Resenhas IME-USP 2000, Vol. 4, No. 3, 327 - 361

Deformation Quantization and Poisson Geometry

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Abstract:The theory of Deformation Quantization has experienced amazing progress in the last few years, culminating with the recent work of Kontsevich proving that every Poisson manifold admits a non-trivial deformation quantization. In this survey we briefly describe the theory of Deformation Quantization and its relation to Poisson Geometry.

Key words: Poisson manifolds, *-products, Deformation quantization.

Contents

[1]Work partially supported by FCT grant PCEX/C/MAT/44/96 and by FCT and PRAXIS XXI through the Research Units Pluriannual Funding Program

Introduction

Morally, "quantization" aims at associating to each *classical system* a *quantum system*. There are several different procedures to achieve this: geometric quantization, asymptotic quantization, deformation quantization. In this survey we will describe the latter procedure, which was born with the paper [1] and grew out of the work of Weyl, Moyal and Vey.

The basic underlying principle of any quantization scheme is that classical and quantum systems are just different realizations of the same abstract object. The two fundamental components of this object are the phase space ("space of states") and the algebra of observables ("physical observables"). In the classical counterpart these concepts are well established: the phase space is a Poisson manifold and the algebra of observables is the algebra of smooth functions on the manifold. In the quantum system these concepts vary depending on the quantization procedure. In geometric quantization the phase space is some Hilbert space naturally associated with the classical system, and the observables are a set of operators on this Hilbert space. This is a dramatic change in the nature of both space of states and observables. In deformation quantization a less radical procedure is adopted: one keeps the nature of the observables and simply deforms the algebraic structure. This is very much related with the modern point of view of non-commutative geometry, for we can think that in the classical picture we have standard, classical, geometry, while as we change scale and go to the quantum level the non-commutative geometry of nature reveals itself (the parameter measuring this non-commutativity being Planck's constant $\hbar$).

Now Poisson geometry is intimately related with deformation quantization and non-commutative geometry. As we will explain, when we start with the usual associative algebra of smooth functions on a manifold M and deform the product in a non-trivial way we obtain to first order, and up to equivalence, a Poisson bracket on M. Thus, Poisson geometry is the first order approximation to non-commutative geometry. Therefore, the language underlying the theory of deformation quantization is the language of Poisson geometry and this was my personal motivation for trying to understand it.

In this paper we will briefly describe some of the developments in deformation quantization that have occurred in the last 25 years, leading to the acknowledgement of this theory as an important branch of Mathematics, as was recognized with the recent award of the Fields medal toculminating with Kontsevich's construction of a non-trivial star product for any Poisson manifold. The paper is organized as follows: In section 1, we start by recalling the hamiltonian formalism of classical mechanics, in the framework of Poisson geometry, and then we introduce the basic notion of a *star product*. In section 2, we discuss existence of star products and explain briefly Kontsevich's approach to deformation quantization. In section 3, we give Fedosov's construction of a canonical star product for a symplectic manifold, and in the final section, we discuss G-invariant quantization and explain how "quantization commutes with reduction".

Acknowledgements

This survey grew out of a set of lectures given at the Department of Mathematics at the University of Algarve, Portugal. I would like to thank the Department, and in particular, Nenad Manojlovic for his kind hospitality.

A detail exposition of some of the ideas discussed in these notes can be found in the recent book [3]. This book also had a strong influence in my exposition and in these notes, and I thank Ana Cannas da Silva for providing a preliminary copy of these lectures.

I also would like to thank J. Stasheff for some suggestions and pointing out several misprints in a preliminary version of this paper.

1 A Star Product is Born

Since the geometry underlying classical mechanical systems is Poisson geometry, it is not surprising that developments in quantization have been influenced by, and parallel to, developments in Poisson geometry. The last 30 years or so have experienced an explosion in the study of symplectic and Poisson geometry, and so this was reflected in great advances in understanding what quantization really means. Deformation quantization is so deeply related to Poisson geometry that in fact it can be described as the study of formal deformations of Poisson structures.

In order to understand how deformation quantization naturally arises we start by recalling the Hamiltonian formalism of classical mechanics.

1.1 Mathematical Model of Classical Mechanics

From the mathematical point of view classical mechanics is the study of Hamiltonian dynamical systems. These are systems defined by certain vector fields on Poisson manifolds.

Recall that a *Poisson manifold* is a smooth manifold M with a Lie bracket $\{\cdot,\cdot\}$ on the space of smooth functions $C^\infty(M)$ such that the following derivation law, also called Leibniz identity, holds

$$\{f, f_1 \cdot f_2\} = f_1 \{f, f_2\} + \{f, f_1\} f_2, \qquad \forall f, f_1, f_2 \in C^\infty(M). \tag{1.1}$$

This identity relates the two algebraic structures on $C^\infty(M)$ namely the usual product "$\cdot$" and the Lie bracket "$\{\cdot,\cdot\}$". This leads to the abstract notion of a *Poisson algebra*, which we leave to the reader to formulate.

There is a useful alternative description of the Poisson bracket. First note that Leibniz identity (1.1) expresses the local character of the Poisson bracket, so there exists a contravariant, alternating, tensor π (for short a *bivector field*) on M such that

$$\{f_1, f_2\} = \pi(df_1, df_2).$$

Now the Jacobi identity for the Poisson bracket $\{\cdot,\cdot\}$ is expressed in terms of the Poisson bivector field by the relation

$$[\pi,\pi]_s = 0,$$

where $[\cdot,\cdot]_s$ is the *Schouten bracket* on the set of multivector fields. This is the unique super Lie algebra product on the set of multivector fields which extends the usual Lie bracket of vector fields and satisfies

$$[\theta, \eta\wedge\zeta]_s = [\theta,\eta]_s \wedge \zeta + (-1)^{(\theta+1)\eta}\eta\wedge[\theta,\zeta]_s.$$

If M is a Poisson manifold with Poisson tensor π, then in local coordinates $(x^1,\cdots,x^n)$ we have

$$\pi = \frac{1}{2}\pi^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$$

(we use the sum convention). One calls the rank of the matrix $(\pi^{ij}(x))$ the *rank* of the Poisson tensor at x. It is easy to see that this definition is independent of local coordinates. The Jacobi identity can be written in local coordinates has a first order semi-linear p.d.e.:

$$[\pi,\pi]_s = \oint_{ijk} \pi^{il}\frac{\partial \pi^{jk}}{\partial x^l} = 0,$$

where $\oint_{ijk}$ means sum over cyclic permutations of i, j and k.

A Poisson manifold with constant rank is called a *regular Poisson manifold*. Note that, on account of skew-symmetry, the rank is always an even number.

Consider a regular Poisson manifold M with rank equal to $\dim M$. Then M has even dimension, and if $(x^1,\ldots,x^n)$ are local coordinates, we have the 2-form $\omega = \omega_{ij}dx\wedge dx^j$ where we introduced the matrix $(\omega_{ij}) = (\pi^{ij})^{-1}$. It is easy to see that this 2-form is well defined (independent of the local coordinates), is non-degenerate and closed. Conversely, any *symplectic manifold*, i. e., a manifold with a closed, non-degenerate, 2-form, gives rise to a non-degenerate Poisson bracket. Accordingly, we will call a regular Poisson manifold with rank equal to $\dim M$ a symplectic manifold.

Examples 1.1.

1. Consider $M = \mathbb{R}^{2n+l}$ with coordinates $(p^1,\ldots,p^n,q^1,\ldots,q^n,c^1,\ldots,c^l)$. Then the we have a Poisson tensor given by:

$$\pi = \sum_{i=1}^{n}\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p^i}.$$

   This tensor has rank $2n$. If $l = 0$ we obtain the *canonical symplectic structure* on $\mathbb{R}^{2n}$.

2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there is a canonical Poisson structure on $M = \mathfrak{g}^*$ defined as follows. If $f : \mathfrak{g}^* \to \mathbb{R}$ is a function and $\xi \in \mathfrak{g}^*$ then $d_\xi f \in \mathfrak{g}^* \simeq \mathfrak{g}$. Therefore we can define

$$\{f_1, f_2\}(\xi) = \langle \xi, [d_\xi f_1, d_\xi f_1] \rangle .$$

Except for the abelian case, this Poisson manifold is not regular. This bracket is called the *Lie-Poisson bracket*.

3. Consider the 3-torus $\mathbb{T}^3$ with coordinates (x, y, z) (mod 1) and the 2-bivector field

$$\pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}$$

where α, β are some non-zero real parameters. Then $\mathbb{T}^3$ becomes a regular Poisson manifold.

Let (M, π) be a Poisson manifold and fix $h \in C^\infty(M)$. Then there is a well defined vector field on M, denoted X_h, and called the *Hamiltonian vector field* associated with h, which is defined by

$$X_h(f) = \{f, h\}, \forall f \in C^\infty(M).$$

One calls h a *Hamiltonian function*. The equations for the integral curves of X_h, which in local coordinates are written

$$\dot{x}^i = \{x^i, h\},$$

are called *Hamilton's equations*.

Examples 1.2.

1. Consider the Poisson manifold $\mathbb{R}^{2n+l}$ as in the example above. Then Hamilton's equations take the usual canonical form:

$$\begin{cases} \dot{q}^i = \frac{\partial h}{\partial p^i}, \\ \dot{p}^i = -\frac{\partial h}{\partial q^i}, \\ \dot{c}^i = 0. \end{cases}$$

(the c's play the role of parameters). For example, if we let $h = \sum_{i=1}^n (p^i)^2 + (q^i)^2$ we obtain the equations for n independent harmonic oscillators.

