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Shod Algebras [1]

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Abstract: An algebra A is called *shod* provided for each indecomposable A-module, either its projective dimension is at most one or its injective dimension is at most one. Tilted and quasitilted algebras are examples of shod algebras. The purpose of these notes is to survey recent results on the class of shod algebras.

Key words: shod algebras, quasitilted algebras, tilted algebras, homological dimensions.

In this paper we want to survey recent results on a special class of algebras which has been much investigated lately. Covered under the general name of shod algebras, this class includes the already classical tilted algebras and the quasitilted algebras.

Before we go on into details in this discussion, let us briefly make some historical comments. We could set here the begining of our comments in the paper [4], where the so-called tilting theory has developed from. There, Bernstein-Gelfand-Ponomarev gave another proof of Gabriel's theorem which characterizes the representation-finite hereditary algebras in terms of their ordinary quivers (see section 1 for definitions). In this proof, they used the so-called Coxeter transformations in order to study the relations between the module categories over two distinct hereditary algebras whose ordinary quivers have the same underlying graph.

In [2], Auslander-Platzeck-Reiten gave a module interpretation of the relations induced by the Coxeter transformations. Roughly speaking, let H be a hereditary algebra and let S be a simple projective H-module. Consider the H-module M given by the sum of all indecomposable projective modules non-isomorphic to S plus $\tau_H^{-1}S$, where τ_H^{-1} denotes the Auslander-Reiten translation TrD. The algebra $H' = \mathrm{End}_H M$ is also hereditary and the functor $\mathrm{Hom}_H(M,-)$ induces an equivalence between the category of all H-modules which do not have S as a direct summand and its image in the category of H'-modules.

This idea was further explored in the 80's by Brenner-Butler in [5] and by Happel-Ringel in [24] in two fundamental works on tilting theory. Happel-Ringel's paper contains the formulation of tilting module and tilted algebras used nowadays.

To go on, let us now recall some notions. Let A be a finite dimensional k-algebra, denote by mod$A$ the category of the finitely generated left A-modules, and by ind$A$ the full subcategory of mod$A$ consisting of one representative of each isoclass of indecomposable module. A module $T \in$ mod$A$ is called a *tilting module* provided:

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(i) its projective dimension $\mathrm{pd}_A T$ is at most one.

(ii) T has no selfextensions, that is, $\mathrm{Ext}_A^1(T,T) = 0$.

(iii) there exists a short exact sequence $0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$, where $T', T'' \in \mathrm{add}T$, that is, the indecomposable summands of T' and of T'' are also summands of T.

The fundamental Brenner-Butler theorem alllows us to pass informations from mod$A$ to mod$B$, which has been a very powerful technique in representation theory of algebras (see Section 3). If the starting algebra A is hereditary, the algebra B constructed as above is called *tilted*. One of the striking feature of a tilted algebra B is the existence of a section Σ in mod$A$, called *complete slice*, which completely divides the category mod$B$ into two parts: for each indecomposable B-module M, either $\mathrm{Hom}_B(M,\Sigma) \neq 0$ or $\mathrm{Hom}_B(\Sigma, M) \neq 0$ (but not both if $M \notin \Sigma$). In the former case, we also have the projective dimension $\mathrm{pd}_B M \leq 1$ and, in the later case, the injective dimension $\mathrm{id}_B M \leq 1$. Also, a tilted algebra has global dimension at most two. Recall that the global dimension of an algebra is the suppremum of the projective dimensions of its modules.

Several attempts were made in order to extend these successful thecniques to other contexts but we will not have the opportunity to discuss all of them here. The interested reader will not have difficulty to find the appropiated references.

The next step in our discussion is the work of Happel-Reiten-Smalø [22], where the class of quasitilted algebras was introduced. The idea was to give a general treatment to tilting theory including not only tilted algebra but also the canonical algebras, introduced by Ringel in [34], under the same construction. They consider tilting objects in some hereditary abelian categories. A quasitilted algebra is then defined to be the endomorphism ring of such an object (see [22] for details).

In the same work, Happel-Reiten-Smalø have shown that a quasitilted algebra A can be characterized by the following properties: (QT1) gl.dim $A \leq 2$; and (QT2) for each indecomposable A-module X, either $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$. These two conditions induce the existence of a trisection in ind$A$. Let $\mathcal{L}_A$ (and $\mathcal{R}_A$) be the full subcategory of ind$A$ consisting of the modules such that its predecessor (respectively, its successors) have the projective (respectively, the injective) dimension at most one. Hence $\mathrm{ind}A = (\mathcal{L}_A \setminus \mathcal{R}_A) \vee (\mathcal{L}_A \cap \mathcal{R}_A) \vee (\mathcal{R}_A \setminus \mathcal{L}_A)$ and the morphisms goes only from left to right in this trisection. Exploring it leads to important informations on the quasitilted algebras, as we shall see.

Quasitilted algebras can also be characterized by the property: $(*)$ any path in ind$A$ from an injective module to a projective module can be refined to a path of irreducible maps and any such path is sectional (see Section 1 for definitions).

In [11], in a joint work with Lanzilotta, we have extended the class of algebras A such that $\mathrm{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$. Indeed, among the two defining properties of quasitilted algebras ((QT1) and (QT2)), the one restricting the global dimension (QT1) does not play any role for the existence of the above trisection. So, in [11], we introduce the notion of shod algebras as being algebras satisfying the property

(QT2). The word shod stands for small homological dimension. Observe that the global dimension of a shod algebra is at most three (see 2.2 bellow).

As we will see along these notes, the following conditions are equivalent for an algebra A:

(a) A is shod.

(b) $\text{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$.

(c) any path in ind$A$ from an injective module to a projective module can be refined to a path of irreducible maps and any such path has at most two hooks, and, in case there are two, they are consecutive (see Section 1 for definitions).

The main purpose of this survey will be to show how these properties are related, and how they can be used to get informations on the class of shod algebras or, particularly, on the Auslander-Reiten quivers of these algebras. We shall mostly discuss the results in our joint works with Lanzilotta [11, 12, 13], with Lanzilotta and Savioli [14], with Savioli [17], with Happel and Unger [10], with Martins and de la Peña [15, 16]. We shall also discuss Reiten-Skowroński's work on shod algebras [33]. We will not provide proofs for all the results. However, in order to show the techniques involved, we shall sketch some of them. We refer the reader to the above papers for the proofs of the results discussed.

This paper is organized in the following way. Section 1 is devoted to some preliminary results that will be used along the paper. In Section 2, we show some basic properties on shod algebras and give some examples of shod algebras of global dimension three (which we will call *strict shod algebras*). Section 3 contains a brief discussion on quasitilted and tilted algebras.

When studying the Auslander-Reiten quiver of a strict shod algebra, we will see that there exists a component which plays a special role dividing the category ind$A$ into two parts. The components on these two parts, we will see, are components of tilted algebras and they are well-understood by now. We shall discuss the properties of this special component in Sections 4 and 5. Also, in 4, we shall relate shod algebras and properties on paths from injective to projective modules.

Another important technique used in our study of shod algebras is the one-point extension. We will see in Section 6 that a strict shod algebra can be written as an iteration of one-point extensions starting at tilted algebras. Sections 7 and 8 are devoted to discuss the consequences of the main result of Section 6. In Section 9, we discuss when an one-point extension of an algebra is shod (or quasitilted). The relations between shod algebras and some class of algebras of global dimension two is the subject of Section 10. The last section is devoted to the work of Reiten and Skowroński on shod [33].

We hope that, at the end of these notes, the reader will have a good idea of the shod algebras and how some techniques are used to get informations on them. We shall mention, however, that this survey is far from being complete in many aspects. We have choosen to discuss mostly one direction of investigation, that is, the one exploring more closely the Auslander-Reiten quiver of a shod algebra, indeed of a strict shod algebra. There are nice works on quasitilted algebras, for instance, which explore more directly the original definition envolving abelian categories (see [23, 21] just to mention two of them). Also, we will not discuss the derived equivalence which plays an essential role in the understanding of tilted and quasitilted algebras (see [19, 22] for instance). The interested reader will not have difficult to get the appropriated references to go further in all the related questions.

One last word. Having in mind some properties discussed here on the Auslander-Reiten quiver, as the existence of bounds on the lengths of paths from injectives to projectives, we can generalize some of the results of these notes. We have done so in a joint work with Lanzilotta [13] where we introduce the so-called *weakly shod algebras.* We will not discuss them here and refer to [13] for details.

1. Preliminaries

Along these notes, k will denote a fixed field. All algebras will be finite dimensional (associative) k-algebras. In fact, some of the results presented here will hold in a more general setting, that is, will hold for Artin algebras (an Artin algebra is a algebra which is finitely generated as a module over its center). For simplicity, we shall, however, restrict here the discussion for finite dimensional k-algebras.

To a given finite quiver Δ, one can assign naturally a k-algebra. Since our examples will be mostly done in this way, we shall now recall this construction. A *quiver* is just an oriented graph. More formally, a quiver Δ is given by two sets Δ_0 whose elements are called *vertices* and Δ_1 whose elements are called *arrows.* To each arrow α it is assigned two vertices: its *start point* $s(\alpha)$ and its *end point* $e(\alpha)$.

The *path algebra* $k\Delta$ *of a quiver* Δ is defined as follows. As a k-vector space, $k\Delta$ has a basis given by all possible paths in Δ, including the *paths of length zero,* associated to each vertex of Δ. The multiplication in $k\Delta$ is defined as the concatenation of paths whenever it makes sense and zero otherwise and then extended linearly. The algebra defined in this way is associative, has unit (just the sum of all paths of length zero). It is finite dimensional if and only if Δ is a finite quiver without oriented cycles.

The next result due to Gabriel justify the importance of studying the quotients of path algebras. Recall that an ideal $I \triangleleft k\Delta$ is *admissible* if there exists an $n \geq 2$ such that $J^n \subset I \subset J^2$, where J denotes the ideal of $k\Delta$ generated by the arrows.

THEOREM 1.1 *Let A be a finite dimensional basic k-algebra, where k is an algebraically closed field. Then there exists a finite quiver Δ_A and an admissible ideal $I \triangleleft k\Delta_A$ such that $A \cong k\Delta_A/I$.*

The quiver Δ_A of the above theorem is called the *ordinary quiver of* A.

For an algebra A, denote by $\text{mod}A$ the category of finitely generated left A-modules and by $\text{ind}A$ its full subcategory consisting of one copy of each isoclass of indecomposable A-modules. Also, denote by Γ_A the *Auslander-Reiten quiver of* A, that is, the quiver where the vertices are associated to the objects of $\text{ind}A$ and the arrows *represent* the so-called *irreducible morphisms of* $\text{ind}A$. By τ_A denote the Auslander-Reiten translation DTr and by τ_A^{-1} its inverse. For unexplained notions on representation theory, we refer the reader to [3].

For $X, Y \in \text{ind}A$, denote by $\text{rad}_A(X,Y)$ the set of the morphisms $f: X \longrightarrow Y$ which are not isomorphisms and by $\text{rad}_A^\infty(X,Y)$ the intersection of all powers $\text{rad}_A^i(X,Y)$, $i \geq 1$, of $\text{rad}_A(X,Y)$. We indicate by $\text{rad}^\infty(\text{mod}A)$ the ideal in $\text{mod}A$ generated by all morphisms in $\text{rad}_A^\infty(X,Y)$ for some $X, Y \in \text{ind}A$. Recall that a component Γ of Γ_A is *generalized standard* provided $\text{rad}^\infty(X,Y) = 0$ for each $X, Y \in \Gamma$. By [39] a generalized standard component has only finitely many τ_A-orbits.

Given $X, Y \in \text{ind}A$, we denote by $X \rightsquigarrow Y$ in case there exists a path

$$(*) \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = Y$$

$(t \geq 0)$, from X to Y in $\text{ind}A$, that is, with $f_1, \cdots, f_t$ being non-zero non-isomorphisms and $X_0, X_1, \cdots, X_t$ being indecomposable modules. By convention, there always exists a path $X \rightsquigarrow X$, for $X \in \text{ind}A$. When all morphisms f_i's in the path $(*)$ are irreducibles, then we say that $(*)$ is a *path of irreducible morphisms* or, simply, a *path in* Γ_A. A path in Γ_A starting and ending at the same module is called an *oriented cycle*.

