi,\eta) \in \bar{X}_{(q,\dot{q})(0)}$, which implies that there exists $\lim_{t\to\infty} b(t) = b_\infty$. Consider the initial condition $\tau' = \tau - \Big(b_\infty N(q(0)), b_\infty \bar{P} \nabla_{\dot{q}} N(q(0)) \Big)$. The corresponding function $b(t)$ satisfies:

$$|b(t)| \leq \left| \int_t^\infty \dot{b}(t) dt \right| \leq C_1 \| (\xi,\eta) \| e^{-\nu t} .$$

But , by equation (18), there exists $\lim_{t\to\infty} a(t) = a_\infty$, for the function $a(t)$ corresponding to the initial condition τ' . For the initial condition $\tau'' = \tau' - (a_\infty \dot{q}(0), -a_\infty N(q(0))$, the corresponding function $a(t)$ also satisfies:

$$|a(t)| \leq \left| \int_t^\infty hb(t)dt \right| \leq hC_1 \parallel (\xi,\eta) \parallel \frac{e^{-\nu t}}{\nu} \leq C_2 \parallel (\xi,\eta) \parallel e^{-\nu t}.$$

So, $i^{-1}_{(q,\dot{q})(0)}(\tau'') \in \mathcal{X}_{(q,\dot{q})(0)}$, and

$$\parallel DT^t_{(q,\dot{q})(0)}(\tau'') \parallel \leq C_3 \parallel (\xi,\eta) \parallel e^{-\nu t} \ , \ \forall t \geq 0 \ ,$$

which implies that

$$\parallel DT^t_{(q,\dot{q})(0)}(\tau'') \parallel \leq C_4 \parallel (\xi,\eta) \parallel e^{-\nu t} \ , \ \forall t \leq 0 \ .$$

Note that if (ξ_i, η_i) , $i = 1, ..., m-2$, is a basis for $\bar{X}_{(q,\dot{q})(0)}$, the corresponding elements in $\mathcal{X}_{(q,\dot{q})(0)}$ are linearly independent, that is $dim\mathcal{X}_{(q,\dot{q})(0)} \geq m-2$.

By the same argument used in lemma 14, $\mathcal{X}_{(q,\dot{q})}$ depends continuously on $(q,\dot{q})$, and as in lemma 15, there is a constant $\tilde{L}$ such that $\parallel \tau'' \parallel \leq \tilde{L} \parallel (\xi,\eta) \parallel$, for any $(\xi,\eta) \in \bar{X}_{(q,\dot{q})(0)}$, and any point $(q,\dot{q})(0) \in \Sigma_1 M$. This shows that $\mathcal{X}_{(q,\dot{q})(0)}$ is uniformly contracting. Analogous arguments for $\mathcal{Y}$ show that $dim\mathcal{Y}_{(q,\dot{q})(0)} \geq m-2$ and that it is uniformly expanding. Since $dim\mathcal{X}_{(q,\dot{q})(0)} + dim\mathcal{Y}_{(q,\dot{q})(0)} = 2m-4$, the two dimensions are equal to $m-2$, which concludes the proof of Theorem 1.

Let us consider the special case $h = 0$. Then the field of lines defined by $\mathcal{W}_{(q,\dot{q})} = i^{-1}_{(q,\dot{q})}[(J_2, \nabla_{\dot{q}} J_2)] = i^{-1}_{(q,\dot{q})}[(N, \bar{P}\nabla_{\dot{q}} N)]$ can be spanned by a globally defined vector field W, that coincides over every trajectory with a special Jacobi field. But , by definition of Jacobi field, $X_{\Sigma_1 M}(q,\dot{q})(t,s)$ commutes with the Jacobi field $W(t,s)$ in the sense described in section 4. Since both are globally defined, they will allways commute, which implies that the Σ-geodesic flow is well defined on the quotient space $\Sigma_1 M/\mathcal{W}$. If this quotient is a regular manifold, then the Jacobi fields of the quotient flow will be the projections of the Jacobi fields on $\Sigma_1 M$. In view of Theorem 1, these arguments complete the proof of Theorem 2.

5 An Example

Let us denote by $\tilde{M}$ the Lie group $SL(2)$, set of all 2×2 real matrices with determinant 1. Then, its Lie algebra is given by:

$$\mathcal{G} = \{A \in \mathcal{M}(2\times 2)/trA = 0\} =$$

$$= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]. \tag{40}$$

Let X, Y, N be the left invariant vector fields on $\tilde{M}$ generated by these three matrices, respectively.