2. Consider the Lie algebra $\mathfrak{so}(3)$ of the rotation group. We can take as basis $\mathbf{v}_1 = y\partial_z - z\partial_y$, $\mathbf{v}_2 = z\partial_x - x\partial_z$, $\mathbf{v}_3 = x\partial_y - y\partial_x$, the infinitesimal rotations around the coordinates axis. Let $\{\omega_1, \omega_2, \omega_3\}$ denote the dual basis for $\mathfrak{so}(3)^*$, defining linear coordinates $\mathbf{u} = u^1\omega_1 + u^2\omega_2 + u^3\omega_3$. Then the Lie-Poisson bracket can be written has

$$\{f_1, f_2\} = -\mathbf{u} \cdot \nabla f_1 \times \nabla f_2,$$

and Hamilton's equations for a Hamiltonian $h \in C^\infty(M)$ read

$$\dot{\mathbf{u}} = \mathbf{u} \times \nabla h(\mathbf{u}).$$

If we take, for example, the Hamiltonian $h = \frac{(u^1)^2}{2I_1} + \frac{(u^2)^2}{2I_2} + \frac{(u^3)^2}{2I_3}$ we obtain Euler's equations for the motion of a rigid body with moments of inertia around the axis (I_1, I_2, I_3):

$$\begin{cases} \dot{u}^1 = \frac{I_2 - I_3}{I_2 I_3} u^2 u^3, \\ \dot{u}^2 = \frac{I_3 - I_1}{I_3 I_1} u^3 u^1, \\ \dot{u}^3 = \frac{I_1 - I_2}{I_1 I_1} u^1 u^2. \end{cases}$$

We saw above that a symplectic manifold is the same as a non-degenerate Poisson manifold. Conversely, it turns out that every Poisson manifold is foliated by symplectic manifolds.

Theorem 1.3. *Let (M, π) be a Poisson manifold. On M consider the following equivalence relation: $p \sim q$ iff there is a piece-wise smooth curve connecting p to q consisting of trajectories of Hamiltonian vector fields. Then the equivalence classes are symplectic submanifolds of M.*

Note that the foliation of M can be singular. For example, for the Lie-Poisson bracket on $\mathfrak{so}(3)^*$ the symplectic leaves are the spheres around the origin (dimension 2) and the origin (dimension 0). Even for regular Poisson manifolds the foliation might not be regular: for the Poisson structure on $\mathbb{T}^3$ above the leaves are 2-dimensional, but they are embedded 2-torii if 1, α and β are $\mathbb{Q}$-linearly dependent, and are dense in the 3-torus, otherwise.

However, locally the structure of a regular Poisson manifold is rather simple. This follows from the following "local splitting":

Theorem 1.4. *Let (M, π) be a Poisson manifold and fix $p \in M$. There are local coordinates $(p^1, \ldots, p^n, q^1, \ldots, q^n, x^1, \ldots, x^l)$ centered at p such that*

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} + \sum_{i<j} \phi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

where $\phi^{ij}(0) = 0$. Moreover, the rank of π at p equals $2n$ and this equals the dimension of the symplectic leaf through p.

Note that the theorem says that locally the Poisson manifold is the direct product of a symplectic manifold and a Poisson manifold whose rank vanishes at the point p.

For more on Poisson geometry we refer to [15].

1.2 Weyl's Quantization

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Henceforth, we denote by $\mathcal{A} = C^\infty(M)$ the Poisson algebra of smooth functions on M. According to Dirac, a *quantization* is a linear map $f \mapsto \hat{f}$ from $\mathcal{A}$ into some set of linear operators $\mathbf{B}(\mathcal{H})$ on some (pre-)Hilbert space $\mathcal{H}$, with the following properties[2]:

I) $\hat{1} = \mathrm{id}$;

II) $\widehat{\{f_1, f_2\}} = \frac{i}{\hbar}[\hat{f}_1, \hat{f}_2]$;

Property I) is a normalization condition, while property II) is often called the *correspondence principle*, which states that under quantization the Poisson bracket should go to the commutator of operators.

Let us consider the simplest case where $M = \mathbb{R}^{2n}$ with the canonical symplectic structure. We label coordinates by $(q^1, \ldots, q^n, p^1, \ldots, p^n) \equiv (q, p)$ and for the quantum space we choose $\mathcal{H} = L^2(\mathbb{R}^n, dq)$. We learn in a first course in quantum physics that

$$
\begin{aligned}
q_i &\longmapsto \hat{q}_i = \text{multiplication by } q_i \\
p_i &\longmapsto \hat{p}_i = i\hbar \frac{\partial}{\partial q_i}.
\end{aligned}
$$

The operators $\hat{q}_i$ and $\hat{p}_i$ do not commute, so when one wants to extend this rule to more general functions, say polynomials of (q_i, p_i), one has to be careful about the ordering of the variables. One possibility is to symmetrize, so for example a product $q_i p_i$ would be mapped to $\frac{1}{2}(\hat{q}_i\hat{p}_i + \hat{q}_i\hat{p}_i)$. The *Weyl quantization* extends this symmetrization for any function of the phase variables (at least in the Schwartz class). It is given by the correspondence

$$
f \longmapsto \hat{f} \equiv \frac{1}{(2\pi)^n} \int_M f^\sharp(u, v) S(u, v) du dv \tag{1.2}
$$

where $f^\sharp$ denotes the Fourier transform of f

$$
f^\sharp(u, v) = \frac{1}{(2\pi)^n} \int_M f(p, q) e^{i((u,p)+(v,q))} dp dq,
$$

and $S(u, v)$ is the family of unitary operators on $\mathcal{H}$

$$
S(u, v) \equiv e^{-i((u,P)+(v,Q))},
$$

[2] Besides this two properties one also requires that $\widehat{f^*} = \hat{f}^*$ and that quantization should not be two big, i. e., that some complete set of elements of $\mathcal{A}$ should be mapped to a complete set of operators. We will not discuss these axioms, but the reader should be aware that especially the last one is behind much of the technical problems underlying geometric quantization, such as polarizations, etc. (see [16]).

where we have set $Q = q\,\cdot$ and $P = i\hbar\partial_q$. One can check that the Weyl correspondence satisfies the normalization condition and the correspondence principle.

In 1949 Moyal interpreted the Weyl correspondence as a deformation of the algebra $\mathcal{A}$. More exactly, he computed the product $*$ on $\mathcal{A}$ such that

$$\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2,$$

and obtained the explicit formula

$$f_1 *_\hbar f_2 = \exp\left\{ \frac{i\hbar}{2} \sum_{j=1}^{n} \left(\partial_{q_1^j}\partial_{p_2^j} - \partial_{q_2^j}\partial_{p_1^j} \right) \right\} f_1(q_1,p_1) f_2(q_2,p_2) \Bigg|_{\substack{q_1=q_2=q \\ p_1=p_2=p}} \tag{1.3}$$

This formula can be written more concisely by regarding the Poisson bracket as a bi-differential operator

$$\Pi : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n})$$

such that:

$$\Pi(f_1, f_2)(x, x) = \{f_1, f_2\}(x).$$

Then setting $\Pi^0(f_1,f_2)(x,y) = f_1(x)f_2(y)$, the *Moyal-Weyl star product* is given by the exponential of π:

$$f_1 *_\hbar f_2 = \exp(\frac{i\hbar}{2}\Pi)(f_1,f_2)\Big|_{x=y}. \tag{1.4}$$

Note that by considering the power series expansion in $\hbar$ we can write this $*$-product in the form

$$f_1 *_\hbar f_2 = \sum_{l=0}^{+\infty} B_l(f_1,f_2)\hbar^l$$

and we see immediately that the coeficients B_l satisfy the following set of properties:

(i) the B_l's are bi-differential operators;

(ii) $B_0(f_1,f_2) = f_1 f_2$;

(iii) $B_1(f_1,f_2) = \frac{i\hbar}{2}\{f_1,f_2\}$;

(iv) $B_l(1,f) = B_l(f,1) = 0$, if $l \geq 1$;

(v) $B_l(f_1,f_2) = (-1)^l B_l(f_2,f_1)$;

This shows that the $*_\hbar$-product is a deformation of the associative product on $\mathcal{A}$ which at order 1 coincides with the Poisson bracket. In this setting, quantization amounts to a deformation of the point-wise commutative algebra of functions in the direction of the Poisson bracket. Moreover, in this theory of deformation quantization everything is algebraic.

1.3 The Axioms of Deformation Quantization

The $*_\hbar$-product of Weyl and Moyal captures the main features of the algebra of quantized observables $\mathbf{B}(\mathcal{H})$, so it is natural to forget about this algebra and consider instead the $*_\hbar$-product. One is led to the following definition of a star product on a manifold.:

Definition 1.5. Let M be a smooth manifold and let $\mathcal{A} = C^\infty(M)$ denote the associative algebra of smooth functions on M. A STAR-PRODUCT on $\mathcal{A}$ is a bi-linear operation $*_\hbar : \mathcal{A}[[\hbar]] \times \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ such that

(i) $*_\hbar$ is $\mathbb{R}[[\hbar]]$-linear: $\left(\sum_{l\geq 0} f_k \hbar^k\right) *_\hbar \left(\sum_{l\geq 0} g_l \hbar^l\right) = \sum_{k,l\geq 0}(f_k *_\hbar g_l)\hbar^{k+l}$;

(ii) $*_\hbar$ is associative: $(f *_\hbar g) *_\hbar h = f *_\hbar (g *_\hbar h)$;

(iii) $*_\hbar$ deforms the usual product: $f *_\hbar g = fg + O(\hbar)$;

(iv) $*_\hbar$ is local: $f *_\hbar g = \sum_{k\geq 0} B_k(f,g)\hbar^k$, where $B_k(f,g)$ are bi-differential operators.

Note that we do not require the manifold to be a Poisson manifold. The relation to Poisson brackets will be explained later.

A *morphism of star products* $D : (M, *_1) \to (M, *_2)$ is a homomorphism of $\mathbb{R}[[\hbar]]$-modules, $D : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$, of the form

$$f \longmapsto \sum_{k\geq 0} D_k(f)\hbar^k$$

where each $D_k : \mathcal{A} \to \mathcal{A}$ is a differential operator, and such that the following commutation relation holds:

$$D(f *_1 g) = D(f) *_2 D(g). \tag{1.5}$$

If $D_0 = I$, then $D = I + \sum_{k>0} D_k\hbar^k$ has a (formal) inverse. In this case we say that D is an *isomorphism of star-products*. In this case we have

$$f *_2 g = D^{-1}(D(f) *_1 D(g)).$$

The set of *automorphisms* of a star product $*_\hbar$ is a group called the *gauge group* of $*_\hbar$. The reader should be aware that one often calls an isomorphism a *change of gauge*.

The fundamental problems in the theory of deformation quantization to be discussed here are:

- Existence of star products;
- Uniqueness of star products;
- Construction of star products;

Before we turn to the discussion of these problems let us clarify the relationship between deformation of associative products, i. e., star products, and Poisson brackets.