A *hook* in $(*)$ is a j, $1 \leq j \leq t-1$, such that $X_{j-1} \xrightarrow{f_j} X_j \xrightarrow{f_{j+1}} X_{j+1}$ satisfies: (i) f_j and f_{j+1} are irreducible maps; and (ii) $\tau_A X_{j+1} = X_{j-1}$. A path of irreducible maps without hooks is called a *sectional path*. We recall two results which will be important in our considerations. The first result is due to Igusa-Todorov, while the second is due to Bautista-Smalø.

THEOREM 1.2 [29](13.4) *Let* $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t$ *be a sectional path in* Γ_A. *Then the composition* $f_t \cdots f_1$ *is non-zero.*

THEOREM 1.3 [6] *Let* $X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = X$ *be an oriented cycle in* Γ_A. *Then* $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t \longrightarrow X_1$ *is not sectional.*

A module $M \in \text{mod}A$ is called *directing* provided there exist no paths $M' \rightsquigarrow \tau_A X$ and $X \rightsquigarrow M''$ where $X \in \text{ind}A$ is not a projective module and M', M'' are indecomposable summands of M. Observe that an indecomposable module is directing if and only if it does not belong to an oriented cycle.

For a module $X \in \text{mod}A$, we shall denote by $\text{pd}_A X$ and by $\text{id}_A X$ the projective and the injective dimensions of X, respectively. We will need the following result whose proof can be found in [3](IV.1.16). Recall that D denotes the usual duality $\text{Hom}_k(-, k)$.

THEOREM 1.4 *Let A be an algebra and X be an indecomposable module. Then*

(a) *$pd_A X \leq 1$ if and only if $Hom_A(D(_AA), \tau_A X) = 0$.*

(b) *$id_A X \leq 1$ if and only if $Hom_A(\tau_A^{-1} X, A) = 0$.*

Also, gl.dim A will denote the *global dimension* of A, that is, the suppremum of the projective dimensions of the A-modules. In the study of quasitilted or shod algebras the following two subcategories of ind$A$ play an important role. Denote

$$\mathcal{L}_A = \{X \in \text{ind}A\text{: if there is a path } Y \rightsquigarrow X, \text{ then } \text{pd}_A Y \leq 1\}$$

$$\mathcal{R}_A = \{X \in \text{ind}A\text{: if there is a path } X \rightsquigarrow Y, \text{ then } \text{id}_A Y \leq 1\}$$

REMARKS 1.5 Let A be an algebra. Then

(a) $\text{pd}_A X \leq 1$ for each $X \in \mathcal{L}_A$.

(b) $\text{id}_A X \leq 1$ for each $X \in \mathcal{R}_A$.

(c) $\mathcal{L}_A$ is closed under predecessors and so $\text{Hom}_A(\mathcal{R}_A \setminus \mathcal{L}_A, \mathcal{L}_A) = 0$.

(d) $\mathcal{R}_A$ is closed under successors and so $\text{Hom}_A(\mathcal{L}_A \setminus \mathcal{R}_A, \mathcal{R}_A) = 0$.

2. Shod algebras

In this section we will discuss some preliminary results concerning the class of shod algebras introduced in [11]. We start with a definition.

DEFINITION 2.1 An algebra A is called *shod* (for small **ho**mological **d**imensions) provided for each indecomposable A-module X, $\text{pd}_A X \leq 1$ or $\text{id}_A X \leq 1$.

PROPOSITION 2.2 [22] *A shod algebra A has global dimension at most three.*

Proof. Let $X \in \text{ind}A$ and assume $\text{pd}_A X \geq 4$. Let

$$P_4 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{f_1} P_0 \longrightarrow X \longrightarrow 0$$

be the start of the minimal projective resolution of X. Denote by $K = \ker f_1$. Clearly, $\text{pd}_A K \geq 2$ and so there is an indecomposable summand Y of K such that $\text{pd}_A Y \geq 2$. It also follows from the above sequence that $\text{Ext}_A^2(X, Y) \neq 0$ and so $\text{id}_A Y \geq 2$, a contradiction to the fact that A is shod. ◇

A shod algebra of global dimension three will be called *strict shod.*

Hereditary algebras are clearly shod since all their modules have projective dimension at most one. As we shall see in the next section, tilted and quasitilted algebras are also examples of shod algebras, all of them of global dimensions at most two. We shall now give some examples of strict shod algebras.

EXAMPLES 2.3 (a) Let A be the k-algebra given by the quiver:

$$\begin{tikzcd}
& & \bullet \arrow[dl, "\beta_1"'] & & \\
\bullet & \bullet \arrow[l, "\alpha"'] & & \bullet \arrow[ul, "\gamma_1"'] \arrow[dl, "\gamma_2"] & \bullet \arrow[l, "\delta"'] \\
& & \bullet \arrow[ul, "\beta_2"] & &
\end{tikzcd}$$

with relations $\alpha\beta_2 = 0$, $\beta_1\gamma_1 = \beta_2\gamma_2$ and $\gamma_1\delta = 0$. It is not difficult to check that A is indeed a strict shod algebra.

(b) The radical square zero k-algebra given by the quiver

$$\begin{tikzcd}
\overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l, shift left] \arrow[l, shift right] & \overset{3}{\bullet} \arrow[l] & \overset{4}{\bullet} \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

is a representation-infinite strict shod algebra.

One of the most important feature for a shod algebra A is the existence of a trisection, induced by the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$, in the category ind$A$. We shall only sketch the proof of this result here, refering the reader to [11, 22], where the details can be found. We observe that the existence of such trisection was first established for quasitilted algebras in [22] and later extended to shod algebras in [11]. We shall need the following lemma.

LEMMA 2.4 [11](1.1,1.2) *Let A be a shod algebra and let $Y \in indA$ such that $pd_A Y \geq 2$.*

(a) If there exists a path $Y \rightsquigarrow X$ in $indA$, then there exists $Z \in indA$, $pd_A Z \geq 2$, and $Hom_A(Z, X) \neq 0$.

(b) If $U \in indA$ satisfies $id_A U \geq 2$, then $Hom_A(Y, U) = 0$.

THEOREM 2.5 [11] *An algebra A is shod if and only if $\mathcal{L}_A \cup \mathcal{R}_A = indA$. Moreover, in this case, both (add $\mathcal{R}_A$, add $(\mathcal{L}_A \setminus \mathcal{R}_A)$) and (add $(\mathcal{R}_A \setminus \mathcal{L}_A)$, add$\mathcal{L}_A$) are split torsion pairs in mod$A$.*

Proof. Clearly, if ind$A = \mathcal{L}_A \cup \mathcal{R}_A$, then each $X \in$ ind$A$ satisfies pd$_A X \leq 1$ or id$_A X \leq 1$ and A is shod.
Assume now that A is shod and let $X \in$ ind$A$. If $X \notin \mathcal{L}_A$, then there exists a path $Y \rightsquigarrow X$ with pd$_A Y \geq 2$. We shall show that $Y \in \mathcal{R}_A$ which will imply that $X \in \mathcal{R}_A$ because $\mathcal{R}_A$ is closed under successors. For this purpose, let $Y \rightsquigarrow X'$

be a path in ind$A$. By Lemma 2.4(a), there exists $Z \in \operatorname{ind} A$ with $\operatorname{pd}_A Z \geq 2$ and $\operatorname{Hom}_A(Z, X') \neq 0$. By Lemma 2.4(b), it yields that $\operatorname{id}_A X' \leq 1$ and so $Y \in \mathcal{R}_A$. This proves the first part of the statement.
Now, $\operatorname{Hom}_A(\mathcal{R}_A \setminus \mathcal{L}_A,\ \mathcal{L}_A) = 0$ by 1.5. If $\operatorname{Hom}_A(\mathcal{R}_A \setminus \mathcal{L}_A, M) = 0$, with $M \in \operatorname{ind} A$, then $M \notin \mathcal{R}_A \setminus \mathcal{L}_A$ and so $M \in \mathcal{L}_A$. Dually, if $\operatorname{Hom}_A(N, \mathcal{L}_A) = 0$, with $N \in \operatorname{ind} A$, then $N \notin \mathcal{L}_A$ and so $N \in \mathcal{R}_A \setminus \mathcal{L}_A$. This proves in fact that (add $(\mathcal{R}_A \setminus \mathcal{L}_A)$, add $\mathcal{L}_A$) is a torsion pair in mod$A$, which clearly splits because $\mathcal{L}_A \cup \mathcal{R}_A = \operatorname{ind} A$. Similarly, one can show that (add $(\mathcal{R}_A \setminus \mathcal{L}_A)$, add$\mathcal{L}_A$) is a split torsion pair in mod$A$. $\diamondsuit$

It follows from the above considerations that

$$\operatorname{ind} A = (\mathcal{L}_A \setminus \mathcal{R}_A) \vee (\mathcal{L}_A \cap \mathcal{R}_A) \vee (\mathcal{R}_A \setminus \mathcal{L}_A)$$

where the non-zero morphisms of ind$A$ goes only from left to right, that is,

$$\operatorname{Hom}_A(\mathcal{R}_A, \mathcal{L}_A \setminus \mathcal{R}_A) = 0 = \operatorname{Hom}_A(\mathcal{R}_A \setminus \mathcal{L}_A, \mathcal{L}_A \cap \mathcal{R}_A).$$

We would like to stress at this point the importance of the existence of this trisection in ind$A$. This *division* of the category will induce most of the time a clear division also in the Auslander-Reiten quiver Γ_A of A, allowing one to give a complete description of it (see Section 7).

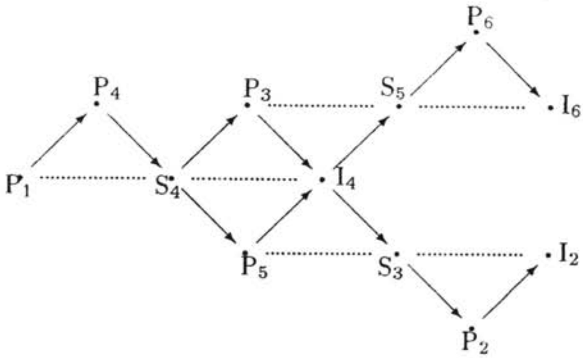
One could wonder how large is the intersection $\mathcal{L}_A \cap \mathcal{R}_A$. In one extreme, A is hereditary if and only if $\mathcal{L}_A \cap \mathcal{R}_A = \operatorname{ind} A$. The next example shows, however, that this intersection can be indeed empty for a strict shod algebra.

EXAMPLE 2.6 Let A be the k-algebra given by the quiver:

$$\begin{tikzcd}
 & & \overset{1}{\bullet} & & \\
\underset{2}{\bullet} \arrow[r, "\alpha"] & \underset{3}{\bullet} \arrow[r, "\beta"] & \underset{4}{\bullet} \arrow[u, "\gamma"'] & \underset{5}{\bullet} \arrow[l, "\delta"'] & \underset{6}{\bullet} \arrow[l, "\epsilon"']
\end{tikzcd}$$

with $\beta\alpha = \delta\epsilon = \gamma\beta = \gamma\delta = 0$

The Auslander-Reiten quiver of A has the following shape:



The dotted lines, indicate, as usual, the Auslander-Reiten translation. Observe that $\mathcal{L}_A = \{P_1, P_4, S_4, P_3, P_5\}$ and $\mathcal{R}_A = \{I_4, S_5, P_6, I_6, S_3, P_2, I_2\}$. Clearly, $\mathcal{L}_A \cap \mathcal{R}_A$ is empty and $\mathcal{L}_A \cup \mathcal{R}_A = \mathrm{ind}A$. Also, gl.dim$A = 3$.

3. Quasitilted algebras

In this section we shall discuss very briefly the class of quasitilted algebras which contains the tilted algebras and the canonical algebras. We start recalling some notions on tilting theory.

Let A be an algebra and $T \in \mathrm{mod}A$ be a tilting module, that is, a module satisfying the following properties:

(T1) $\mathrm{pd}_A T \leq 1$.

(T2) $\mathrm{Ext}^1_A(T,T) = 0$.

(T3) there exists a short exact sequence $0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$, where $T', T'' \in \mathrm{add}T$.

The next result, due to Brenner and Butler is essential, in tilting theory since it relates two module categories allowing the transfer of informations from one to the other. For a proof, we refer to one of the following papers [1, 5, 24].