Then:

$$[X, N] = -2Y, \tag{41}$$

$$[Y, N] = 2X, \tag{42}$$

$$[X, Y] = -2N. \tag{43}$$

Consider on the group $\tilde{M}$ the left invariant metric defined by :

$$< X, X >=< Y, Y >=< N, N >= 1$$

and

$$< X, Y >=< X, N >=< Y, N >= 0.$$

Let $\Sigma_q \tilde{M}$ be the subspace spanned by $X(q)$, $Y(q)$, $\forall q \in \tilde{M}$. Then, it is clear from (43), that Σ is a non-integrable distribution. Using the expression for the Levi-Civita connection, it easy to show that $\nabla_N N = 0$ that is, the volume is conserved.

Note now that, if $S \in \Sigma_q \tilde{M}$ with $\|S\| = 1$, then $S = aX(q) + bY(q)$, with $a^2 + b^2 = 1$ and $[N, \tilde{S}](q) = 2(-bX(q) + aY(q))$ where $\tilde{S}$ is the left invariant extension of S. Then

$$< B(S, S), N(q) >= - < S, [N, \tilde{S}](q) >= -2(-ab + ab) = 0,$$

that is,

$$B^s(X, Y) = 0.$$

This means that Σ is constant umbilical. In fact it is totally geodesic.

Note that if $\dot{q} \in \Sigma_{q1} \tilde{M}$, $\dot{q} = aX + bY$, then a basis for $\Sigma_{q1} \tilde{M} \cap [\dot{q}]^{\perp}$ is $z = -bX + aY$ and :

$$< B(\dot{q}, z), N >=< B(X, Y), N >= -1 \ , \ \forall \dot{q} \in \Sigma_{q1} \tilde{M} \ , \ \forall \ q \in \tilde{M}.$$

After some computations, it is possible to show that

$$-K(\Sigma)(q) = -K(X, Y)(q) = +7 \geq \ 0 + 4 \times 1,$$

that is, the sectional curvature satisfies the condition imposed in the statement of Theorem 1.

Using equation (31) with $Z(t) = f(t)z(t)$ one obtains $f'' = 4f$. From equations (18) and (19) we get $a' = 0$ and $b' = -2f$, respectively. Then it is not difficult to show that the variational fields that generate the subspaces $\mathcal{X}_{(q,\dot{q})}$, $\mathcal{Y}_{(q,\dot{q})}$, $\mathcal{Z}_{(q,\dot{q})}$ and $\mathcal{W}_{(q,\dot{q})}$ are given respectively by:

$$i_{(q,\dot{q})}(A_1(t)) = e^{-2t}(z+N, -z+N)(q(t)),$$

$$i_{(q,\dot{q})}(A_2(t)) = e^{2t}(z-N, 3z-N)(q(t)),$$

$$i_{(q,\dot{q})}(A_3(t)) = (\dot{q}, 0)(q(t)),$$

$$i_{(q,\dot{q})}(A_4(t)) = (N, -z)(q(t)).$$

Let $\mathcal{D}$ be a uniform subgroup of $SL(2)$, that is, a discrete subgroup such that $M = SL(2)/\mathcal{D}$ is compact (they do exist, see [Bo]). The Σ-geodesic flow is well defined on $\Sigma_1 M$, and all the other hypoteses of Theorem 2, including compacity, are already verified.

Explicit computations show that the integrables curves of the Jacobi field $A_4 = W$ are all periodic orbits of the same period. In this case, the quotient space $\Sigma_1 M/[A_4]$ is a C^∞ regular manifold.

Then, by Theorem 2, the Σ-geodesic flow is defined in this last quotient manifold, so, we have an Anosov flow induced by a Σ-geodesic flow, where Σ is a non-integrable distribution.

As a final observation, it is interesting to see that it occurs a left action of the compact group $SO(2)$ on $SL(2)$, and, moreover, $SO(2)$ leaves invariant the metric and the distribution. Then, $SO(2)$ provides a momentum map for the Σ-geodesic flow. So, the final reduced Anosov system can be identified with the geodesic flow of a compact surface of negative curvature; that compact manifold is diffeomorphic to the quotient $SO(2)\backslash SL(2)/\mathcal{D}$.

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