We start with a smooth manifold M, and assume that $*_\hbar$ is a star product on $\mathcal{A} = C^\infty(M)$ which we write in the form

$$f *_\hbar g = \sum_{k \geq 0} B_k(f,g)\hbar^k.$$

We have:

Theorem 1.6. *The star product $*_\hbar$ is gauge equivalent to a star product $\tilde{*}_\hbar$ with $\tilde{B}_1(f,g)$ a Poisson bracket on M.*

We divide the proof of this theorem into a few lemmas. The reader may skip this proof since a natural cohomological interpretation of this results will be given in the next section.

Lemma 1.7. *The bi-linear map $B_1 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfies*

$$fB_1(g,h) - B_1(fg,h) + B_1(f,gh) - B_1(f,g)h = 0 \tag{1.6}$$

Proof. We compute

$$\begin{aligned}(f *_\hbar g) *_\hbar h &= fgh + (B_1(fg,h) + B_1(f,g)h)\,\hbar + O(\hbar^2),\\ f *_\hbar (g *_\hbar h) &= fgh + (B_1(f,gh) + fB_1(g,h))\,\hbar + O(\hbar^2).\end{aligned}$$

Therefore the associativity of $*_\hbar$ gives the desired relation (1.6). □

Now let us decompose B_1 into its symmetric and skew-symmetric parts:

$$B_1 = B_1^+ + B_1^-, \qquad \text{with} \qquad B_1^\pm(f,g) = \frac{B_1(f,g) \pm B_1(g,f)}{2}.$$

Lemma 1.8. *Let $D = I + D_1\hbar + O(\hbar^2)$ be a change of gauge from $*_\hbar$ to $\tilde{*}_\hbar$. Then B_1 is changed to $\tilde{B}_1$ with*

$$\tilde{B}_1(f,g) = B_1(f,g) + D_1(fg) - fD_1(g) - D_1(f)g.$$

Proof. We have $D^{-1} = I - D_1\hbar + O(\hbar^2)$, so we compute:

$$\begin{aligned}f\tilde{*}_\hbar g &= D(D^{-1}(f) *_\hbar D^{-1}(g))\\ &= D\left(fg - fD_1(g)\hbar - D_1(f)g\hbar + B_1(f,g)\hbar + O(\hbar^2)\right)\\ &= fg + (B_1(f,g) + D_1(fg) - fD_1(g) - D_1(f)g)\,\hbar + O(\hbar^2).\end{aligned}$$

Therefore, $\tilde{B}_1$ is as claimed. □

Note that if B_1 satisfies (1.6) then $B_1^{\pm}$ also satisfy (1.6). Next one shows that it is possible to kill the symmetric part:

Lemma 1.9. *Given a symmetric bidifferential operator B_1^+ satisfying (1.6) there exists a differential operator D_1 such that*

$$B_1^+(f,g) = D_1(fg) - fD_1(g) - D_1(f)g.$$

Finally we check that B_1^- is a Poisson bracket:

Lemma 1.10. *B_1^- is a Poisson bracket.*

Proof. If we cyclic permute f, g and h in (1.6) we obtain:

$$\begin{aligned}
fB_1^-(g,h) - B_1^-(fg,h) + B_1^-(f,gh) - B_1^-(f,g)h &= 0,\\
gB_1^-(h,f) - B_1^-(gh,f) + B_1^-(g,hf) - B_1^-(g,h)f &= 0,\\
hB_1^-(f,g) - B_1^-(hf,g) + B_1^-(h,fg) - B_1^-(h,f)g &= 0.
\end{aligned}$$

If we subtract the second equation from the other two equations, we obtain, using skew-symmetry:

$$B_1^-(fg,h) = fB_1^-(g,h) + B_1^-(f,h)g,$$

so B_1^- is a derivation in each entry. All that remains to show is that B_1^- satisfies the Jacobi identity. If we consider the commutator

$$[f,g] = f *_\hbar g - g *_\hbar f,$$

we compute:

$$[[f,g],h] = 4B_1^-(B_1^-(f,g),h)\hbar^2 + O(\hbar^3).$$

The associativity of $*_\hbar$ implies the Jacobi identity for the commutator. At order $\hbar^2$ we obtain the Jacobi identity for B_1^-. □

2 Existence of Star Products

In the previous section we saw that a star product is a deformation of the usual product on the algebra $\mathcal{A} = C^\infty(M)$. The natural question arises if one can always deform this product in a non-trivial way. The general study of deformation of multiplicative structures was pursued by Gerstenhaber in the 50's and 60's. For most of this section we will follow Gerstenhaber approach [8].

2.1 Formal Deformations and Hochschild Cohomology

Let V be a vector space over a field k. The candidates for products of k-factors on V are members of the set of all k-multilinear maps on V:

$$C^k(V) = \{c : V \times \cdots \times V \to V \mid c \text{ is linear in each argument}\}$$

With an eye towards studying the associative property, for $a \in C^k(V)$ and $b \in C^l(V)$ let $a \circ b \in C^{k+l-1}(V)$ be the element defined by

$$\begin{aligned} a \circ b(x_1, \dots, x_{k+l-1}) &= \\ &= \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} a(x_1, \dots, x_{i-1}, b(x_i, \dots, x_{i+l-1}), x_{i+l}, \dots, x_{k+l-1}) \end{aligned}$$

and let $[a, b] \in C^{k+l-1}(V)$ be the *Gerstenhaber bracket* defined by:

$$[a, b] \equiv a \circ b - (-1)^{(k-1)(l-1)} b \circ a.$$

Then, if we declare an element of $C^k(V)$ to have degree $k-1$, the Gerstenhaber bracket defines a structure of super Lie algebra on $C^\bullet(V) = \bigoplus_k C^k(V)$.

In the case of binary operations, i.e., elements $a, b \in C^2(V)$, we compute

$$[a, b](x, y, z) = a(b(x, y), z) - a(x, b(y, z) + b(a(x, y), z) - b(x, a(y, z)),$$

so if we set $a = b$ and write $a(x, y) = x \cdot y$ we see that:

$$[a, a](x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

Therefore we have:

Proposition 2.1. *An element $a \in C^2(V)$ defines an associative product on V iff the Gerstenhaber bracket $[a, a]$ vanishes.*

Now suppose that $\mathcal{A}$ is an associative algebra and denote by $a \in C^2(\mathcal{A})$ the product on $\mathcal{A}$ so $[a, a] = 0$. We define a map $d_a : C^\bullet(\mathcal{A}) \to C^{\bullet+1}(\mathcal{A})$ by setting $d_a(b) = [a, b]$. The super Jacobi identity gives

$$d_a^2 = 0,$$

so d_a is a coboundary map. The complex $(C^k(\mathcal{A}), d_a)$ is called the *Hochschild complex* of the associative algebra $\mathcal{A}$. The associated cohomology is called the *Hochschild cohomology* and is denoted by $\mathrm{HH}^\bullet(A)$.

Let us find out the meaning of the lowest Hochschild cohomology groups. If we write $a(x, y) = x \cdot y$ we have:

- If $b \in C^0(\mathcal{A}) = \mathcal{A}$ and $d_a(b) = [a, b] = 0$ then

  $$[a, b](x) = a(b, x) - a(x, b) = b \cdot x - x \cdot b = 0$$

  so b lies in the center of $\mathcal{A}$. Hence,

  $$\mathrm{HH}^0(\mathcal{A}) = \text{ center of } \mathcal{A}.$$

- We just saw that if $b \in C^0(\mathcal{A})$ then $d_a(b)(x) = [b, x] = \operatorname{ad} b \cdot x$ is an inner derivation of the algebra $\mathcal{A}$. On the other hand, if $b \in C^1(\mathcal{A})$ is such that $d_a(b) = 0$ we have

$$[a, b](x, y) = a(b(x), y) + a(x, b(y) - b(a(x, y)) = 0,$$

which can be written as

$$b(x \cdot y) = x \cdot b(y) + b(x) \cdot y.$$

This just means that b is a derivation of $\mathcal{A}$. Therefore,

$$\mathrm{HH}^1(\mathcal{A}) = \frac{\text{derivations of } \mathcal{A}}{\text{inner derivations of } \mathcal{A}}$$

- Let $a(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$ be a formal deformation of the associative product $a_0 = a$. According to proposition 2.1, $a(\varepsilon)$ defines an associative product iff

$$0 = [a(\varepsilon), a(\varepsilon)] = [a_0, a_0] + 2[a_0, a_1]\varepsilon + (2[a_0, a_2] + [a_1, a_1])\varepsilon^2 + \cdots \quad (2.1)$$

Since $a_0 = a$ is associative, the constant term vanishes. We say that a_1 is an *infinitesimal deformation* of a if

$$d_a(a_1) = [a, a_1] = 0,$$

and we call a_1 a *trivial infinitesimal deformation* of a if for some $b \in C^1(\mathcal{A})$ one has

$$a_1 = d_a(b) = [a, b].$$

We conclude that

$$\mathrm{HH}^2(\mathcal{A}) = \frac{\text{infinitesimal deformations of } \mathcal{A}}{\text{trivial infinitesimal deformations of } \mathcal{A}}$$

Let us consider now the higher order terms in equation (2.1). If we assume that $a = a_0$ is associative and a_1 is an infinitesimal deformation of a, in order to eliminate the ε^2-term, we need

$$2[a_0, a_2] + [a_1, a_1] = 0 \qquad \Longleftrightarrow \qquad d_a(a_2) = \frac{1}{2}[a_1, a_1].$$

Note also that, by super-Jacobi, we have always

$$d_a([a_1, a_1]) = [a, [a_1, a_1]] = 0.$$

We conclude that $[a_1, a_1]$ determines an element of $\mathrm{HH}^3(\mathcal{A})$ and that we can continue infinitesimal deformations iff this element is zero.