THEOREM 3.1 [5] *Let A be an algebra, T_A be a tilting module, and $B = End_A T$. Then*

(i) *${}_BT$ is a tilting B-module and $A \cong End_B T$.*

(ii) *The categories*

$$\mathcal{T}(T) = \{M : Ext^1_A(T,M) = 0\} \ \ and \ \ \mathcal{Y}(T) = \{N : Tor_1^B(N,T) = 0\}$$

are equivalent, the equivalence being given by the functor $Hom_A(T,-)$ and its inverse $- \otimes_B T$.

(iii) *The categories*

$$\mathcal{F}(T) = \{M : Hom_A(T, M) = 0\} \quad and \quad \mathcal{X}(T) = \{N : N \otimes_B T = 0\}$$

are equivalent, the equivalence being given by the functor $Ext^1_A(T, -)$ *and its inverse* $Tor^B_1(-, T)$.

For further details in tilting theory, we refer the reader to [1, 5, 7, 24].

Let now $H = k\Delta$ be a hereditary algebra, where Δ is a finite quiver without oriented cycles, and $T \in \operatorname{mod}H$ be a tilting module. Since $\operatorname{gl.dim}H \leq 1$, then the condition (T1) above is naturally satisfied. In this case, the algebra $B = \operatorname{End}_H T$ is called a *tilted algebra of type* Δ.

The class of tilted algebras has been much investigated since its introduction in [24]. On one hand, the class of hereditary algebras is by now well-understood and so, it is possible to get much information on the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$. On the other hand, a tilting module over an hereditary algebra H is splitting, that is, the induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ splits, and so each indecomposable B-module (where $B = \operatorname{End}_H T$) lies either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$. Therefore, each $X \in \operatorname{ind}B$ is either of the form $\operatorname{Hom}_H(T, M)$ for some $M \in \mathcal{T}(T)$ or of the form $\operatorname{Ext}^1_H(T, N)$ for some $N \in \mathcal{F}(T)$.

Using homological techniques, one can show that $\operatorname{pd}_B M \leq 1$ for each $M \in \mathcal{X}(T)$ and $\operatorname{id}_B N \leq 1$ for each $N \in \mathcal{Y}(T)$. Also, $\operatorname{gl.dim}B \leq 2$. In resume, the following holds for a tilted algebra (recall that $(\mathcal{X}(T), \mathcal{Y}(T))$ splits).

Proposition 3.2 [24] *Let B be a tilted algebra. Then $gl.dimB \leq 2$ and for each indecomposable module X, $pd_B X \leq 1$ or $id_B X \leq 1$.*

By now, the Auslander-Reiten quiver of a tilted algebra is well-understood. Let B be a tilted algebra of type Δ. Then Γ_B has a special component Γ containing a so-called *complete slice* Σ. The underlying graph of a complete slice is the same of Δ. Such a component Γ is called *connecting component* and plays an important role in the study of Γ_B. A connecting component is generalized standard and has no oriented cycles. There are at most two connecting components in Γ_B and, if two, they are a postprojective and a preinjective component and, in this case, the algebra is called *concealed.* The complete Auslander-Reiten quiver of a tilted algebra has been described in [31]. The components of Γ_B are of the following form:

(i) postprojective component(s).

(ii) preinjective component(s).

(iii) a connecting component (if neither postprojective nor preinjective).

(iv) stable tubes.

(v) components of type $\mathbf{Z}\mathbf{A}_\infty$.

(vi) components constructed from tubes or from $\mathbf{ZA}_\infty$ by ray or coray insertions.

As mentioned in the introduction, quasitilted algebras arise as endomorphim algebras of tilting objects in hereditary abelian categories [22]. We are, however, more interested in an equivalent definition, also given in [22].

DEFINITION 3.3 An algebra A is called *quasitilted* provided it satisfies the following two properties: (QT1) gl.dim $A \leq 2$; and (QT2) for each $X \in \operatorname{ind}A$, $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$.

Clearly, quasitilted algebras are shod. By 3.2, tilted algebras are quasitilted. Another important class of quasitilted algebras is given by the canonical algebras introduced by Ringel [34]. Recently, Happel [21] has characterized the hereditary categories with tilting objects and, as a consequence, it follows that a quasitilted algebra is derived equivalent either to a tilted algebra or to a canonical algebra.

It is also worth mentioning that Skowroński has characterized the tame quasitilted algebras. He shows, for instance, that a quasitilted algebra A is tame if and only if A is either a tame tilted or a tame semiregular branch enlargement of a tame concealed algebra (see [40] for details).

We finish this section recalling the following result from [8].

THEOREM 3.4 [8] *Let A be a quasitilted algebra and let Γ be a component of Γ_A consisting of directing modules. Then Γ is postprojective, preinjective or connecting.*

4. Paths from injective modules to projective modules

We have seen in 1.4 that there exists a strong relation between the existence of non-zero morphisms from injective modules to the translation $\tau_A X$ of a module $X \in \operatorname{ind}A$ and its projective dimension. More especifically, $\operatorname{Hom}_A(\mathrm{D}({}_AA), \tau_A X) = 0$ if and only if $\operatorname{pd}_A X \leq 1$. Dually, one has $\operatorname{Hom}_A(\tau_A^{-1} X, A) = 0$ if and only if $\operatorname{id}_A X \leq 1$.

If now A is a shod algebra, then there is no indecomposable modules X with both $\operatorname{pd}_A X \geq 2$ and $\operatorname{id}_A X \geq 2$ and so no modules $X \in \operatorname{ind}A$ has simultaneously $\operatorname{Hom}_A(\mathrm{D}({}_AA), \tau_A X) \neq 0$ and $\operatorname{Hom}_A(\tau_A^{-1} X, A) \neq 0$. Intuitively, there is no much *room* for modules which are successors of injective modules and predecessors of projective modules. In this section, we want to explore this idea. The results here have been mostly proven in [11].

Let $X, Y \in \operatorname{ind}A$ and let

$$(*) \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = Y$$

($t \geq 0$) be a path in $\operatorname{ind}A$. A *refinement* of $(*)$ is a path

$$X = Z_0 \xrightarrow{g_1} Z_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{u-1}} Z_{u-1} \xrightarrow{g_u} Z_u = Y$$

in ind$A$ from X to Y such that there exists an order-preserving function σ from $\{1, \cdots, t-1\}$ to $\{1, \cdots, u-1\}$ such that $X_i \cong Z_{\sigma(i)}$ for each $1 \leq i \leq t-1$.

PROPOSITION 4.1 *Let A be a shod algebra, and suppose there exists a path*

$$I = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots X_{t-1} \xrightarrow{f_t} X_t = P \tag{$*$}$$

in ind$A$, where I and P are, respectively, an injective module and a projective module.

(a) *If for each $1 \leq i \leq t-1$, X_i is neither projective nor injective, then $t \leq 2(n+1)$, where n stands for the number of simple modules.*

(b) [11](1.3) *None of the f_i's lie in rad^∞(mod$A$). In particular, $(*)$ has a refinement of irreducible maps.*

Proof. (a) Suppose $t > 2n+2$. Then there exists a refinement of $(*)$ as follows:

$$I = Y_0 \xrightarrow{g_1} Y_1 \longrightarrow \cdots Y_{m-1} \xrightarrow{g_m} Y_m \overset{(\xi)}{\rightsquigarrow} Z_l \xrightarrow{h_l} Z_{l-1} \longrightarrow \cdots Z_1 \xrightarrow{h_1} Z_0 = P$$

where $g_1, \cdots, g_m, h_1, \cdots, h_l$ are irreducible morphisms, (ξ) is a path in ind$A$, and at least $n+1$ modules among $Y_0, \cdots, Y_m$ are non-injectives and at least $n+1$ modules among $Z_0, \cdots, Z_l$ are non-projectives.

Claim. $Y_m \notin \mathcal{L}_A$.
Let $(**) : I = Y_0 \xrightarrow{g_1} Y_1 \longrightarrow \cdots Y_{m-1} \xrightarrow{g_m} Y_m$ be the begining of the above refinement, and suppose first that it is not sectional. Then there exists $j < m$ such that $\tau_A Y_{j+1} \cong Y_{j-1}$. Choose it minimal. By 1.2, $\mathrm{Hom}_A(I, Y_{j-1}) \neq 0$ and so, by 1.4, $\mathrm{pd}_A Y_{j+1} \geq 2$ implying the claim. Suppose now that $(**)$ is sectional. Hence $\mathrm{Hom}_A(I, Y_i) \neq 0$, for each i (1.2) and so, by 1.4, $\mathrm{pd}_A \tau_A^{-1} Y_i \geq 2$, whenever Y_i is not injective. If $Y_i \cong Y_j$ for distinct i and j, then one can easily construct a non-sectional path $I \rightsquigarrow Y_m$ (using 1.3) and, as above, $Y_m \notin \mathcal{L}_A$. So, we can assume that $(**)$ pass through at least $n+1$ non-isomorphic non-injective modules. By [38], there exist $1 \leq p, q \leq n$ such that $\mathrm{Hom}_A(\tau_A^{-1} Y_p, Y_q) \neq 0$ and so $Y_q \notin \mathcal{L}_A$ because $\mathrm{pd}_A \tau_A^{-1} Y_p \geq 2$. The claim is now proven because $\mathcal{L}_A$ is closed under predecessors.
Similarly, one can show that $Z_l \notin \mathcal{R}_A$ and so $Y_m \notin (\mathcal{L}_A \cup \mathcal{R}_A)$, a contradiction to the fact that A is shod.

(b) Suppose $f_l \in \mathrm{rad}^\infty(\mathrm{mod}A)$ for some l. Then, by [37], for each $r \geq 0$, there exists a path

$$X_{l-1} = U_0 \xrightarrow{g_1} U_1 \longrightarrow \cdots U_{r-1} \xrightarrow{g_r} U_r \xrightarrow{h_r} X_l$$

where $g_1, \cdots, g_r$ are irreducible maps, $h_r \in \mathrm{rad}^\infty(\mathrm{mod}A)$ and $h_r g_r \cdots g_1 \neq 0$. Choosing conveniently r, one gets at least $2n+3$ non-isomorphic modules among $U_0, \cdots, U_r$ which are neither projectives nor injectives. This leads to a contradiction to (a). ◇

THEOREM 4.2 [11] *An algebra A is shod if and only if any path in ind$A$ from an injective module to a projective module can be refined to a path of irreducible maps and any such refinement has at most two hooks, and in case there are two, they are consecutive.*

Proof. Assume that A is shod and suppose there exists a path $I \rightsquigarrow P$ in ind$A$ where I and P are, respectively, an injective module and a projective module. Using the 4.1, it yields that there exists a path of irreducible maps

$$(*) \qquad I = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots X_{t-1} \xrightarrow{f_t} X_t = P$$

Assume that $(*)$ has at least two hooks. Hence, there are j and l such that $\tau_A^{-1}X_j = X_{j+2}$, $\tau_A X_l = X_{l-2}$ and the paths $I \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{j+1}$ and $X_{l-1} \longrightarrow \cdots \longrightarrow P$ are sectionals. Since there are at least two hooks, we infer that $j+1 < l-1$. By 1.2, $\mathrm{Hom}_A(I, \tau_A X_{j+2}) \neq 0$ and $\mathrm{Hom}_A(\tau_A^{-1}X_{l-2}, P) \neq 0$. Hence $\mathrm{pd}_A X_{j+2} \geq 2$ and $\mathrm{id}_A X_{l-2} \geq 2$. If now $j+2 < l-1$, we get a path $X_{j+2} \rightsquigarrow X_{l-2}$, a contradiction to Lemma 2.4. So $j+2 = l-1$ and in this case the path $(*)$ has only two hooks and they are consecutives.
Suppose now that A is not shod. Then there exists an indecomposable module M with $\mathrm{pd}_A M \geq 2$ and $\mathrm{id}_A M \geq 2$. Hence $\mathrm{Hom}_A(\mathrm{D}(A), \tau_A M) \neq 0$ and $\mathrm{Hom}_A(\tau_A^{-1}M, A) \neq 0$, and so there exists a path in ind$A$

$$I \xrightarrow{f_1} \tau_A M \xrightarrow{f_2} E \xrightarrow{f_3} M \xrightarrow{f_4} F \xrightarrow{f_5} \tau_A^{-1}M \xrightarrow{f_6} P \qquad (*)$$

where I is an indecomposable injective module, P is an indecomposable projective module, and f_i is irreducible for $i = 2, 3, 4, 5$. By 4.1, $(*)$ can be refined to a path of irreducible maps which clearly contains two non-consecutive hooks. $\diamondsuit$

COROLLARY 4.3 *Let A be a shod algebra. Then there exists a positive integer n_0 such that any path in ind$A$ from an injective module to a projective module has length at most n_0.*

For our next result, recall that for a given non-projective $X \in \mathrm{ind}A$, $\alpha(X)$ denotes the number of indecomposable modules in the middle term of an Auslander-Reiten sequence ending at X.

PROPOSITION 4.4 *Let A be a strict shod algebra and suppose there exists a non-sectional path $(*)$ in Γ_A from an injective I to a projective P. Then*

(a) [11](2.3) *there exists a path from I to P in Γ_A with just one hook.*

(b) *there exists a projective module P' in the path $(*)$ having a submodule X with $pd_A X \geq 2$.*

Proof. (a) Let

$$(*) \qquad I = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = P$$

be a path of irreducible maps in ind$A$ from I to P with two (consecutive) hooks. Then there exists an i such that $\tau_A^{-1}X_{i-1} = X_{i+1}$ and $\tau_A^{-1}X_i = X_{i+2}$. If $\alpha(X_{i+1}) = 1$ and $\alpha(X_{i+2}) = 1$, then

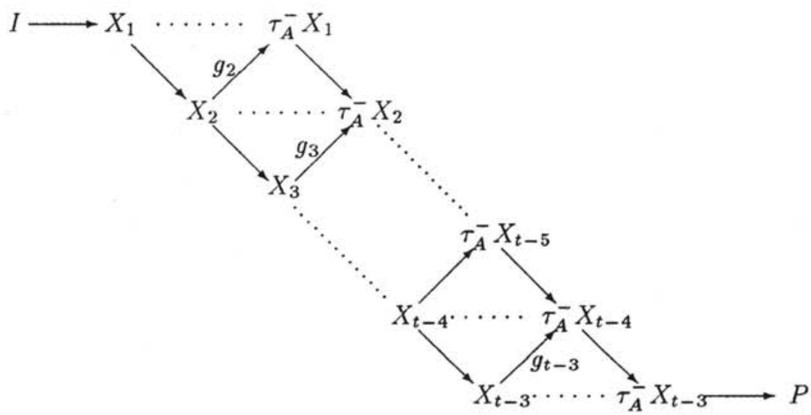
$$0 \longrightarrow X_{i-1} \longrightarrow X_i \xrightarrow{f_{i+1}} X_{i+1} \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow X_i \xrightarrow{f_{i+1}} X_{i+1} \longrightarrow X_{i+2} \longrightarrow 0$$

are Auslander-Reiten sequences and so f_{i+1} is an isomorphism, a contradiction. So, either $\alpha(X_{i+1}) > 1$ or $\alpha(X_{i+2}) > 1$. Also, if X_{i+3} is not projective, then $\alpha(X_{i+2}) \geq 2$ because X_{i+1} and $\tau_A X_{i+3}$ are non-isomorphic summands of the middle term of the Auslander-Reiten sequence ending at X_{i+2}. In this case,

$$I \to X_1 \to \cdots \to X_i \to \tau_A X_{i+3} \to \tau_A^{-1}X_i \to X_{i+3} \to X_{i+4} \to \cdots \to P$$

is also a path from I to P with two hooks. Thus, without loss of generality, we can assume that $i = t-3$, and that X_j is not injective for $j > 0$. Clearly,



is a subquiver of Γ_A. Suppose now that $\alpha(\tau_A^{-1}(X_i)) \geq 3$ for some $2 \leq i \leq t-4$. Then, the middle term of the Auslander-Reiten sequence ending at $\tau_A^{-1}X_i$ has an indecomposable summand Y which is neither isomorphic to $\tau_A^{-1}X_{i-1}$ nor to X_{i+1}. Therefore the path of irreducible maps

$$I = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow Y \longrightarrow \tau_A^{-1}X_i \longrightarrow \cdots \longrightarrow X_t = P$$

has exactly one hook.
A similar argument can be done if $\alpha(\tau_A^{-1}X_1) \geq 2$ or if $\alpha(\tau_A^{-1}X_{t-3}) \geq 2$. It remains to consider the case where $\alpha(\tau_A^{-1}X_1) = 1 = \alpha(\tau_A^{-1}X_{t-3})$ and $\alpha(\tau_A^{-1}(X_i)) = 2$ for $i = 2, \cdots, t-2$. In this case, g_2 should be an epimorphism because

$$0 \longrightarrow X_1 \longrightarrow X_2 \xrightarrow{g_2} \tau_A^{-1}X_1 \longrightarrow 0$$

is an Auslander-Reiten sequence. So g_i is an epimorphism for each $i = 2, \cdots, t-3$. However, g_{t-3} cannot be epimorphism because

$$0 \longrightarrow X_{t-3} \xrightarrow{g_{t-3}} \tau_A^{-1} X_{t-4} \longrightarrow \tau_A^{-1} X_{t-3} \longrightarrow 0$$

is an Auslander-Reiten sequence. This proves (a).

(b) Suppose $(*)$: $I = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = P$ is a paht in Γ_A with just one hook i. So $\tau_A X_{i+1} \cong X_{i-1}$. Let $r > i+1$ such that $P' = X_r$ is a projective module and if $i+1 < l < r$, then X_l is not projective. By applying conveniently τ_A if necessary in $(*)$ one gets a path

$$(**) \qquad I = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{r-3} \longrightarrow Y_{r-2} \longrightarrow \tau_A^{-1} Y_{r-3} \longrightarrow Y_r = P'$$

in Γ_A. Since A is shod, if $(**)$ has another hook, it has to be $r-3$. In any case, $I = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{r-3}$ is sectional and so $\mathrm{Hom}_A(I, Y_{r-3}) \neq 0$ (1.4). This yields that $\mathrm{pd}_A \tau_A^{-1} Y_{r-3} \geq 2$ and the result is proven since $\tau_A^{-1} Y_{r-3}$ is a direct summand of $\mathrm{rad} P'$. $\diamondsuit$

The next result has been established in [22]. We shall give here an alternative proof.

COROLLARY 4.5 [22] *An algebra A is quasitilted if and only if any path in ind$A$ from an injective module to a projective module can be refined to a path of irreducible maps and any such refinement is sectional.*

Proof. Suppose A is quasitilted. Then, by 4.1, any path in ind$A$ from an injective to a projective can be refined to a path in Γ_A. Assume now that there exists a non-sectional path $I \rightsquigarrow P$ in Γ_A where I is an injective and P is a projective. By Proposition 4.4, there exists a projective $P' \in \mathrm{ind} A$ having a submodule X with $\mathrm{pd}_A X \geq 2$. Clearly, the quotient $\frac{P'}{X}$ is indecomposable and has projective dimension at least 3, a contradiction to the fact that A is quasitilted.
Conversely, if any path in ind$A$ from an injective module to a projective module can be refined to a path of irreducible maps and any such refinement is sectional, then A is shod by 4.2. If A is not quasitilted, then $\mathrm{gl.dim} A = 3$. In particular, there exists a simple module S with $\mathrm{pd}_A S = 3$. Let $P_S \in \mathrm{ind} A$ be the projective module associated to S. Then $\mathrm{rad} P_S$ has an indecomposable summand X with $\mathrm{pd}_A X = 2$. Hence, there exists a path

$$I \longrightarrow \tau_A X \xrightarrow{f} Y' \xrightarrow{g} X \longrightarrow P_S$$

in ind$A$ where I is an injective module and f, g are irreducible morphisms. By 4.1, such a path can be refined to a path in Γ_A which is clearly not sectional, a contradiction. $\diamondsuit$

COROLLARY 4.6 [11] *The following statements are equivalent for a shod algebra A:*

(a) *A is strict shod.*

(b) *There exists a non-sectional path in indA from an injective module to a projective module.*

(c) $\mathcal{R}_A \setminus \mathcal{L}_A$ *has a projective module.*

(d) $\mathcal{L}_A \setminus \mathcal{R}_A$ *has an injective module.*

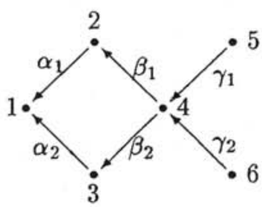
Proof. The equivalence between (a) and (b) follows directly from Corollary 4.5.
(b) $\Rightarrow$ (c) and (d). Assume there exists a non-sectional path $(*)$ in Γ_A from an injective module I to a projective module P. Since $(*)$ is not sectional but it has at most two (consecutive) hooks, then there exists a non-injective module X such that $(*)$ has the form:

$$I \overset{(\epsilon_1)}{\rightsquigarrow} X \longrightarrow Y \longrightarrow \tau_A^{-1}X \overset{(\epsilon_2)}{\rightsquigarrow} P$$

where (ϵ_1) and (ϵ_2) are sectional paths. In particular, $\mathrm{pd}_A\tau_A^{-1}X \geq 2$ and $\mathrm{id}_A X \geq 2$. Since A is shod, we infer that $X \in \mathcal{L}_A \setminus \mathcal{R}_A$ and $\tau_A^{-1}X \in \mathcal{R}_A \setminus \mathcal{L}_A$ and so $I \in \mathcal{L}_A \setminus \mathcal{R}_A$ and $P \in \mathcal{R}_A \setminus \mathcal{L}_A$.
(c) $\Rightarrow$ (b) Assume there exists a projective $P \in \mathcal{R}_A \setminus \mathcal{L}_A$. Since $P \notin \mathcal{L}_A$, there exists a path $X \rightsquigarrow P$ in ind$A$ with $\mathrm{pd}_A X \geq 2$. As before, one gets a path $I \longrightarrow \tau_A X \longrightarrow E \longrightarrow X \rightsquigarrow P$ in ind$A$, where I is an injective. Hence (b) holds. The proof of (d)$\Rightarrow$ (c) is similar. $\Diamond$

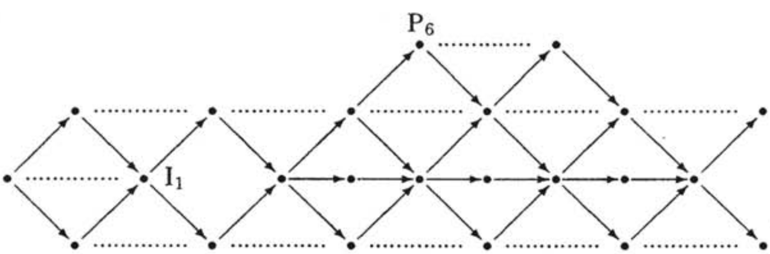
We finish this section with two examples.

EXAMPLES 4.7 (a) Let A be the k-algebra given by the quiver



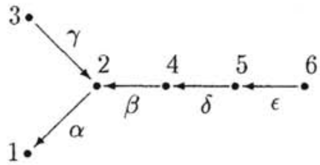
with $\alpha_1\beta_1 = \alpha_2\beta_2 = \beta_2\gamma_2 = 0$

This is a representation-finite iterated tilted algebra of type $\tilde{\mathbf{A}}_n$. Its Auslander-Reiten quiver has the following shape:



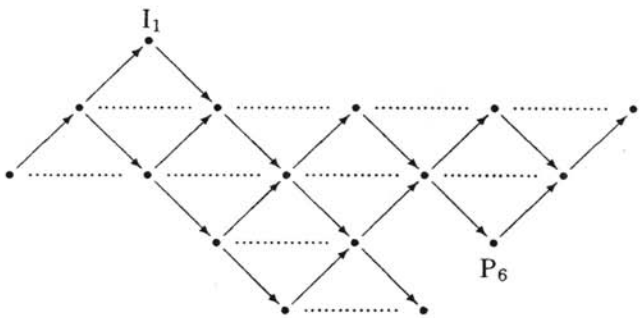
An easy calculation shows that A is shod. Also, since $\mathrm{gl.dim}A = 3$, A is indeed strict shod. Observe that from I_1 to P_6 there are both a path with one hook and a sectional path.

(b) Let A be the k-algebra given by:



with $\beta\delta\epsilon = \alpha\beta = 0$

Its Auslander-Reiten quiver is:



Observe that from I_1 to P_6 there is one path with one hook and another with two hooks. Clearly, A is strict shod.