In general, the coefficient of the ε^n-term in equation (2.1) takes the form

$$n[a_0, a_n] + (\text{quadratic terms in the } a_i\text{'s with } i < n)$$

so the vanishing of this term is equivalent to:

$$d_a(a_n) = \text{quadratic expression in the } a_i\text{'s with } i < n$$

and we conclude that:

Proposition 2.2. *The obstructions to formal deformation of the associative product in $\mathcal{A}$ lie in $\mathrm{HH}^3(\mathcal{A})$.*

2.2 Deformations of $C^\infty(M)$.

We are interested in the case of deformations of the algebra $\mathcal{A} = C^\infty(M)$, i.e., star products. In this case we have:

- $\mathrm{HH}^0 = C^\infty(M)$ since the center is everything;
- $\mathrm{HH}^1 = \mathcal{X}(M)$ since every derivation of $C^\infty(M)$ can be identified with a vector field;

In the case of star products we are not interested in *arbitrary* deformations but rather *local* deformations where at each step the a_i's are differential operators (see section 1). Therefore we need to consider the subcomplex $C^k_{\mathrm{diff}}(\mathcal{A}) \subset C^k(\mathcal{A})$ consisting of k-multilinear maps on $\mathcal{A}$ which are differential operators on $\mathcal{A}$. Then, as before, one can construct the Hochschild cohomology $\mathrm{HH}^k_{\mathrm{diff}}(\mathcal{A})$ and one has (see [2]):

Theorem 2.3. *Let $\mathcal{A} = C^\infty(M)$. In the complex $C^k_{diff}(\mathcal{A})$ we have:*

(i) Every k-cocycle is cohomologous to a skew-symmetric cocycle;

(ii) Every skew-symmetric cocycle is given by a k-vector field;

(ii) A k-vector field is a coboundary iff it is zero.

In other words,

$$\mathrm{HH}^k_{diff}(C^\infty(M)) \simeq \Gamma(\wedge^k(M)).$$

Note that lemmas 1.8 and 1.9 are special cases of this theorem. We can also reinterpret lemmas 1.7 and 1.10 in cohomological terms: suppose that

$$f *_\hbar g = \sum_{k>0} B_k(f,g)\hbar^k$$

is a formal deformation of B_0, and B_1 is skew-symmetric (which can be achieved if B_1 is local). Then, for $*_\hbar$ to be associative, we have at the lowest orders:

- $[B_0, B_0] = 0$ iff B_0 is associative;
- $[B_0, B_1] = 0$ iff B_1 satisfies Leibniz identity;
- $2[B_0, B_2] + [B_1, B_1] = 0$ iff B_1 satisfies Jacobi identity;

As we saw above, higher orders obstructions lie in the third Hochschild cohomology group $\mathrm{HH}^3_{\mathrm{diff}}(C^\infty(M))$. One can show that, in the case of a symplectic manifold, these obstructions actually lie in the third de Rham cohomology group $H^3(M;\mathbb{C})$.

For a symplectic manifold with third Betty number $b_3 = \dim H^3(M;\mathbb{C}) = 0$ there are no obstructions to deformation quantization. It is easy to give examples (forming quocients of manifolds with $b_3 = 0$) where the obstructions to deformation quantization are only apparent. So the question naturally arose if these obstructions are always apparent. In 1983, De Wilde and Lecomte [4] proved that every sympletic manifold admits a formal deformation. More geometric proofs were later given by Karasev and Maslov[9], Omori, Maeda and Yoshioka [14], and Fedosov [5]. We will explain Fedosov's proof in the next section.

2.3 Kontsevich's Approach to Deformations of $C^\infty(M)$

The question of whether every Poisson manifold admits non-trivial formal deformation quantizations has been settled recently by Kontsevich [10]. He showed that this follows from his *formality conjecture* for which he was able to give a proof. We will now briefly describe Kontsevich's results.

Recall that a *differential graded Lie algebra* (abrev. *DGLA*), is a $\mathbb{Z}$-graded Lie superalgebra

$$L = \bigoplus_{k\geq 0} L^k$$

together with a degree one map $\mathrm{d} : L^k \to L^{k+1}$ which satisfies:

$$\mathrm{d}[a,b] = [\mathrm{d}\, a, b] + (-1)^k [a, \mathrm{d}\, b].$$

The reader should keep in mind the case $L^k = C^{k+1}_{\mathrm{diff}}(C^\infty(M))$ with the Gestenhaber bracket, the differential d being induced by standard multiplication, and the case of multivector fields $L^k = \Gamma(\wedge^{k+1}(M))$, with the Schouten bracket, and zero differential.

For a DGLA we have an action of L^0 on L^1 by setting:

$$\rho(a) : b \longmapsto [a,b] + \mathrm{d}\, a \qquad (a \in L^0, b \in L^1).$$

This action can be exponentiated to a formal action of $\exp(L^0)$ on L^1:

$$\exp(ta) : b \longmapsto \exp(t\,\mathrm{ad}\, a)(b) + \frac{I - \exp(t\,\mathrm{ad}\, a)}{\mathrm{ad}\, a}\,\mathrm{d}\, b.$$

Also, there is a quadratic map $Q : L^1 \to L^2$ given by:

$$Q(b) = \mathrm{d}\, b + \frac{1}{2}[b, b].$$

One can check that the action of $\rho(a)$ perserves $\operatorname{Ker} Q$, so the action of $\exp(L^0)$ also preserves $\operatorname{Ker} Q$.

One says that two DGLA L and L' are *quasi-isomorphic* if there exists homomorphisms of DGLA

$$L \longrightarrow L_1 \longleftarrow L_2 \longrightarrow \cdots \longleftarrow L_n \longrightarrow L'$$

which induce isomorphisms on cohomology (the directions of the arrows are not important).

Theorem 2.4. *If L and L' are quais-isomorphic DGLA's then the corresponding actions of L_0 on L_1 and of L'_0 on L'_1 are equivalent.*

A DGLA is called *formal* is it is isomorphic to the DGLA of its cohomology with the trivial differential. Kontsevich's formality conjecture states:

Conjecture 2.5. *$C^{\bullet}_{diff}(C^{\infty}(M))$ is a formal DGLA.*

A proof of this conjecture has appeared recently ([10]). A remarkable fact is that this conjecture implies the existence of deformation quantization for any Poisson manifold M.

Theorem 2.6. *There exists a non-trivial deformation quantization for every Poisson manifold.*

Sketch of the proof. Consider $L^k = \Gamma(\wedge^{k+1}(M))$, with the Schouten bracket, and zero differential. If π is the Poisson tensor on M then $\pi \in \operatorname{Ker} Q$, where $Q : L^1 \to L^2$ is the quadratic map defined above. Consider also the path $\gamma = \varepsilon\pi \in \operatorname{Ker} Q$. By the formality conjecture and theorem 2.3, L is quasi-isomorphic to $L' = C^{\bullet+1}_{\mathrm{diff}}(C^{\infty}(M))$ and we can apply theorem 2.4 to obtain a corresponding path $\gamma' \in {L'}^1$ such that

$$\gamma' \in \operatorname{Ker} Q' \Longleftrightarrow \mathrm{d}\, \gamma' + \frac{1}{2}[\gamma, \gamma'] = 0.$$

If we denote the 2-cycle corresponding to multiplication on $C^{\infty}(M)$ by $a_0 \in {L'}^1$ and use the fact that $\mathrm{d} = [a_0, \cdot]$, we obtain

$$[a_0 + \gamma', a_0 + \gamma'] = 0,$$

so $a = a_0 + \gamma'$ is a star product deforming a_0 in the direction of π. □

In fact, Kontsevich gives an explicit quasi-isomorphism between $\Gamma(\wedge^{\bullet+1}(M))$ and $C^{\bullet+1}_{\text{diff}}(C^\infty(M))$ and, as a consequence, he obtains a formula in local coordinates for the star product. Kontsevich formula is

$$f *_\hbar g = \sum_{n=0}^{\infty} \sum_{\Gamma \in G_n} \hbar^n \omega_\Gamma B_{\Gamma,\pi}(f,g),$$

where G_n is a certain class of graphs for each integer n, ω_Γ are weights obtained by integration over each graph $\Gamma \in G_n$, and $B_{\Gamma,\pi}$ are bidifferential operators given by a certain rule for each graph Γ and depending on the Poisson tensor π (see [10] for more details). Other quasi-isomorphisms give rise to other deformation quantizations.

3 Constructions of Star Products

Of the many explicit constructions of star products for symplectic manifolds discovered in the last 15 years, Fedosov's construction [5] is the most geometric, and lies at the base of much of the recent work in deformation quantization. We will now describe this construction.

3.1 Weyl Structures

Let V be a vector space with a constant (i. e., translation invariant) Poisson tensor. Then V has a natural star product, namely the Moyal-Weyl $*_\hbar$-product given by (see section 1):

$$\begin{aligned} f_1 *_\hbar f_2 &= \exp(\frac{i\hbar}{2}\Pi)(f_1, f_2)\Big|_{x=y} \\ &= \sum_{n=0}^{+\infty} \frac{1}{n}\left(\frac{i\hbar}{2}\right)^n P_n(f,g), \end{aligned} \tag{3.1}$$

where

$$P_n(f,g) = \pi^{i_1 j_1} \cdots \pi^{i_n j_n} \partial_{i_1} \cdots \partial_{i_n} f \partial_{j_1} \cdots \partial_{j_n} g.$$

The algebra $W(V) = (C^\infty(V)[[\hbar]], *_\hbar)$ is called the *Weyl-Moyal algebra* of (V, π).

For a general regular Poisson manifold (i. e., π with constant rank), around a point $x \in M$, (M, π) is locally isomorphic to the Poisson vector space $(T_x M, \pi_x)$. Therefore, any regular Poisson manifold admits locally $*_\hbar$-products. The problem is to patch together these $*_\hbar$-products in order to quantize the algebra $\mathcal{A} = C^\infty(M)$. This will be done through the use of special connections.

Definition 3.1. Let (M, π) be a Poisson manifold. A linear connection ∇ on M is called a *Poisson connection* if its torsion vanishes ($T = 0$) and

$$\nabla \pi = 0.$$

Note that the existence of a Poisson connection places restrictions on a Poison manifold: since π is preserved under parallel transport, its rank is constant, so M is a regular Poisson manifold. Conversely, any regular Poisson manifold admits a Poisson connection.

If (M, π) is a Poisson manifold with a Poisson connection ∇ we might try to define a star-product like the Moyal-Weyl product, but where we replace the usual derivatives by covariant derivatives:

$$P_n(f,g) = \pi^{i_1 j_1} \cdots \pi^{i_n j_n} \nabla_{i_1} \cdots \nabla_{i_n} f \nabla_{j_1} \cdots \nabla_{j_n} g. \tag{3.2}$$

This corresponds to the following observation. The Weyl-Moyal product on a Poisson vector space (M, π) is invariant under affine symplectic maps: if $L : V \to V$ is a linear map that preserves the Poisson bracket then $L^* : C^\infty(V)[[\hbar]] \to C^\infty(V)[[\hbar]]$ is an automorphism of $*_\hbar$. This means that the Weyl-Moyal product passes to any Poisson manifold locally modeled on V as long as only affine symplectic coordinate changes are allowed. This of course is possible if M has a flat Poisson connection without torsion. In fact, we have:

Proposition 3.2. *Let (M, π) be a Poisson manifold with a Poisson connection ∇. Then (3.1) with P_n given by (3.2) defines a star-product on M iff ∇ is flat.*

Note that the only condition that needs to be verified is that $*_\hbar$ is associative. The proposition says that this is equivalent to ∇ being flat:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0.$$

For a proof see [1].

Of course, there are many examples of Poisson manifolds that are not flat, and so this formula in general will not give a star product. We need a more sophisticated approach. Omori, Maeda and Yoshioka[14] and Fedosov [5] independently proposed the following idea: each tangent space of a regular Poisson manifold M can be considered as a Weyl algebra with the Moyal-Weyl star product, so the tangent bundle TM becomes a Poisson manifold with fiberwise Poisson bracket and which is quantizable with the fiberwise Moyal-Weyl star product. To quantize M itself we can try to look in the quantized algebra $C^\infty(TM)[[\hbar]]$ for a *subalgebra* isomorphic to the vector space $C^\infty(M)[[\hbar]]$ such that the $*_\hbar$-product induced on $C^\infty(M)[[\hbar]]$ is a deformation quantization of M. One calls such a subalgebra a *Weyl structure* on M.

Example 3.3.

As a simple example consider a Poisson vector space (V, π). If $(x^1, \ldots, x^n)$ are coordinates on V let $(y^1, \ldots, y^n)$ be the corresponding linear coordinates on $T_x V$. Then (x^i, y^i) are coordinates on TV and we have the fiberwise Poisson bracket

$$\{y^k, y^l\} = \pi^{kl}, \qquad \{x^k, x^l\} = 0, \qquad \{x^k, y^l\} = 0,$$

and the fiberwise $*_\hbar$-product

$$f(x,y,\hbar) *_\hbar g(x,y,\hbar) = \sum_{n=0}^{+\infty} \frac{1}{n}\left(\frac{i\hbar}{2}\right)^n P_n(f(x,\cdot,\hbar), g(x,\cdot,\hbar)).$$

where the P_n act on the y's. Now if we consider the class of functions $f(x,y,\hbar)$ satisfying

$$f(x+c, y-c, \hbar) = f(x,y,\hbar), \qquad c \in V,$$

we see that it is in fact a subalgebra, and the projection $C^\infty(TV)[[\hbar]] \to C^\infty(V)[[\hbar]]$

$$f(x,y,\hbar) \longmapsto f(x,0,\hbar)$$

restricts to an isomorphism on this subalgebra. Its inverse is the map

$$g(x,\hbar) \longmapsto g(x,y.\hbar) \equiv g(x+y,\hbar).$$

Therefore this class of functions gives a Weyl structure on M.

In the next paragraphs, following Fedosov, we will show how to construct a Weyl structure on any symplectic manifold.

3.2 Formal Weyl Algebras

Let (M,π) be a Poisson manifold. If $x \in M$ we introduce the *formal Weyl algebra*, denoted W_x, whose elements are formal series

$$a(y) = \sum_{2k+l\geq 0} \hbar^k a_{ki_1\cdots i_l} y^{i_1}\cdots y^{i_l},$$

where $(y^1,\ldots,y^m)$ are the coordinates on T_xM determined by some set of local coordinates $(x^1,\ldots,x^m)$ on M around x, and $a_{ki_1\cdots i_l}$ are covariant tensors for each fixed value of the first index. The product on W_x is the fiberwise Weyl-Moyal $*_\hbar$-product (3.1). The *formal Weyl bundle* $W \to M$ is the bundle over M whose total space is

$$W = \bigcup_{x\in M} W_x,$$

and whose sections $C^\infty(M,W)$ when restricted to a coordinate chart are formal sums

$$a(x,y,\hbar) = \sum_{2k+l\geq 0} \hbar^k a_{ki_1\cdots i_l}(x) y^{i_1}\cdots y^{i_l},$$

with $a_{ki_1\cdots i_l}(x)$ smooth functions[3]. We can think of these sections as smooth functions on the quantized tangent bundle TM^Q.

[3]The indices mean that we assign weight 2 to the variable $\hbar$ and weight 1 to each linear function on T_xM.

The sections $C^\infty(M,W)$ of the formal Weyl bundle form an algebra for the fiberwise Weyl-Moyal $*_\hbar$-product. Its center $\mathcal{Z}$ consists of sections of the form

$$a(x,\hbar)=\sum_{2k\geq 0}\hbar^k a_k(x),$$

and hence we can identify $\mathcal{Z}\simeq C^\infty(M)[[\hbar]]$. There is also a natural Lie algebra structure on the formal Weyl bundle, namely the "quantum commutator":

$$[a,b]=\frac{1}{i\hbar}\left(a*_\hbar b-b*_\hbar a\right).$$

If we denote by W_r the ideal in W generated by terms of weight r then we have a filtration for $*_\hbar$

$$W\supset W_1\supset W_2\supset\cdots\supset W_r\supset\cdots$$

which satisfies:

$$[W_r,W_s]\subset W_{r+s-2}.$$

We are interested in picking a subalgebra of $C^\infty(M,W)$. If this subalgebra is the annihilator of some Lie algebra of derivations of W, it should correspond geometrically to some kind of foliation of the quantized tangent bundle TM^Q. This foliation is transverse to the fibers, when the derivations are of the form D_X, with $X\in\mathcal{X}(M)$, and D a connection on W. Finally, when this foliation is transverse to the zero section, parallel sections of W are in 1:1 correspondence with $C^\infty(M)[[\hbar]]$, thus giving a Weyl structure on M.

In order to study connections on the Weyl bundle (see next paragraph) we need to consider differential forms on M with values in W, i. e., sections of $\mathcal{W}\equiv W\otimes\Lambda=\oplus_{q=0}^m W\otimes\Lambda^q$. A typical section is:

$$a(x,y,dx,\hbar)=\sum\hbar^k a_{ki_1\cdots i_l j_1\cdots j_q}(x)y^{i_1}\cdots y^{i_l}\otimes dx^{j_1}\wedge\cdots\wedge dx^{j_q}.$$

The $*_\hbar$-product and quantum commutator $[\cdot,\cdot]$ extend to differential forms: we define $*_\hbar$ using Moyal on the y^i-part and $\wedge$ on the dx^j-part. For the commutator we set for $a\in W\otimes\Lambda^p$ and $b\in W\otimes\Lambda^q$

$$[a,b]=\frac{1}{i\hbar}\left(a*_\hbar b-(-1)^q b*_\hbar a\right).$$

Note that the center of $\mathcal{W}$ is $\mathcal{Z}\otimes\Lambda$ and we have projections:

$$\begin{array}{l}\mathcal{W}\ni a(x,y,dx,\hbar)\longrightarrow a_0=a(x,0,dx,\hbar)\in\mathcal{Z}\otimes\Lambda\\ \qquad\qquad\qquad\searrow\\ \qquad\qquad\qquad\qquad a_{00}=a(x,0,0,\hbar)\in\mathcal{Z}\end{array}$$

In the case where $a \in W$ we write $a_0 = a_{00} \equiv \sigma(a)$, the so called *symbol* of a.

On the space $\mathcal{W}$ of W-valued forms there is an exterior derivative type operator $\delta : W_r \otimes \Lambda^q \to W_{r-1} \otimes \Lambda^{q+1}$ defined by:

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}$$

(recall the sum convention). Note that:

$$\delta(y^i \otimes 1) = 1 \otimes dx^i, \quad \delta(\hbar \otimes 1) = 0, \quad \delta(1 \otimes dx^i) = 0,$$

so the effect of δ is replacing one by one each y^i by dx^i. The operator δ is a derivation of $\mathcal{W}$ and $\delta^2 = 0$. In the case where π is symplectic there is a nice description of δ as an inner derivation:

Lemma 3.4. *If π is symplectic with symplectic form $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$ then for all $a \in \mathcal{W}$*

$$\delta(a) = -\frac{i}{\hbar}[\omega_{ij} y^i \otimes dx^j, a] \tag{3.3}$$

Proof. If $a \in \mathcal{Z}$ note that

$$\frac{1}{i\hbar}[y^i, a] = \{y^i, a\} = \pi^{ij} \frac{\partial a}{\partial y^j},$$

so one has $\frac{\partial a}{\partial y^i} = -\frac{i}{\hbar}[\omega_{ij} y^i, a]$. Therefore we conclude that

$$\delta(a \otimes 1) = -\frac{i}{\hbar}[\omega_{ij} y^i \otimes dx^j, a \otimes 1].$$

Since δ is a derivation, (3.3) follows. □

There exists another operator which has the effect of replacing one by one each dx^j by $(-1)^j y^j$. It is defined by:

$$\delta^* a = y^k \imath_{\frac{\partial}{\partial x^k}} a.$$

For a monomial $y^{i_1} \cdots y^{i_l} \otimes dx^{j_1} \wedge \cdots \wedge dx^{j_q}$, we have the easily proved relation

$$\delta \delta^* + \delta^* \delta = (l + q) I.$$

Therefore, if we let

$$\delta^{-1} = \begin{cases} \frac{1}{l+q} \delta^* & \text{if } l + q \neq 0 \\ 0, & \text{if } l + q = 0 \end{cases}$$

we obtain a Hodge type decomposition for W-valued forms:

Proposition 3.5. *Let $a \in \mathcal{W}$. Then, setting $\mathcal{H}(a) = a_{00} \in \mathcal{Z}$, the following decomposition of a is valid*

$$a = \delta \delta^{-1} a + \delta^{-1} \delta a + \mathcal{H} a \tag{3.4}$$

Note that the operator δ^* is not a derivation.