5. Non-semiregular components for shod algebras

The results in the last section indicate that the paths in Γ_A from an injective module to a projective module when A is shod can play an important role in the understanding of the category $\mathrm{mod}A$, or particularly, on the Auslander-Reiten quiver of A.

We have seen that if A is strict shod, then there always exist a path in $\mathrm{ind}A$ from an injective to a projective. On the other hand, if A is quasitilted but not tilted, there is no such path and if A is tilted, there could exist such path. In any case, it follows from 4.1 that if there exists a path in $\mathrm{ind}A$ from an injective I to a projective P, it can be refined to a path of irreducible morphisms. In particular, I and P lie in the same component of Γ_A. We shall now study some properties of such component. It will, indeed, play the same role for strict shod algebra as the so-called connecting component for tilted algebras, *dividing* the category $\mathrm{ind}A$ into two parts, the left one lying in $\mathcal{L}_A$ and the right one in $\mathcal{R}_A$.

Our first aim is to establish that a non-semiregular component Γ of Γ_A for a shod algebra A has no oriented cycles. Indeed, such a result can be generalized and we have done so in [12] in a joint work with Lanzilotta, where we have studied the so-called *pip-bounded components*. However, we shall provide here a different proof for the non-existence of oriented cycles than the one given in [12]. We shall need the following lemma proved in [18]. For the convenience of the reader, we shall give here a proof.

LEMMA 5.1 [18](1.4) *Let A be an algebra, $X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = X$ be an oriented cycle in Γ_A, and $r \geq 1$. If $\tau_A^i X_j \neq 0$ for each $1 \leq i \leq r$ and $j = 0, \cdots, t$, then there exists a path in Γ_A from X to $\tau_A^r X$.*

Proof. By 1.3, the path $(*) : X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t \longrightarrow X_1 = X_{t+1}$ is not sectional and so there exists an l, $1 \leq l \leq t$, such that $\tau_A X_{l+1} \cong X_{l-1}$. By hypothesis, one can apply τ_A to $(*)$ to get the following path

$$(**) \qquad \tau_A X = \tau_A X_0 \longrightarrow \tau_A X_1 \longrightarrow \cdots \longrightarrow \tau_A X_t \longrightarrow \tau_A X_1$$

in Γ_A. Observe that the module $\tau_A X_{l+1} \cong X_{l-1}$ appears in both $(*)$ and $(**)$ and hence, there exists a path

$$X = X_0 \longrightarrow \cdots \longrightarrow X_{l-1} \cong \tau_A X_{l+1} \longrightarrow \tau_A X_{l+2} \longrightarrow \cdots \longrightarrow \tau_A X_t = \tau_A X$$

in Γ_A from X to $\tau_A X$. Iterating this procedure, one gets the desired result. $\diamond$

PROPOSITION 5.2 *Let Γ be a component of Γ_A and let $Z \in \Gamma$ be a module lying in an oriented cycle.*

(a) *If Γ has projective modules, then there is a path in Γ_A from Z to a projective.*

(b) *If Γ has injective modules, then there is a path in Γ_A from an injective to Z.*

Proof. We shall only prove (a) since the proof of (b) is dual.
(a) Let $\theta : Z = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_t = Z$ be an oriented cycle in Γ containing Z. Suppose first that there exists an j such that $\tau_A^r Z_j$ is a projective module for some $r \geq 0$. Without lost of generality we can assume that $\tau_A^l Z_i$ is not projective for each $l < r$ and each $i = 0, \cdots, t$. By 5.1, there exists a path from Z_j to $\tau_A^r X_j$ as required.
Suppose now that each $Z_0, \cdots, Z_t$ is left stable, that is, $\tau_A^n Z_i$ is not a projective for each $n \geq 0$ and each $i = 0, \cdots, t$. Since Γ contains projective modules and it is connected, there exists a walk

$$(*) \qquad Z' = X_0 - X_1 - \cdots - X_m = P$$

in Γ_A of minimal length, where P is a projective module in Γ and Z' is a module in the τ_A-orbit of Z. It follows from the imposed minimality on $(*)$ that each of $X_0, \cdots, X_{m-1}$ is left stable. Therefore, by applying τ_A if necessary one gets a path $Z'' \longrightarrow \cdots \longrightarrow P$, where $Z'' = \tau_A^s Z$ for some s. If $s < 0$, then clearly there exists a path $Z \rightsquigarrow \tau_A^s Z$ in Γ. Otherwise, if $s > 0$, then by 5.1, there exists a path $Z \rightsquigarrow \tau_A^s Z$ in Γ. In both cases, we get a path $Z \rightsquigarrow P$ as required. $\diamondsuit$

In order to show our main result of this section, we recall the following lemma from [12].

LEMMA 5.3 [12] *Let A be a shod algebra and let Γ be a non-semiregular component of Γ_A. Then Γ has only finitely many τ_A-orbits.*

THEOREM 5.4 [12] *Let A be a shod algebra and Γ be a non-semiregular component of Γ_A. Then Γ is generalized standard and has no oriented cycles.*

Proof. (a) Suppose that $(*)$: $X = X_0 \longrightarrow \cdots \longrightarrow X_t = X$ is an oriented cycle in Γ. Since Γ is non-semiregular, by 5.2, there exist an injective I and a projective P and paths $I \rightsquigarrow X$ and $X \rightsquigarrow P$. Using $(*)$, one can produce paths $I \rightsquigarrow P$ of arbitrary length, a contradiction to 4.3.

(b) Suppose there exists a non-zero morphism $f \in \mathrm{rad}_A^\infty(X, Y)$ with X and Y in Γ. By [37], there exists an infinite path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \xrightarrow{f_l} X_l \longrightarrow \cdots$$

of irreducible maps and non-zero maps $g_i \in \mathrm{rad}^\infty(X_i, Y)$, for each i. By 5.3, Γ has only finitely many τ_A-orbits, and so there exists a positive integer n_0 and an infinite set of integers J such that if $j \in J$, then the τ_A-orbits of X_j and of X_{n_0} coincide. Without lost of generality, we can assume that $n_0 = 0$. Observe also that, since Γ has no oriented cycles, if $j_1, j_2 \in J$, $j_1 \geq j_2$, then $\tau_A^s X_{j_1} \cong X_{j_2}$ for some $s \geq 0$. In particular, the modules X_j, with $j \in J$, are right stable. Since Γ has injective modules, it is not difficult to see that any right stable τ_A-orbit has a module which is a successor of injectives. Let then l be such that X_l is a successor of an injective I. Hence, we have shown the existence of a path $I \overset{(*)}{\rightsquigarrow} X_l \xrightarrow{g_l} Y$ from the injective I to Y passing through a morphism g_l in $\mathrm{rad}_A^\infty(X_l, Y)$. Using a dual argument and the morphism g_l, one gets a path $(**)$: $X_l \xrightarrow{g'} Y' \rightsquigarrow P$ with P an indecomposable projective and $g' \in \mathrm{rad}_A^\infty(X_l, Y')$. The glueing of the paths $(*)$ and $(**)$ yields a contradiction to 5.1 and the result is proven. $\diamondsuit$

COROLLARY 5.5 *Let A be a connected quasitilted algebra and let Γ be a non-semiregular component of Γ_A. Then A is tilted and Γ is a connecting component.*

Proof. We shall only sketch the proof here. By 5.4, Γ has no oriented cycles and it is generalized standard. Therefore, Γ is directing. Suppose Γ is not connecting. Then, by 3.4, Γ is either postprojective or preinjective. Suppose the former. If Γ contains all the indecomposable projective modules, then it is connecting, contrary to our assumption. Therefore, there exist projective modules not lying in Γ. Since A is connected, there are projectives $P, P' \in \operatorname{ind}A$ such that $P \in \Gamma$, $P' \notin \Gamma$ and $\operatorname{Hom}_A(P, P') \neq 0$. Clearly, $\operatorname{Hom}_A(P, P') = \operatorname{rad}_A^\infty(P, P')$. Lifting any non-zero morphism of $\operatorname{Hom}_A(P, P')$ through paths of irreducible morphisms in Γ, one gets a path from an injective $I \in \Gamma$ to P' passing through a morphism in $\operatorname{rad}^\infty(\operatorname{mod}A)$, a contradiction to 5.1. Dually, one gets a contradiction if assuming that Γ is preinjective. $\diamondsuit$

COROLLARY 5.6 *A representation-finite connected quasitilted algebra is tilted.*

Proof. Since A is connected, then Γ_A is also connected (see [3](VII.2.1)). Clearly, it is a non-semiregular component and so the result follows from 5.5. $\diamondsuit$

In [12], in a joint work with Lanzilotta, we have considered an special kind of component in the Auslander-Reiten quiver Γ_A which includes the one discussed above. A component $\Gamma \subset \Gamma_A$ is called *pip-bounded* provided there exists a positive integer n_0 such that any path in $\operatorname{ind}A$ from an injective in Γ to a projective in Γ has length at most n_0. We have shown that a pip-bounded component is generalized standard and has no oriented cycles (see [12]). The following result follows easily from 4.3.

COROLLARY 5.7 *Let A be a shod algebra. Then any non-semiregular component of Γ_A is pip-bounded.*

The study of pip-bounded components has been also useful in [13] where we consider a more general class of algebras, the so-called weakly shod algebras. We just recall its definition and refer the reader to [13] for details. An algebra A is called *weakly shod* provided there exists a positive integer n_0 such that any path in $\operatorname{ind}A$ from an injective to a projective has length at most n_0.

6. Shod algebras as an iteration of one point extensions

We have seen that if A is a strict shod algebra, then Γ_A has a non-semiregular generalized standard component Γ with no oriented cycles. We will see below that in fact, if A is connected, then there exists a unique such a component which is, in addition, faithful. Moreover, it resembles a connecting component for tilted algebras in many aspects since it divides Γ_A into two parts. In order to prove this, we shall make use of a technique called *one point extension*. Being more precise, let B be an algebra and $M \in \operatorname{mod}B$. The algebra $A = B[M] = \begin{pmatrix} k & 0 \\ M & B \end{pmatrix}$ is called the *one point extension algebra of B by M*. In this case, the A-modules can be described as triples (k^t, X, f), where X is a B-module and $f\colon k^t \otimes_k M \longrightarrow X$ is a B-homomorphism. Observe then that $\operatorname{mod}B$ can be naturally embedded into $\operatorname{mod}A$. The indecomposable projective A-modules can be described as: (i) $(0, P, 0)$ where P is an indecomposable projective B-module; and (ii) the *extended* *projective* A-module $P_\omega = (k, M, id_M)$, whose radical is the module $(0, M, 0)$. Also, if I is an injective B-module, then $(0, I, 0)$ is an injective A-module if and only if $\mathrm{Hom}_B(M, I) = 0$. When there is no danger of confusion, we shall also indicate the A-module $(0, X, 0)$ simply by X.

Suppose $A = B[M]$. If one denotes by n the number of simple A-modules, then the number of simple B-modules is $n - 1$. Also, the ordinary quiver Δ_A of A is an extension of the ordinary quiver Δ_B of B in the folllowing way: $(\Delta_A)_0 = (\Delta_B)_0 \cup \{\omega\}$ and $(\Delta_A)_1$ has all the arrows of Δ_B plus some extra arrows going from the vertex ω to vertices in $(\Delta_B)_0$. For more details on this construction we refer the reader, for instance, to [3].

Observe that the construction above will allow us to transport some informations from the algebra B to A using induction on the number of simple modules. The next result is very useful for this purpose.

LEMMA 6.1 *If $A = B[M]$ is a shod algebra, so is B.*

Proof. Suppose B is not shod and let $X \in \mathrm{ind}B$ with $\mathrm{pd}_B X \geq 2$ and $\mathrm{id}_B X \geq 2$. It is not difficult to see that $\mathrm{pd}_A(0, X, 0) \geq 2$ and $\mathrm{id}_A(0, X, 0) \geq 2$, which contradicts our assumption on A. $\diamond$

Using this *reduction* procedure, we will see that any strict shod algebra is in fact an iteration of one point extensions starting at a product of tilted algebras and one can then use the knowledgment on this later class of algebras to study the class of strict shod algebras. This is the aim of this section. For complete proofs of the material discussed here, we refer to [13, 30].

Our main result here can be stated as follows.