3.3 Connections on the Weyl Bundle

We will now motivate the introduction of a certain class of connections on the Weyl bundle of a Poisson manifold (M, π) with Poisson connection ∇. The corresponding connection 1-form is denoted[4]:

$$\phi = \Gamma_{ij}^{k} E_{k}^{i} \otimes dx^{j}.$$

Note that ∇ has no torsion iff it is symmetric, i. e., $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$.

We begin by lifting the connection to the Weyl bundle. We need:

Lemma 3.6. *∇ is a symplectic connection iff Γ_{ij}^{k} is symmetric and ϕ takes values in $\mathfrak{sp}(m, \mathbb{R})$.*

Now note that we have identifications

$$\begin{array}{ccccc} \mathfrak{sp}(m,\mathbb{R}) & \simeq & \begin{array}{c}\text{linear Hamiltonian}\\ \text{vector fields}\end{array} & \simeq & \text{quadratic Hamiltonians} \\ (JA + AJ = 0) & & (a_i^j y^i \frac{\partial}{\partial y^j}) & & (H = \omega_{rj} a_i^j y^i y^r) \end{array}$$

so, under these identifications, the connection 1-form can be written as

$$\phi = \Gamma_{ijk} y^i y^j \otimes dx^k,$$

where we have set $\Gamma_{ijk} = \omega_{ir} \Gamma_{jk}^{r}$.

If we reinterpret the y^i's as formal variables on the Weyl bundle, we see that ϕ defines a connection 1-form on W. This connection on W is precisely the lift to the Weyl bundle of the given linear connection ∇ on the tangent bundle.

Example 3.7.

Consider the case $M = V$ with constant symplectic form ω. As we saw above, a Weyl structure is formed by those sections satisfying:

$$a(x + c, y - c, \hbar) = a(x, y, \hbar).$$

The flat canonical connection is symplectic and has trivial Christoffel symbols $\Gamma_{ijk} = 0$. This would give $\phi = 0$, and the sections above are not flat for this connection. In fact we have to consider instead

$$\phi = \frac{i}{\hbar} \omega_{kj} y^k \otimes dx^j.$$

To check this let a be a flat section. In Darboux coordinates we compute, using lemma 3.4,

$$D_\phi a = da + \frac{i}{\hbar}[\omega_{kj} y^k \otimes dx^j, a] = da - \delta a.$$

[4]For notations concerning connections see the appendix.

Therefore, we see that $da = \delta a$. If $\sigma(a) = a_0$ is the symbol of a, the Hodge decomposition gives

$$a = a_0 + \delta^{-1}\delta a = a_0 + \delta^{-1} da,$$

since $\delta^{-1}a = 0$. Iterating this equation gives

$$\begin{aligned} a &= a_0 + \delta^{-1} da_0 + \delta^{-1} d\delta^{-1} da_0 + \cdots \\ &= \sum_{k=0}^{+\infty} (\delta^{-1}d)^k a_0 = \sum_{k=0}^{+\infty} \frac{1}{k!} (\partial_{i_1} \cdots \partial_{i_k} a_0) y^{i_1} \cdots y^{i_k}. \end{aligned}$$

This shows that a satisfies the condition $a(x+c, y-c, \hbar) = a(x, y, \hbar)$.

This example suggests considering connections on W of the form

$$\phi_0 = \frac{i}{\hbar}\omega_{kj} y^k \otimes dx^j + \frac{i}{\hbar}\Gamma_{ijk} y^i y^j \otimes dx^k$$

where Γ_{ijk} are the Christoffel symbols of a symplectic connection on M. In order for the flat sections to form a Weyl structure, we need the connection to be abelian. Unfortunately, this is not the case for the connection ϕ_0:

Lemma 3.8. *Let (M, ω) be a symplectic manifold and let $\phi_0 = \frac{i}{\hbar}\omega_{kj} y^k \otimes dx^j + \frac{i}{\hbar}\Gamma_{ijk} y^i y^j \otimes dx^k$ be a connection 1-form on the Weyl bundle W associated with a symplectic connection ∇ on M. Then its curvature 2-form $\frac{i}{\hbar}\Omega_0$ in local Darboux coordinates is given by*

$$\Omega_0 = -\frac{1}{2}\omega_{ij} dx^i \wedge dx^j + \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l,$$

where R is the curvature tensor of the symplectic connection.

Note that if a is any section, we have:

$$D^2_{\phi_0} a = \frac{i}{\hbar}[\Omega_0, a] = \frac{i}{\hbar}[R, a],$$

so if R is not a central form, i. e., if $R \neq 0$, we do not have an abelian connection as it is required for producing a Weyl structure consisting of flat sections. One is led to consider "perturbations" of ϕ_0:

$$\phi = \phi_0 + \varepsilon = \phi_0 + \frac{i}{\hbar} r, \tag{3.5}$$

where $r \in W_3 \otimes \Lambda$. The covariant derivative for this connection 1-form is given by:

$$Da = da - \delta a + \frac{i}{\hbar}[\Gamma_{ijk} y^i y^j \otimes dx^k + r, a],$$

so it is clear that we can add to r any central factor. The *Weyl normalization condition* states that

$$r_0 = 0. \tag{3.6}$$

Using iteration, the properties of δ and δ^{-1}, and the Hodge decomposition, we obtain

Theorem 3.9. *There exists a unique 1-form $r \in W_3 \otimes \Lambda$ satisfying*

$$\delta^{-1} r = 0$$

(which implies the Weyl normalization condition (3.6)) and such that the connection D with connection 1-form (3.5) has curvature $\frac{i}{\hbar}\Omega_\phi$ where

$$\Omega_\phi = -\omega.$$

In particular D is an abelian connection.

Proof. If one calculates the curvature of a connection ϕ of type (3.5) one gets

$$\Omega_\phi = -\frac{1}{2}\omega_{ij} dx^i \wedge dx^j + R - \delta r + \nabla r + \frac{i}{\hbar} r^2.$$

Therefore, the abelian property will be fulfilled provided

$$\delta r = R + \nabla r + \frac{i}{\hbar} r^2. \tag{3.7}$$

Assume that r satisfies $\delta^{-1} r = 0$, so its Hodge decomposition is $r = \delta^{-1}\delta r$. Applying δ^{-1} to (3.7) we get

$$r = \delta^{-1} R + \delta^{-1}(\nabla r + \frac{i}{\hbar} r^2). \tag{3.8}$$

Since ∇ preserves the filtration and δ^{-1} raises it by 1, iteration of this equation gives a unique solution.

Conversely, one shows that if r is a solution of (3.8) then it satisfies $\delta^{-1} r = 0$ and (3.7) so it determines an abelian connection with curvature $\Omega_\phi = -\omega$. □

Iterating (3.8) we can construct explicitly the form r, and hence ϕ, up to any order. The first two terms are

$$r = \frac{1}{8} R_{ijkl} y^i y^j y^k \otimes dx^l + \frac{1}{20}\partial_m R_{ijkl} y^i y^j y^k y^m \otimes dx^l + \mathcal{O}(\hbar).$$

To define the Weyl structure let $W_D \subset W$ be the subspace of flat sections for $D = D_\phi$ where ϕ is as in the theorem:

$$W_D = \{a \in W : Da = 0\}.$$

We have:

Theorem 3.10. *The space W_D is a subalgebra of W. Moreover, for any $z \in \mathcal{Z}$ there exists a unique $a \in W_D$ such that $\sigma(a) = z$.*

Proof. Since δ, ∇ and $[r, \cdot]$ act as derivations of W we see that:

$$D(a *_\hbar b) = D(a) *_\hbar b + a *_\hbar D(b)$$

so W_D is a subalgebra. On the other hand, the equation $Da = 0$ can be written in the form

$$\delta a = \nabla a + \frac{i}{\hbar}[r, a] \tag{3.9}$$

If we apply to this equation δ^{-1} and use Hodge decomposition we get

$$a = a_0 + \delta^{-1}(\nabla a + \frac{i}{\hbar}[r, a]), \tag{3.10}$$

where $a_0 = \sigma(a)$. Iteration gives a unique solution, since δ^{-1} increases the filtration.

Conversely, one can show that if a is a solution of (3.10) then a is a solution of (3.9) so a is a flat section with $\sigma(a) = a_0$.

□

Again, iterating (3.10) we can construct explicitly the section a corresponding to some a_0 up to any order. The first few terms are:

$$\begin{aligned} a = a_0 + \partial_i a_0 y^i + \frac{1}{2}\partial_i\partial_j a_0 y^i y^j + \frac{1}{6}\partial_i\partial_j\partial_k a_0 y^i y^j y^k + \\ -\frac{1}{24} R_{ijkl}\omega^{lm}\partial_m a_0 y^i y^j y^k + \dots \end{aligned}$$

When $R = 0$ we recover the expression we had before (see example 3.7):

$$a = \sum_{k=0}^{+\infty} \frac{1}{k!}(\partial_{i_1} \cdots \partial_{i_k} a_0) y^{i_1} \cdots y^{i_k}.$$

Now we can construct the $*_\hbar$-product on $C^\infty(M)[[\hbar]] = \mathcal{Z}$. We simply consider the push forward by the map $\sigma : W_D \to \mathcal{Z}$ of the fiberwise Weyl-Moyal $*_\hbar$-product:

$$a *_\hbar b \equiv \sigma(\sigma^{-1}(a) *_\hbar \sigma^{-1}(b)). \tag{3.11}$$

The inverse map $Q = \sigma^{-1} : \mathcal{Z} \to W_D$ is called the *canonical quantization* map for M.

3.4 Uniqueness of Deformation Quantization

We finish this section with some brief comments on the question of uniqueness of star products.

In the previous sections we saw that for any symplectic manifold (M, π), with $\omega = \pi^{-1}$, one can construct an abelian connection on the Weyl bundle W whose curvature 2-form is $\Omega = -\omega$. In [6], Fedosov also showed that this construction can be extended so that the curvature becomes $\Omega = \sum_{j\geq 0} \hbar^j \omega_j$, for any sequence of closed 2-forms $\{\omega_j\}$ such that $\omega_0 = \omega$ is the original symplectic form. Moreover, the isomorphism classes of these generalized Fedosov star products only depend on the cohomology classes $[\omega_j] \in H^2(M, \mathbb{R})$ $(j > 0)$. Conversely, in [11, 12], Nest and Tsygan showed that any deformation of (M, π) is isomorphic to a generalized Fedosov star product.