THEOREM 6.2 [13] *Let A be an strict shod algebra. Then there are algebras $B = A_0, A_1, \cdots, A_t = A$ and A_i-modules M_i, $i = 0, \cdots, t-1$, such that:*

(a) *B is a product of tilted algebras.*

(b) *For each $i = 1, \cdots, t$, $A_i = A_{i-1}[M_{i-1}]$.*

(c) *For each $i = 1, \cdots, t$, there are no paths from the extended projective A_i-module to any other projective A_i-module.*

We shall now discuss only the main steps of the proof of the above theorem. Let A be a strict shod algebra and denote by $\mathcal{P}^g_A$ the set of all indecomposable projective A-modules lying in $\mathcal{R}_A \setminus \mathcal{L}_A$. By 4.6, $\mathcal{P}^g_A \neq \emptyset$ and, clearly, an indecomposable projective A-module P lies in $\mathcal{P}^g_A$ if and only if there exists a non-sectional path in $\mathrm{ind}A$ from an injective module I to P.

Let now $P, P' \in \mathcal{P}^g_A$ and let $I \in \mathrm{ind}A$ be an injective module such that there exists a non-sectional path $I \rightsquigarrow P$. If P' is a successor of P, then there is no path $P' \rightsquigarrow P$, since otherwise one would get a non-trivial path $P \rightsquigarrow P$ and so paths of arbitrary length from I to P, contradicting 4.3. Hence, one can define the following order in $\mathcal{P}_A^g$: for $P, P' \in \mathrm{ind}A$, $P \preceq P'$ if and only if there exists a path $P \rightsquigarrow P'$.

Under this order, a maximal element $P \in \mathcal{P}_A^g$ also satisfies the following $\mathrm{Hom}_A(P, P') = 0$ for each indecomposable projective P' not isomorphic to P. In terms of the ordinary quiver of A, the vertex associated to P is a source. Therefore, A can be seen as one-point extension $C[M]$, where P is the extended projective A-module.

Write $C = C_1 \times \cdots \times C_t$ and $M = M_1 \times \cdots \times M_t$ where for each $i = 1, \cdots, t$, C_i is a connected algebra and $M_i \in \mathrm{mod}C_i$. We know, by 6.1, that each C_i is a shod algebra. In fact, each C_i is either a strict shod algebra or a tilted algebra, the possibility of C_i being quasitilted but not tilted excluded (see [13] for details). Moreover, if Γ' is a component of Γ_{C_i} containing an indecomposable summand of M_i, then Γ' is either a pip-bounded component (in case C_i is strict shod) or a connecting component (in case C_i is tilted). One can now repeat this procedure for each summand of C which is strict shod. An induction on the number of indecomposable projective modules gives now the main result.

This procedure also shows that a pip-bounded component Γ of a strict shod algebra can be built up from connecting components of some connected tilted algebras. Since a connecting component of a connected tilted algebra is faithful, one gets the following result (see [13](5.4) for a complete proof).

THEOREM 6.3[13] *Let A be a connected strict shod algebra. Then Γ_A has a unique pip-bounded component which is, in addition, faithful.*

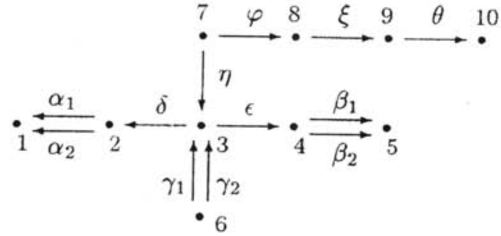
Next result is a direct consequence of 6.2 and shows that the ordinary quiver of strict shod algebra has no oriented cycles. For an alternative proof, we refer the reader to [11](2.2) and [22](III.1.1).

COROLLARY 6.4 *The ordinary quiver of a shod algebra A is directed.*

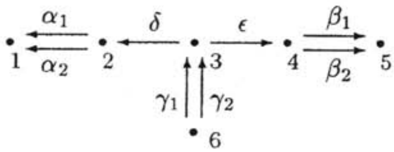
Proof. (a) If A is quasitilted, then the the result follows from [22]. Suppose A is strict shod. As we have seen, A is then built up from a (product of) tilted algebra(s) by iterating one-point extensions. The result will now follow from the following easily verified remarks: (i) the ordinary quiver of a tilted algebra is directed; and (ii) the process of one-point extending an algebra does not produce cycles in its ordinary quiver. $\diamondsuit$

There exists a dual notion of one-point coextension. It is not difficult to see that 6.2 can be dualized using this notion. We finish this section with an example to illustrate the above procedure.

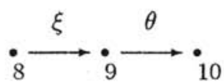
EXAMPLE 6.5 Let A be the k-algebra given by the quiver Δ:



with relations $\theta\xi = \xi\varphi = \delta\eta = 0$, $\alpha_i\delta = 0$, $\beta_i\epsilon = 0$, $\delta\gamma_i = 0$ and $\gamma_i\epsilon = 0$, for each $i = 1, 2$. Observe that $\mathcal{P}_A^g$ consists of the projectives P_6 and P_7 and both are maximal there. Choosing P_7 as the extended projective, A can be written as the one-point extension $A = (C_1' \times C_1'')[S_8 \oplus N]$, where: (i) N is the indecomposable module of length 2 such that $\mathrm{rad}N \cong S_4$ and $N/\mathrm{rad}N \cong S_3$; (ii) C_1' is the radical square zero algebra given by the quiver:

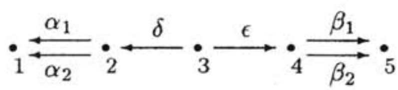


and (iii) C_1'' is the algebra given by the quiver



with $\theta\xi = 0$

Observe that C_1'' is a tilted algebra while C_1' is a strict shod algebra. Now $\mathcal{P}_{C_1'}^g = \{P_6\}$ and so we can write $C_1' = C_2[S_3]$. Observe that C_2 is the radical square zero algebra given by the quiver:



which is a tilted algebra. In the notations of 6.2, we have $B = A_0 = C_2 \oplus C_1''$, $A_1 = C_1' \oplus C_1''$, $M_0 = S_3$ and $M_1 = S_0 \oplus N$.

7. The Auslander-Reiten quiver of a shod algebra

It follows from our considerations in the last section that if A is a strict shod algebra, then Γ_A has a unique non-semiregular component which is, in addition, faithful, generalized standard and has no oriented cycles. In [36], Skowroński, studied the algebras whose AR-quivers have a generalized standard component without oriented cycles. We shall now combine these two results to further understand the AR-quiver of a strict shod algebra. The following result is a consequence of Skowroński's results of [36].

THEOREM 7.1 *Let A be an Artin algebra and let Γ be a connected component of Γ_A. If Γ is faithful, generalized standard and without oriented cycles, then there exist tilted algebras $A^{(l)}$ and $A^{(r)}$ such that any component Γ' of Γ_A different of Γ, satisfies one and only one of the following conditions:*

(a) *Γ' is a component of $\Gamma_{A^{(l)}}$ and $Hom_A(X,Y) \neq 0$ for some $X \in \Gamma'$ and $Y \in \Gamma$; or*

(b) *Γ' is a component of $\Gamma_{A^{(r)}}$ and $Hom_A(X,Y) \neq 0$ for some $X \in \Gamma$ and $Y \in \Gamma'$.*

Applying this to our context, we have the following result. We refer to [13] for a proof.

PROPOSITION 7.2 [13] *Let A be a strict shod algebra and let Γ be the unique non-semiregular component of Γ_A.*

(a) *if Γ' is a component of Γ_A different of Γ, then Γ' is semiregular and either $\Gamma' \subset \mathcal{L}_A \setminus \mathcal{R}_A$ or $\Gamma' \subset \mathcal{R}_A \setminus \mathcal{L}_A$.*

(b) *the intersection $\mathcal{L}_A \cap \mathcal{R}_A$ is finite and it is contained in Γ.*

Let A be a connected strict shod algebra, and let Γ be the unique non-semiregular component of Γ_A. As we have seen above, if Γ' is a component of Γ_A different from Γ, then it is a component of a tilted algebra. Using now the well-known description of the Auslander-Reiten quiver of tilted algebras, we have the following. For a strict shod algebra A, the components of Γ_A are of the following shape (using the notation of (7.1)):

(i) postprojective component(s) (those of $\Gamma_{A^{(l)}}$).

(ii) preinjective component(s) (those of $\Gamma_{A^{(r)}}$).

(iii) a unique and faithful pip-bounded component which is the unique non-semiregular component.

(iv) stable tubes.

(v) components of type $\mathbf{ZA}_\infty$.

(vi) components constructed from tubes or from $\mathbf{ZA}_\infty$ by ray or coray insertions.

Observe moreover that the components of $\Gamma_{A^{(l)}}$ (or $\Gamma_{A^{(r)}}$) which are embedded in Γ_A are semiregular without injective (respectively, projective) modules and are contained in $\mathcal{L}_A \setminus \mathcal{R}_A$ (respectively, in $\mathcal{R}_A \setminus \mathcal{L}_A$) (see (7.2)).

We finish this section with some further results concerning the connection of components of Γ_A and the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ for a strict shod algebra A. We first recall the following result which has been proven in [15] and [22].

LEMMA 7.3 *Let A be a tilted algebra with connecting component Γ which is neither postprojective nor preinjective.*

(a) [15](3.1) *Then $\mathcal{L}_A \cap \mathcal{R}_A \subset \Gamma$.*

(b) [22](II.3.1) *If Γ is regular, then $\mathcal{L}_A \cap \mathcal{R}_A = \Gamma$.*

PROPOSITION 7.4 *Let A be a connected shod algebra.*

(a) *If $\mathcal{L}_A \cap \mathcal{R}_A$ contains a component of Γ_A, then A is quasitilted.*

(b) *Assume A is not hereditary. Then $\mathcal{L}_A \cap \mathcal{R}_A$ contains a directing component if and only if A is a tilted algebra with a regular connecting component Γ. Moreover, in this case, $\mathcal{L}_A \cap \mathcal{R}_A = \Gamma$.*

Proof. (a) It follows from 7.2(b).
(b) If A is a tilted algebra with a regular connecting component Γ, then by 7.3 $\Gamma = \mathcal{L}_A \cap \mathcal{R}_A$ and $\mathcal{L}_A \cap \mathcal{R}_A$, in particular, contains a directing component.
Suppose now that $\mathcal{L}_A \cap \mathcal{R}_A$ contains a directing component Γ'. Observe first that if $P \in \Gamma'$ is a projective module, then rad$P$ is also projective. Indeed, if rad$P$ has an indecomposable summand X which is not projective, then $\mathrm{id}_A \tau_A X \geq 2$ and so $\Gamma' \not\subset \mathcal{L}_A \cap \mathcal{R}_A$.
Suppose Γ' is postprojective. The remark above implies that each projective in Γ' is hereditary. Since the algebra A is not hereditary, there exist projective modules not lying in Γ'. Since now A is connected, there exist projective modules $P, P' \in \mathrm{ind}A$ with $P' \in \Gamma'$ and $P \notin \Gamma'$ and a non-zero morphism $f: P' \longrightarrow P$. Lifting now f through the irreducible morphisms of Γ' it is not difficult to see that $\mathrm{Hom}_A(X, P) \neq 0$ for some non-projective module $X \in \Gamma'$. As above, $\mathrm{id}_A \tau_A X \geq 2$ and so $\Gamma' \not\subset \mathcal{L}_A \cap \mathcal{R}_A$, a contradiction. A similar argument shows that Γ' cannot be preinjective. Then, by 3.4, Γ' is a connecting component of a tilted algebra. Clearly, if Γ' is not regular, then it would contain either a projective P such that rad$P$ is not projective or, dually, an injective I such that $I/\mathrm{soc}I$ is not injective, a contradiction and the result is proven. ◇

We refer the reader to [13, 18], where further results relating the components of Γ_A and the categories $\mathcal{L}_A$ and $\mathcal{R}_A$, for A shod or quasitilted, are proven.

8. Hochschild cohomology of strict shod algebras

In this section, we want to show how Theorem 6.2 can be used to calculate the Hochschild cohomology of a strict shod algebra. The results below appear in a joint work with Lanzilotta and Savioli [14].

For an algebra A, denote by $\mathrm{H}^i(A)$ its i-th Hochschild cohomology group (see [20, 27] for details). We want to show here that, for a strict shod algebra A, $\mathrm{H}^i(A) = 0$ for each $i \geq 2$. The next results, due to Happel, will be useful in our considerations. For a proof of them, we refer to [20].