It follows that the relevant data for characterizing (and constructing) isomorphism classes of star products are sequences

$$\omega,\ [\omega_1],\ [\omega_2], \ldots,\ [\omega_j], \ldots$$

where $[\omega_j] \in H^2(M, \mathbb{R})$ $(j > 0)$. Therefore, the moduli space of star products on a symplectic manifold is $\frac{1}{\hbar}(\omega + H^2(M)[[\hbar]])$. Nest and Tsygan have developed this result into a theory of characteristics classes for star products. They also gave a new version of the Index Theorem of Atiyah and Singer in the context of deformation quantization.

In the 60's, Moser had shown that the deformations of symplectic structures are classified by their cohomology classes. Therefore we can state:

Theorem 3.11. *For a symplectic manifold, the isomorphism classes of star products are in 1:1 correspondence with isomorphism classes of formal deformations of the symplectic structure.*

This result was extended to the Poisson case by Kontsevitch.

As a final note we remark that, up to isomorphism, there is a unique deformation quantization whose characteristic class is independent of $\hbar$, namely the one corresponding to the star product we constructed in the previous sections. This is the reason for the name "canonical quantization". However, this name is a bit misleading for there is strong evidence that other isomorphism classes of star products play a significant role.

4 G-Invariant Deformation Quantization

It is well known that symmetry plays a fundamental role in Physics. In this final section we consider deformation quantization in the presence of symmetry.

Symmetry is well understood at the classical level. There is a well define procedure (which we recall below) that, starting from a Poisson manifold on which a Lie group acts respecting the Poisson bracket, produces a reduced Poisson manifold. Moreover, if on the original phase space a hamiltonian system is given by an invariant hamiltonian function, one obtains a reduced hamiltonian system. This is the celebrated Meyer-Marsden-Weinstein reduction procedure.

The question naturally arises if there is a similar reduction procedure on the quantum level, and if "reduction" commutes with "quantization". We will see that the answer to this question in the case of deformation quantization is positive, at least in the symplectic case, as was shown recently by Fedosov [7].

4.1 Reduction of Poisson Manifolds

We shall recall the usual reduction procedure for Poisson manifolds. For a more detailed exposition in the same spirit as presented here see [13].

Recall that given a Poisson manifold M and a Lie group G acting on M, we say that it is a *Poisson action* if for each $g \in G$ the map

$$M \ni m \longmapsto g \cdot m \in M$$

is a Poisson map.

Given an action of G on M, we let $\mathfrak{g}$ be the Lie algebra of G, and for each $x \in \mathfrak{g}$ we denote by $X_x \in \mathcal{X}(M)$ the corresponding infinitesimal generator:

$$X_x(m) = \frac{d}{dt} \exp(tx) \cdot m \bigg|_{t=0}.$$

If the action is a Poisson action then each vector field X_x is a *locally hamiltonian vector field*[5].

We say that a Poisson action is a *hamiltonian action* if each infinitesimal generator X_x ($x \in \mathfrak{g}$), is a (global) hamiltonian vector field. This means that there exists a function $h_x \in C^\infty(M)$ such that

$$X_x(f) = \{f, h_x\}, \qquad f \in C^\infty(M).$$

Moreover, the family h_x ($x \in \mathfrak{g}$) satisfies

$$\{h_x, h_y\} = -h_{[x,y]}.$$

For a hamiltonian action of a Lie group G on a Poisson manifold M we have a *moment map* $P : M \to \mathfrak{g}^*$ defined by

$$\langle P(m), x\rangle = h_x(m), \qquad m \in M, x \in \mathfrak{g}.$$

Here we have denoted by $\langle \cdot, \cdot \rangle$ the natural paring between $\mathfrak{g}$ and $\mathfrak{g}^*$. The main important features of the moment map are given in the following proposition:

[5] A vector field X on a Poisson manifold (M, π) is called locally hamiltonian if $\mathcal{L}_X \pi = 0$, so its flow preserves the bracket. In this case, $X = \pi^\sharp \alpha$ for some closed 1-form α.

Proposition 4.1. *For the moment map $P : M \to \mathfrak{g}^*$ of a hamiltonian action of G on a Poisson manifold M one has:*

(i) P is a Poisson map when we consider $\mathfrak{g}^*$ as a Poisson manifold with the Lie-Poisson bracket;

(ii) P is equivariant for the coadjoint action of G on $\mathfrak{g}^*$:

$$P(g \cdot m) = \mathrm{Ad}^* g \cdot P(m), \qquad g \in G, m \in M.$$

If we start with a Poisson manifold with a Poisson action we can reduce the dimension of our phase space. In fact, we have the following important result (see [13]):

Theorem 4.2. *Let $\mu \in \mathfrak{g}^*$ be a regular value of the moment map $P : M \to \mathfrak{g}^*$ for a Poisson action of G on M. Assume that G acts regularly on M and that $G_\mu = \{g \in G : \mathrm{Ad}^* g \cdot \mu = \mu\}$ acts regularly on $P^{-1}(\mu)$. Then we have the commutative diagram:*

$$\begin{tikzcd}
& M \arrow[dr, "\phi"] & \\
P^{-1}(\mu) \arrow[ur, "i_\mu"] \arrow[dr, "\hat{\phi}"'] & & M/G \\
& P^{-1}(\mu)/G_\mu \arrow[ur, "\hat{i}_\mu"'] &
\end{tikzcd}$$

where M/G and $P^{-1}(\mu)/G_\mu$ are Poisson manifolds, and ϕ and $\hat{i}_\mu$ are Poisson maps. Moreover, if M is symplectic, $P^{-1}(\mu)/G_\mu$ is a symplectic leaf of M/G.

Classically, the name "reduction" refers to the procedure of starting with a hamiltonian system and using symmetry to reduce the number of degrees of freedom, thereby improving one's chance of integrating the system (for example, this procedure was well known to Jacobi). In the modern formulation, the precise meaning of a system with symmetry is express by the following data: (i) a hamiltonian action of a Lie group G on a Poisson manifold M and (ii) a hamiltonian function $h : M \to \mathbb{R}$ invariant under the action:

$$h(g \cdot m) = h(m), \qquad g \in G, m \in M.$$

Now, according to the theorem above, the quotient space M/G ("reduced space") is a Poisson manifold. Since h is G-invariant it factors through a function ("reduced hamiltonian") $\tilde{h} : M/G \to \mathbb{R}$. By the theorem above, the hamiltonian vector fields X_h on M and $X_{\tilde{h}}$ on M/G are ϕ-related. It follows that:

Corollary 4.3. *Trajectories of a G-invariant hamiltonian system on M are projected by ϕ onto trajectories of the reduced hamiltonian system on G/M.*

For certain classes of Lie groups actions this result gives rise to the classical theory of integrable systems.

4.2 G-Invariant Quantization

We would like to quantize the reduction procedure just described and, in particular, the diagram of theorem 4.2.

As a starting point, let us consider the corresponding diagram of associative algebras of smooth functions:

$$\begin{tikzcd}
 & C^{\infty}(M) \arrow[dl, "\iota_\mu{}^*"'] & \\
C^{\infty}(P^{-1}(\mu)) & & C^{\infty}(M/G) \arrow[ul, "\phi^*"'] \arrow[dl, "\hat{\iota}_\mu^*"] \\
 & C^{\infty}(P^{-1}(\mu)/G_\mu) \arrow[ul, "\hat{\phi}^*"] &
\end{tikzcd}$$

In this diagram, $C^\infty(M)$, $C^\infty(M/G)$ and $C^\infty(P^{-1}(\mu)/G_\mu)$ are Poisson algebras, and the maps ϕ^* and $\hat{\iota}_\mu^*$ are homomorphisms of Poisson algebras.

Suppose we have a star product on $C^\infty(M)[[\hbar]]$. Is $C^\infty(M/G)[[\hbar]]$ a subalgebra of $C^\infty(M)[[\hbar]]$, i. e., is $C^\infty(M/G)[[\hbar]]$ closed for this star product? In general, the answer is *no* for the following reason: a function on M/G can be identified with a G-invariant function on M and, in fact, what the map ϕ^* does is to realize this identification. So we can recast the question above in the following form: given G-invariant functions f_1 and f_2 is the product $f_1 *_\hbar f_2$ also G-invariant? Clearly, this will not be the case for a general star product on M.

Let $g \in G$, so we have the pull-back action of g on $C^\infty(M)$:

$$g^* f(m) \equiv f(g \cdot m), \qquad m \in M.$$

If the action of G on M is Poisson we have:

$$g^* \{f_1, f_2\} = \{g^* f_1, g^* f_2\}.$$

Now, assuming the manifold to be a symplectic manifold, consider the Weyl bundle W of M, so we have the Weyl-Moyal star product on W:

$$a *_\hbar b \equiv \sum_{n \geq o} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^2 \pi^{i_1 j_1} \ldots \pi^{i_n j_n} \partial_{i_1 \cdots i_n} a \partial_{j_1 \cdots j_n} b.$$

From this formula it is clear that the induced action of g^* on sections of W preserves the star product:

$$g^*(a *_\hbar b) = (g^* a) *_\hbar (g^* b).$$

The star product on functions was defined by (cf. 3.11)

$$f *_\hbar g = Q^{-1}(Q(f) *_\hbar Q(g))$$

so, to get a positive answer to the question posed, we would like to have:

$$Q(g^* f) = g^* Q(f),$$

where Q is the canonical quantization map for M. As the following result shows, this in fact will hold if we use a G-invariant Poisson connection.

Proposition 4.4. *Let M be a symplectic manifold and suppose given a Poisson action of a Lie group G on M. Given a G-invariant Poisson connection ∇ on M let $D = D_\phi$ be the corresponding flat connection on W defining a quantization map $Q : Z \to W_D$. Then*

$$g^* Q = Q g^*.$$

Proof. Since ∇ is G-invariant, so is D, and we find:

$$D(g^* Q(f)) = g^* DQ(f) = g^* 0 = 0.$$

since $Q(f) \in W_D$ is a flat section. On the other hand, for the symbols we also find

$$\sigma(g^* Q(f)) = g^* \sigma(Q(f)) = g^* f,$$

and by uniqueness of quantization we conclude that

$$g^* Q(f) = Q(g^* f).$$

□

Therefore, for G-invariant quantization, we can identify $C(M/G)[[\hbar]]$ with the subalgebra of $C^\infty(M)[[\hbar]]$ of G-invariant functions, and the map ϕ^* becomes a homomorphism of star algebras.