THEOREM 8.1 [20]. *Let B be a connected tilted algebra of type Δ. Then*

(a) $\mathrm{H}^0(B) = k$;

(b) $\mathrm{H}^1(B) = 0$ *if and only if* Δ *is a tree*;

(c) $\mathrm{H}^i(B) = 0$ *for each* $i \geq 2$.

THEOREM 8.2 [20]. *Let $A = B[M]$. Then there exists a long exact sequence*

$$0 \longrightarrow H^0(A) \longrightarrow H^0(B) \longrightarrow (\, End_A M)/k \longrightarrow H^1(A) \longrightarrow$$
$$\longrightarrow H^1(B) \longrightarrow Ext^1_B(M,M) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^i(A) \longrightarrow H^i(B) \longrightarrow Ext^i_B(M,M) \longrightarrow \cdots$$

Let A be a strict shod algebra. The strategy of the proof of our main result is to show that at each step in the iteration of one-point extension given in 6.2, the modules M_i's satisfy $\mathrm{Ext}^j_{A_i}(M_i, M_i) = 0$, for $j = 1, 2$ (using the notations of 6.2) and then use Happel's long exact sequence given in 8.2. This will follow from the next proposition.

PROPOSITION 8.3 [14] *Let $A = B[M]$ be a strict shod algebra and assume that the extended projective A-module P_ω is a maximal element in $\mathcal{P}^g_A$. Then $Ext^i_B(M,M) = 0$, for each $i = 1, 2$.*

Proof. We shall not give a complete proof here, see [14] for it. However, we shall sketch the proof that $\mathrm{Ext}^2_A(M,M) = 0$ for the taste of it.
Suppose $\mathrm{Ext}^2_B(M,M) \neq 0$. Then there exists an indecomposable summand M_1 of M such that $\mathrm{Ext}^2_B(M, M_1) \neq 0$. Clearly, then, $\mathrm{Ext}^2_A((0,M,0),(0,M_1,0)) \neq 0$. Denote by Z the quotient of the extended projective A-module $P_\omega = (k, M, id_M)$ by $(0, M_1, 0)$. Applying now $\mathrm{Hom}_A((0,M,0), -)$ to the short exact sequence

$$0 \longrightarrow (0, M_1, 0) \longrightarrow P_\omega \longrightarrow Z \longrightarrow 0$$

one gets

$$\cdots \longrightarrow \mathrm{Ext}^1_A((0,M,0),Z) \longrightarrow \mathrm{Ext}^2_A((0,M,0),(0,M_1,0)) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}_A^2((0,M,0),P_\omega) \longrightarrow \cdots$$

Observe that $\operatorname{id}_A P_\omega \leq 1$. Indeed, if $\operatorname{id}_A P_\omega \geq 2$, there would exist a nonzero morphism from $\tau_A^{-1} P_\omega$ to a projective A-module leading to a contradiction to the fact that P_ω is maximal in $\mathcal{P}_A^g$. Therefore, $\operatorname{Ext}_A^2((0,M,0),P_\omega) = 0$. Since $\operatorname{Ext}_A^2((0,M,0),(0,M_1,0)) \neq 0$, we then infer that $\operatorname{Ext}_A^1((0,M,0),Z)$ is nonzero. Consequently, $\operatorname{Hom}_A(Z,\tau_A(0,M,0)) \neq 0$ (recall that $\operatorname{Ext}_A^1((0,M,0),Z) = \mathrm{D}\,\overline{\operatorname{Hom}}_A(Z,\tau_A(0,M,0))$, see [3](IV.4.6)). In particular, there exists an indecomposable direct summand N of M such that $\operatorname{Hom}_A(Z,\tau_A(0,N,0)) \neq 0$. We obtain then an oriented cycle

$$(*) \qquad P_\omega \longrightarrow Z \longrightarrow \tau_A(0,N,0) \longrightarrow * \longrightarrow (0,N,0) \longrightarrow P_\omega$$

in Γ_A. Since $P_\omega \in \mathcal{P}_A^g$ there exists a path $(**) : I \rightsquigarrow P_\omega$ in $\operatorname{ind}A$ where I is an injective module. Using the paths $(*)$ and $(**)$ we would get paths $I \rightsquigarrow P_\omega$ of arbitrary length, a contradiction to 4.3. $\diamondsuit$

We can now show the main result of this section.

THEOREM 8.5 [14] *If A is a strict shod algebra, then* $\mathrm{H}^i(A) = 0$ *for each* $i \geq 2$.

Proof. By 6.2, there are algebras $B = A_0, A_1, \cdots, A_t = A$ and A_i-modules M_i for each $i = 0,\cdots,t-1$ such that: (i) B is a product of tilted algebras; (ii) $A_{i+1} = A_i[M_i]$ for each $i = 0,\cdots,t-1$; and (iii) the extended projective A_{i+1}-module (k, M_i, id_{M_i}) is a maximal element in $\mathcal{P}_{A_{i+1}}^g$. We shall use induction on $t \geq 1$ to get our result. First observe that, since gl.dim $A \leq 3$, we get $\mathrm{H}^i(A) = 0$, for each $i \geq 4$. Suppose $t = 1$, that is, $A = B[M]$, where B is a product of tilted algebras and the extended indecomposable projective A-module is maximal in $\mathcal{P}_A^g$. Then, by 8.1, $\mathrm{H}^i(B) = 0$, for each $i \geq 2$. Since $\operatorname{Ext}_B^i(M,M) = 0$ for each $i = 1,2$, we get from Happel's long exact sequence that $\mathrm{H}^2(A) = \mathrm{H}^3(A) = 0$. The above argument can be indeed made at each step of the iteration of one-point extensions described in 6.2 in order to get the desired result. $\diamondsuit$

We first observe that 8.5 cannot be generalized to shod algebras since there are quasitilted algebras A with $\mathrm{H}^2(A) \neq 0$. The second remark we want to make concerns the first Hochschild cohomology group of a strict shod algebra. The group $\mathrm{H}^1(A)$ will clearly depend on the types of the tilted algebras which are components of B and properties of the modules M_i (using the notations of 6.2) as shown in the next result.

PROPOSITION 8.6 [14] *Let A be a strict shod algebra. Using the notations of 6.2,* $H^1(A) = 0$ *if and only if*

(a) *B is a product of connected tilted algebras of tree type;*

(b) *for each $i > 0$, the extended projective A_{i+1}-module is separating; and*

(c) *for each $i > 0$, the module M_i is a direct sum of pairwise orthogonal bricks.*

We refer to [14] for more details of this discussion. We finish this section with some examples.

Examples. (a) Let B be the k-algebra given by the quiver:

$$\begin{tikzcd} & & 5 \arrow[d, "\gamma"] & \\ 1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] & 4 \arrow[l] \end{tikzcd} \qquad \text{with } \alpha\beta\gamma = 0$$

It is not difficult to see that B is a tilted algebra of type $\mathbf{D}_5$. Therefore, by 8.1, $\mathrm{H}^1(B) = 0$. Consider $M = \tau^{-2}P_3$, that is, the indecomposable B-module of dimension vector $\underline{\dim}M = (0,0,1,0,1)$ and $A = B[M]$. Then A is the k-algebra given by the quiver

$$\begin{tikzcd} & & 5 \arrow[d, "\gamma"] & 6 \arrow[l, "\delta"'] \\ 1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] & 4 \arrow[l] \end{tikzcd} \qquad \text{with } \alpha\beta\gamma = \beta\gamma\delta = 0$$

Clearly, A is a strict shod algebra, and since M is a brick and the extended projective A-module is separating we infer that $\mathrm{H}^1(A) = 0$.

(b) Let B be the k-algebra given by the quiver

$$\begin{tikzcd} 1 & 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left] & 3 \arrow[l, "\beta"'] \end{tikzcd} \qquad \text{with } \alpha\beta = 0$$

The algebra B is tilted of type $\tilde{\mathbf{A}}_3$ (with a complete slice in its preinjective component) and therefore by 8.1, $\mathrm{H}^1(B) \neq 0$. Consider the one-point extension $A = B[S_3]$ of B by the simple B-module S_3 associated to the vertex 3 which is indeed the unique indecomposable B-module of projective dimension 2. It is not difficult to see that there are then only two indecomposable A-modules which have projective dimension greater than 2, namely, S_3 and S_4. Since $\mathrm{pd}_A S_3 = 2$, $\mathrm{pd}_A S_3 = 3$, $\mathrm{id}_A S_3 = 1$, and $\mathrm{id}_A S_4 = 0$, we infer that A is a strict shod algebra. Also, it follows from the above considerations that $\mathrm{H}^1(A) \neq 0$.

9. Shod extensions of algebras

We have seen that if A is a shod algebra, then there exists an algebra B and a B-module M such that $A = B[M]$ and this allowed us to understand better the algebra A. In this section, we shall discuss the converse problem, that is, given an algebra B and a B-module M, when is the one-point extension $B[M]$ a shod algebra ? As we have seen in Section 6, if $A = B[M]$ is strict shod, then B is a product of algebras which are either strict shod or tilted. In other words, if B has a summand which is a quasitilted algebra but not tilted, then there is no connected one-point extension of B which is strict shod. So, it is sensible to divide our discussion into two parts: (i) quasitilted extensions of algebras; and (ii) strict shod extensions of algebras.

Quasitilted extension of algebra. The results below are part of a joint work with M. I. Martins and J. A. de la Peña and their proofs can be found in [15, 16].

As we have seen, the canonical algebras are quasitilted. On the other hand, they are one-point extensions of hereditary algebras by indecomposable modules lying in regular components. It is not difficult to see that these modules are not directing. Our first result implies that, in a sense, the above construction is an exception, that is, if a one-point extension $B[M]$ is quasitilted and M decomposes, then M is directing.

THEOREM 9.1 [15] *Let B be a connected algebra and let M be a non-zero decomposable B-module such that the one-point extension $B[M]$ is quasitilted. Then M is directing. Moreover, A is a tilted algebra.*

The following results provide complete characterizations of the properties of a decomposable B-module M for $B[M]$ to be quasitilted in case B is indecomposable. Our first theorem deals with the case where $M \in \mathrm{add}(\mathcal{L}_B \cap \mathcal{R}_B)$.

THEOREM 9.2 [16] *Let B be an indecomposable quasitilted algebra and M be a non-zero decomposable B-module in $add(\mathcal{L}_B \cap \mathcal{R}_B)$. The following are equivalent:*

(a) *$B[M]$ is tilted.*

(b) *$B[M]$ is quasitilted.*

(c) *M is directing.*

In the situation complementary to the above theorem, since $M \in \mathrm{add}\mathcal{L}_B$, we have to consider non-zero decomposable modules M with non-zero direct summands in $\mathcal{L}_B \setminus \mathcal{R}_B$. We shall divide it in the next two theorems.

THEOREM 9.3 [16] *Let B be an indecomposable quasitilted algebra and $M = M_1 \oplus M_2$ be a B-module such that $0 \neq M_1$ is an indecomposable module in $\mathcal{L}_B \setminus \mathcal{R}_B$ and $0 \neq M_2 \in add(\mathcal{L}_B \cap \mathcal{R}_B)$. Then the one-point extension $B[M]$ is quasitilted if and only if the following conditions hold:*

(a) *$B[M_1]$ is quasitilted;*

(b) *M_2 is a hereditary projective module and $Hom_B(M_2, \mathcal{R}_B \setminus \mathcal{L}_B) = 0$;*

(c) *M is directing.*

THEOREM 9.4 [16] *Let B be an indecomposable quasitilted algebra and M a decomposable B-module in $add\mathcal{L}_B$ such that it contains at least two non-zero indecomposable direct summands in $\mathcal{L}_B \setminus \mathcal{R}_B$. The following conditions are equivalent:*

(a) *$B[M]$ is a quasitilted algebra;*

(b) *M is a hereditary projective B-module and $Hom_B(M, \mathcal{R}_B \setminus \mathcal{L}_B) = 0$;*

(c) *M is a projective B-module and $Hom_B(M, \mathcal{R}_B \setminus \mathcal{L}_B) = 0$.*

There has been some further work in order to characterize the modules M in mod$B$ such that an extension $B[M]$ is quasitilted, see [26, 32, 41] for instance.