So far we have assumed that the action of G on M is Poisson. For hamiltonian actions we get:

Proposition 4.5. *Let M be a symplectic manifold with a hamiltonian action of a Lie group G on M. Then, for G-invariant quantization, we have*

$$[h_x, h_y] = h_{[x,y]}, \qquad x, y \in \mathfrak{g}$$

where on the left side $[\cdot,\cdot]$ denotes the quantum commutator

$$[a, b] \equiv \frac{1}{i\hbar}(a *_\hbar b - b *_\hbar a).$$

Proof. For any section a of the Weyl bundle W one finds:

$$\mathcal{L}_{X_x} a = i_{X_x} Da - [Q(h_x), a].$$

In particular, since $Q(a)$ is a flat section, we obtain

$$\mathcal{L}_{X_x} Q(h_y) = -[Q(h_x), Q(h_y)].$$

The action being hamiltonian, we also have

$$\{h_x, h_y\} = -h_{[x,y]} \Longleftrightarrow \mathcal{L}_{X_x} h_y = -h_{[x,y]}.$$

So quantization leads to

$$Q(\mathcal{L}_{X_x} h_y) = -\mathcal{L}_{X_x} Q(h_y) = -Q(h_{[x,y]}),$$

since we know that $g^* Q = Q g^*$. This gives

$$[Q(h_x), Q(h_y)] = Q(h_{[x,y]}),$$

which is equivalent to the the proposition. □

This proposition can also be stated in the following form. Recall that for an hamiltonian action we have the moment map $P : M \to \mathfrak{g}^*$, which is a Poisson map. If we dualize, we get a Poisson algebra map $P^* : C^\infty(\mathfrak{g}^*) \to C^\infty(M)$, which is just pull-back by P. The proposition states that the restriction of this map to $\mathfrak{g}$ is a Lie algebra homomorphism. This leads also to the following question: Is it possible to quantize $C^\infty(\mathfrak{g}^*)$ in some natural way so that P^* becomes a star algebra homomorphism?

4.3 Quantization Commutes with Reduction

Let M be a symplectic manifold with a hamiltonian action of a Lie group G, and write A for the algebra $C^\infty(M)[[\hbar]]$ with the star product obtained by canonical G-invariant quantization. Then we have the subalgebra $A_G \subset A$ which is the image of ϕ^*.

Classically, the image $\phi^*(C^\infty(M/G))$ is just the subspace of $C^\infty(M)$ consisting of G-invariant functions:

$$\phi^*(C^\infty(M/G)) = \{f \in C^\infty(M) : \{P(x), f\} = 0,\ \forall x \in \mathfrak{g}\}.$$

On the quantum side, the algebra A_G admits a similar interpretation as the subalgebra of A of G-invariant elements:

$$A_G = \{a \in : [P(x), a] = 0, \forall x \in \mathfrak{g}\}.$$

Now assume that all conditions of the classical reduction (theorem 4.2) are satisfied, and let $R = C^\infty(P^{-1}(\mu)/G_\mu)[[\hbar]]$, so the map $\iota_\mu{}^* : A_G \to R$ is surjective. There are two ways of obtaining a star product on R:

(i) Since $P^{-1}(\mu)/G_\mu$ is a symplectic manifold (cf. theorem 4.2) we can endow it with a symplectic connection obtained by reduction from the G-invariant connection on M. Then canonical quantization gives a product $*_\hbar$ on R;

(ii) Let N be the kernel of the map $\iota_\mu{}^* : A_G \to R$. Classically, the map $\iota_\mu{}^* : C^\infty(M/G) \to C^\infty(P^{-1}(\mu)/G_\mu)$ has kernel

$$\operatorname{Ker} \iota_\mu{}^* = \left\{ f \in C^\infty(M/G) : f = \sum_i b_i(P_i - \mu_i) \right\},$$

where we write $P = \sum_i P_i\omega^i$ and $\mu = \sum_i \mu_i\omega^i$ in some basis $\{\omega^1, \ldots, \omega^n\}$ of $\mathfrak{g}^*$. Now, when we turn to the quantum picture, we have

$$N = \operatorname{Ker} \iota_\mu{}^* = \left\{ a \in A_G : a = \sum_i b^i *_\hbar (P_i - \mu_i) \right\},$$

and we have:

Lemma 4.6. *N is an ideal for $*_\hbar$ in A_G and $A_G/N \simeq R$.*

Clearly, the identification $A_G/N \simeq R$ also gives a product $*_\hbar$ on R.

It turns out that these two different ways of furnishing a product on R actually coincide, as follows from the following result:

Theorem 4.7. *The map $\iota_\mu{}^* : C^\infty(M/G)[[\hbar]] \to C^\infty(P^{-1}(\mu)/G_\mu)[[\hbar]]$ is a homomorphism of $*_\hbar$-algebra.*

This theorem gives a precise meaning to the statement that "deformation quantization commutes with reduction", in the case of a symplectic manifold. Similar results hold for geometric quantization. A general result of this sort should also be true for general Poisson manifolds and deformation quantization in the spirit of Kontsevich. For a proof of theorem 4.7 we refer the reader to the recent paper [7].

Appendix.
Notations for Connections on Vector Bundles

Recall that a connection on a vector bundle $E \to M$ over M is a first order differential operator $D : C^\infty(E) \to C^\infty(E \otimes \Lambda^1)$ which is $\mathbb{R}$-linear and satisfies the Leibniz identity:

$$D(fu) = (df)u + fDu, \text{ if } u \in C^\infty(E), f \in C^\infty(M).$$

If $X \in \mathcal{X}(M)$ is a vector field and we set

$$D_X = \imath_X D$$

we obtain the usual covariant derivative operator. We can extend D to an operator $D : C^\infty(E \otimes \Lambda^q) \to C^\infty(E \otimes \Lambda^{q+1})$ by requiring that

$$D(u \otimes \alpha) = Du \otimes \alpha + (-1)^k u \otimes d\alpha,$$

for all k-forms $\alpha \in \Lambda^k(M)$. One can also extend this connection in the usual way to any bundle associated with E.

Given local coordinates $(x^1, \ldots, x^m)$ and a local frame $(e_1, \ldots, e_r)$ for E, we have

$$D(e_i) = \Gamma_i^k e_k,$$

where Γ_i^k are 1-forms defined in the coordinate domain:

$$\Gamma_i^k = \Gamma_{ij}^k dx^j.$$

Denoting by E_k^i the canonical basis for $\mathfrak{gl}(r, \mathbb{R})$ induced by the local frame, one defines the *connection 1-form*

$$\phi = \Gamma_{ij}^k E_k^i \otimes dx^j.$$

Note that ϕ is a 1-form with values in $\mathrm{Hom}(E, E)$, and that it does not depend on local coordinates.

The *curvature 2-form* of the connection is defined by refering to the following lemma:

Lemma 5.1. *For any section $a \in C^\infty(E)$ we have*

$$D^2(a) = \Omega a,$$

where $\Omega \in C^\infty(\mathrm{Hom}(E, E) \otimes \Lambda^2)$ is a certain 2-form on M, with values in the bundle $\mathrm{Hom}(E, E)$, given by[6]*:*

$$\Omega = d\phi + [\phi, \phi].$$

The 2-form Ω is called *the curvature of the connection.* A connection on E is called *flat* if its curvature vanishes: $\Omega = 0$. It is called ***abelian*** if the associated connection on $\mathrm{Hom}(E, E)$ is flat. Since for any section $u \in C^\infty(\mathrm{Hom}(E, E) \otimes \bigwedge^2)$ one has

$$D^2 u = [\Omega, u],$$

we see that for an abelian connection the curvature is $\Omega = \omega I$ with ω a scalar 2-form.

[6] The commutator of two $\mathrm{Hom}(E, E)$-valued forms ω_p and ω_q, with degree p and q, is defined by

$$\binom{p+q}{p} [\omega_p, \omega_q](\tau_1, \ldots, \tau_{p+q}) = \sum_\sigma \operatorname{sgn} \sigma [\omega_p(\tau_1, \ldots, \tau_p), \omega_p(\tau_{p+1}, \ldots, \tau_{p+q})].$$

A connection on the tangent bundle is called a *linear connetion*. These are denoted in the text by the symbol ∇, rather than D. For linear connections there is an interplay between coordinates on M and on TM. In particular one can define the *torsion tensor* to be the symmetric 2-tensor:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics **111** (1978), no. 1, 61–151.

[2] M. Cahen, S. Gutt, and M. De Wilde, *Local cohomology of the algebra of C^∞ functions on a connected manifold*, Lett. Math. Phys. **4** (1980), no. 2, 157–167.

[3] A. Cannas da Silva, K. Hartshorn, and A. Weinstein, *Lectures on geometric models for noncommutative algebras*, University of California at Berkley, 1998.

[4] M. De Wilde and P. Lecomte, *Existence of star-products and of formal deformations of the Poisson-Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. **7** (1983), no. 2, 487–496.

[5] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), no. 2, 213–238.

[6] ———, *Deformation quantization and index theory*, Mathematical Topics, vol. 9, Akademie Verlag, Berlin, 1996.

[7] ———, *Non-abelian reduction in deformation quantization*, Lett. Math. Phys. **43** (1998), no. 2, 137–154.

[8] M. Gerstenhaber, *On the deformations of rings and algebras*, Ann. of Mathematics **79** (1964), 59–103.

[9] M. Karasev and V. Maslov, *Pseudodifferential operators and the canonical operator in general symplectic manifolds*, Math. USSR Izvestia **23** (1984), 277–305.

[10] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, q-alg/9709040, 1997.

[11] R. Nest and B. Tsygan, *Algebraic index theorem*, Comm. Math. Phys. **172** (1995), no. 2, 223–262.

[12] ———, *Algebraic index theorem for families*, Adv. Math. **113** (1995), no. 2, 151–205.

[13] P. Olver, *Applications of Lie groups to differential equations*, 2 ed., GTM, vol. 107, Springer-Verlag, New-York, 1986.

[14] H. Omori, Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. Math. **85** (1991), no. 2, 224–255.

[15] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Birkhäuser, Berlin, 1994.

[16] N. Woodhouse, *Geometric quantization*, Clarendon Press, Oxford, 1980.

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