Strict shod extensions of algebras. We shall now comment the main results from our joint work with A. Savioli (see [17] for details).

We have seen that if $A = B[M]$ is shod then so is B. The informations one gets on M are somehow not so straight. The next results give some partial informations on M.

THEOREM 9.5 [17, 13] *Let $A = B[M]$ be a strict shod algebra such that the extended projective A-module lies in $\mathcal{P}^g_A$. Then:*

(a) *each indecomposable summand of M lies in a component of Γ_B which is generalized standard and without oriented cycles.*

(b) *M is a directing module.*

Proof. (a) If the extended projective module is a maximal element in $\mathcal{P}^g_A$ the result has been established in [13], where we refer the reader to for a proof. However, the result can be extended to the case considered here. Indeed, suppose the extended projective module P_ω is a non-maximal element in $\mathcal{P}^g_A$. So, there exists a projective A-module P' which is maximal element in $\mathcal{P}^g_A$ and a path $(*)$ from P_ω to P'. Observe that since P_ω is the extended projective, then $\mathrm{Hom}_A(P_\omega, P') = 0$ and the path $(*)$ is not sectional. On the other hand, since $P_\omega \in \mathcal{P}^g_A$, there exists a path $(**)$ from an indecomposable injective module to P_ω. Glueing now the paths $(*)$ and $(**)$ we get a path from an injective to P' which can be refined to a path in Γ_A and so P_ω and P' lie in the same component in Γ_A. The result now follows from the description of the components containing P_ω (5.4).
(b) Clearly then each indecomposable direct summand of M is directing. By [25], we infer that M is itself directing. $\diamondsuit$

PROPOSITION 9.6 [17] *Let $A=B[M]$ be a shod algebra where M is a directing module. Then either M is a projective B-module or $\tau_B M \in add\mathcal{L}_B$.*

We shall consider first the situation where M is a projective B-module and characterize when $A = B[M]$ is (strict) shod. We observe that the equivalence of conditions (a) and (c) of the next result was first established by Huard in [28]. For a complete proof, we refer the reader to [17].

THEOREM 9.7 [17, 28] *Let B be an algebra and let M be a projective B-module. The following statements are equivalent:*

(a) *$A = B[M]$ is shod.*

(b) *For each $(k^t, X, f) \in indA$, either $X \in add\mathcal{L}_B$ or $X \in add\mathcal{R}_B$.*

(c) *For each $(k^t, X, f) \in indA$, either $pd_B X \leq 1$ or $id_B X \leq 1$.*

As a consequence we have the following.

PROPOSITION 9.8 [17] *Let $A = B[M]$ be a shod algebra where M is a projective B-module.*

(a) *If $M \in add\mathcal{R}_B$, then A is shod.*

(b) *A is strict shod if and only if B is strict shod.*

Proof. (a) Let $Z = (k^t, X, f)$ be an indecomposable A-module. We want to show that $\mathrm{pd}_B X \leq 1$ or $\mathrm{id}_B X \leq 1$. If X is indecomposable, this is the case because B is shod (by 6.1). If X is not indecomposable, then in particular $f \neq 0$. Moreover, the image of f intersects each indecomposable summand of X. Since, by hypothesis $M \in \mathrm{add}\mathcal{R}_B$, it yields that $X \in \mathrm{add}\mathcal{R}_B$ and, in particular, $\mathrm{id}_B X \leq 1$. The result now follows using 9.7.
(b) In general, if $A = B[M]$ then $\mathrm{gl.dim} A = \max\{ \mathrm{gl.dim} B, pd_B M + 1\}$ (see [3](III.2.7)). Now, since in our case A is a shod algebra we have that $\mathrm{gl.dim} A \leq 3$. Since $\mathrm{pd}_B M = 0$, we infer that $\mathrm{gl.dim} A = 3$ if and only if $\mathrm{gl.dim} B = 3$ and the result is proven. $\diamond$

The next result deals with the second possibility discussed in 9.6.

THEOREM 9.9 [17] *Let B be a shod algebra and let M be a directing B-module in $add\mathcal{R}_B$ such that $\tau_B M \in add\mathcal{L}_B$.*

(a) *Then $B[M]$ is shod.*

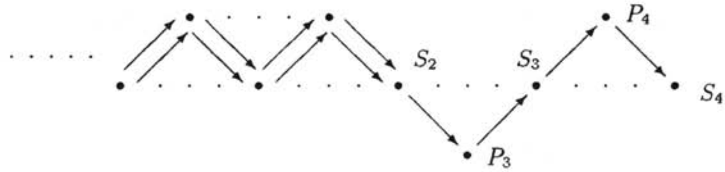
(b) *The algebra $B[M]$ is strict shod if and only if B is strict shod or if $pd_B M = 2$.*

We borrow the following example from [17] (see also [35]) to show that there exists an extension $B[M]$ which is strict shod but with M not directing.

EXAMPLE 9.10 Let B be the radical square zero k-algebra given by the quiver Δ:

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift left] \arrow[l, shift right] & \underset{3}{\bullet} \arrow[l] & \underset{4}{\bullet} \arrow[l] \end{tikzcd}$$

The Auslander-Reiten quiver Γ_B of B consists of a postprojective component and a family of homogeneous tubes corresponding to the algebra given by the full subquiver of Δ containing the vertices 1 and 2, and a component Γ as follows:



The subcategory $\mathcal{L}_B$ consists of all indecomposable B -modules but S_3, P_4 and S_4, while $\mathcal{R}_B$ consists of the modules P_3, S_3, P_4 and S_4. So B is shod. Moreover, it is strict shod because $\mathrm{pd}_B S_4 = 3$. Let now $M = \ (k \overset{1}{\underset{0}{\leftleftarrows}} k \leftarrow 0 \leftarrow 0)$ in $\mathrm{ind}B$. It is not difficult to see that M belongs to one of the homogeneous tubes of Γ_B and so M is not directing. Consider now $A = B[M]$, that is, the k-algebra given by the quiver Δ':

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] & \underset{3}{\bullet} \arrow[l, "\beta"'] & \underset{4}{\bullet} \arrow[l, "\gamma"'] \\ & 5\ \bullet \arrow[u, "\delta"'] & & \end{tikzcd}$$

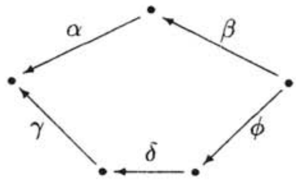
with $\alpha_2\delta = \alpha_1\beta = \alpha_2\beta = \beta\gamma = 0$. Observe that the k-algebra B' given by the full subquiver of Δ' consisting only of the vertices 1, 2 and 5 is tilted. Also, it is not difficult to see that the algebra A is strict shod. Hence, there are strict shod algebras which are one-point extensions by non-directing modules.

10. Tilting up shod algebras

In this section we will comment very quickly some results from our joint work with Happel and Unger [10], where we refer to for details.

The starting point of our considerations was the problem whether or not there is a relantionship between hereditary abelian categories and strict shod algebras via the tilting process. The following example, borrowed from [35], shows that there are strict shod algebras which do not come in this way.

EXAMPLE 10.1 Let A be the k-algebra given by the quiver Δ:



with $\alpha\beta = 0$, $\gamma\delta = 0$ and $\delta\phi = 0$. This is a strict shod algebra. Observe however that $\mathrm{Ext}^2_A(S, S') \neq 0$ and $\mathrm{Ext}^3_A(S', S) \neq 0$ if S is the unique simple injective module and S' is the unique simple projective. Therefore, by [19](IV.1.11), A cannot be a piecewise hereditary.

However, there is a nice relation between the class of strict shod algebras and a class of algebras of global dimension 2 admiting a special tilting torsion pair which we shall now describe. We start discussing the so-called canonical tilting module.

Let A be a strict shod algebra. Let now P' (respectively, I') be the sum of all indecomposable projective (respectively, injective) modules lying in $\mathcal{R}_A \setminus \mathcal{L}_A$ (respectively, in $\mathcal{L}_A \setminus \mathcal{R}_A$). By 4.6, both P' and I' are non-zero. Let finally J be the sum of I' with all indecomposable modules $X \in \mathcal{L}_A$ such that $\tau_A^{-1} X \notin \mathcal{L}_A$. In fact, J is the sum of all indecomposable Ext-injective modules of $\mathcal{L}_A$. Recall that a module X is an Ext-injective in a subcategory $\mathcal{C}$ of mod$A$ if $X \in \mathcal{C}$ and $\mathrm{Ext}^1_A(X, C) = 0$ for each $C \in \mathcal{C}$. Dually, one can define Ext-projective modules in $\mathcal{C}$.

THEOREM 10.2 [10](3.6) *If A is a strict shod algebra, then $T = P' \oplus J$ is a tilting module.*

The (tilting) module T as in 10.2 is called the *canonical tilting module* for A. Observe that a similar version of this module has been considered by Savioli in [35] in connection with the so-called separating slice of Assem.

Let A be a strict shod algebra and let $T = P' \oplus J$ be the canonical tilting module. Clearly, P' is an Ext-projective in $\mathcal{R}_A \setminus \mathcal{L}_A$. Also, it is not difficult to see that the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T is

$$\mathcal{T}(T) = \text{ add } ((\mathcal{R}_A \setminus \mathcal{L}_A) \cup \text{ ind } J) \quad \text{and} \quad \mathcal{F}(T) = \text{ add } (\mathcal{L}_A \setminus \text{ ind } J)$$

Consider now the following set $\mathcal{S}$ consisting of all pairs (A, T) where:

(i) A is a strict shod algebra.

(ii) $T = T_l \oplus T_r$ is a cotilting module, T_l is Ext-injective in add$(\mathcal{L}_A \setminus \mathcal{R}_A)$.

(iii) the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ splits.

(iv) $\mathrm{pd}_A X \leq 1$ for each indecomposable $X \in \mathcal{Y}(T)$ which is not a direct summand of T.

For a strict shod algebra A, the pair (A, T), where T is the canonical tilting module, belongs to $\mathcal{S}$. The main result of [10] is the following theorem.

THEOREM 10.3 [10] *There exists a bijective correspondence between the set $\mathcal{S}$ and the set of all pairs (B, T') where:*

(a) *B is an algebra of global dimension two.*

(b) *T' is a tilting module.*

(c) *$id_B X \leq 1$ for each $X \in \mathcal{F}(T')$ and $pd_B Y \leq 1$ for each non-injective Y in $ind\mathcal{T}(T')$.*

(d) *$Ext^2_B(\mathcal{F}(T'), \mathcal{T}(T')) \neq 0$.*

11. Double tilted algebras

In an independent work, Reiten and Skowroński [33] have proven some similar results to the main results of Section 6. There, they introduce the notion of double section and double tilted algebras and relate them to strict shod algebras. In this section, we want to discuss this work very briefly, refering to [33] for details.

DEFINITION 11.1 Let A be an algebra and Γ be a component of Γ_A. A *double section* Δ in Γ is a full connected subquiver of Γ such that

(i) Δ has no oriented cycles.

(ii) Δ is convex.

(iii) Δ crosses each τ_A-orbit of Γ at least once and at most twice.

(iv) If Δ crosses two modules X, X' in the same τ_A-orbit then (without lost of generality) $X' = \tau_A X$ and there are sectional paths $I \rightsquigarrow \tau_A X$ and $X \rightsquigarrow P$, where I is an injective module and P is a projective module.

This double section can be seen as the glueing of a *left section* Δ_l and a *right section* Δ_r (with a possible intersection). A *double tilted algebra* A is an algebra such that Γ_A has a component with a faithful double section Δ which induces two tilted algebras $A^{(l)}$ and $A^{(r)}$ which are factors of A (we refer to [33] for a precise definition). The main result of [33] is the following.

THEOREM 11.2 [33] *The following statements are equivalent for a connected algebra A:*

(a) *A is a strict shod algebra.*

(b) *A is an iterated strict shod extension of a tilted algebra.*

(c) *A is an iterated strict shod coextension of a tilted algebra.*

(d) *A is a double tilted algebra.*

(e) *Γ_A admits a component Γ with a faithful strict double section Δ such that $Hom_A(X, \tau_A Y) = 0$ for all modules X from Δ_r and Y from Δ_l.*

Also in [33], Reiten and Skowroński studied the tame strict shod algebras. In particular, they extend Skowroński's characterization of tame quasitilted algebras to shod (see [33](9.4)).

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