Resenhas IME-USP 2001, Vol. 5, No. 1, 63 - 84.

Orders and Relative Pythagorean Closures

M. Dickmann and F. Miraglia

Introduction

Our main topic here is the structure of orders on the Pythagorean closures of a non-Pythagorean formally real field. Following §3 of Chapter II of [Be2], if Ω is a prime-closed extension of a formally real field F, then the intersection of all Ω-real closures of F in Ω, $(\Omega|F)*$, is the largest Galois extension of F in Ω, to which every order in F can be extended (Theorem II.3.6, p. 81). When Ω is the algebraic closure of F, $(\Omega|F)*$ is the intersection of all real closures of F; when Ω is the maximal 2-extension of F, $(\Omega|F)*$ is the Pythagorean closure of F (see also [Be1], Satz 10). The field $(\Omega|F)*$ is called the Ω-Pythagorean closure of F. Since a field with a finite Pythagorean extension is itself Pythagorean (see [L1], Exercise VIII.17, p.254), $(\Omega|F)*$ is infinite dimensional over F, whenever F is not Pythagorean. It seems natural to investigate the relations between the space of orders of F, that of $(\Omega|F)*$ and the Galois group of latter over the former.

Section 1 discusses the natural bijective correspondence between orders on a field L and characters of the special group associated to L. We also present criteria guaranteeing that the morphism of special groups induced by the embedding of a field in an extension be injective and isometry preserving.

In section 2 it is shown that, if L/F is a formally real algebraic extension, then the group of field automorphisms of L leaving F fixed, can be embedded in the group of automorphisms of the reduced special group of L.

In section 3 we prove that if L is a formally real Galois extension of F, then the Galois group of L over F operates freely and continuously on the space of orders of L, and that the space of orbits of this action is closely related to the space of orders of the ground field. Moreover, if L is infinite dimensional over F and every order on F can be extended to L, then the orbits of the orders on L under this action, as well as the space of orders of L, are perfect compacts. (Theorem 3.10). When F is countable, we can do better : if L is a Galois extension of F, with Galois group T, and the canonical restriction map from the space of orders of L to the space of orders of X_F, $X_L \xrightarrow{\iota_*} X_F$, is surjective, then there is an isomorphism of T-spaces, between $X_L \xrightarrow{\iota_*} X_F$ and the trivial T-bundle over X_F (Theorem 3.11).

In section 4 we prove an isotropy reflection principle between F and $(\Omega|F)*$, phrased in terms of the reduced special groups naturally associated to these fields (Theorem 4.17).

In all that follows, F denotes a fixed formally real field, often referred to as the ground field. We work inside a fixed algebraic closure of F, written F^a; thus, unless explicitly stated otherwise, all fields herein are extension fields of F and subfields of F^a.

Let Ω be a prime-closed extension of F. To simplify exposition, we write

- F^Ω for$(\Omega|F)*$, the Ω-Pythagorean closure of F;

- F^π for the intersection of all Pythagorean subfields of F^a, that contain F. Thus, $F^\pi = (\Omega|F)*$, where Ω is the maximal 2-extension of F inside F^a;

- F^Σ for the intersection of all real closures of F in F^a; hence $F^\Sigma = (\Omega|F)*$, where $\Omega = F^a$.

Clearly, $F \subseteq F^\pi \subseteq F^\Omega \subseteq F^\Sigma$. Moreover, if L is a field, then

$$F \subseteq L \subseteq F^\Omega \ \Rightarrow\ L^\Omega = F^\Omega.$$

For a field L, let $\dot{L}$ be the multiplicative group of non-zero elements in L. As usual, write

$$\dot{L}^2 = \{a^2 : a \in \dot{L}\} \quad \text{and} \quad \Sigma\dot{L}^2 = \{\Sigma_{i=1}^n a_i^2 : n \geq 1, \{a_1, \ldots, a_n\} \subseteq \dot{L}\},$$

for the subgroups of $\dot{L}$ consisting of squares and sums of squares, respectively.

Our basic reference for the Theory of Special Groups is [DM1], although [DM2] contains the most of the needed material on these structures.

1 Orders, SG-characters and Complete Embeddings

We start collecting some basic facts about the reduced special groups associated to formally real fields, the majority of which appear in section 3 of chapter 1 of [DM1].

Let $G_{red}(F) = \dot{F}/\Sigma\dot{F}^2$; we write an element of $G_{red}(F)$ as $\overline{a}_F$, $a \in \dot{F}$. Whenever context allows, we drop the reference to F from the notation. We define a relation $\equiv$ in $G_{red}(F) \times G_{red}(F)$, called binary isometry by the rule

$$(\equiv) \qquad \langle \overline{u}, \overline{v} \rangle \equiv \langle \overline{x}, \overline{y} \rangle \quad \text{iff} \quad \overline{uv} = \overline{xy} \quad \text{and} \quad \left\{ \begin{array}{c} \exists\, s,\, t \in \Sigma\dot{F}^2, \\ \text{such that } ux = s + (xy)t. \end{array} \right.$$

We write $\overline{1}$ and $\overline{-1}$ in $G_{red}(F)$ as $1, -1$, respectively.

For $x \in \dot{F}$, define

$$D(1, \overline{x}) = \{\overline{u} \in G_{red}(F) : \exists\, s,\, t \in \Sigma\dot{F}^2, \text{ such that } u = s + xt\},$$

called the set of elements represented by $\langle 1, \overline{x} \rangle$ in $G_{red}(F)$. With this in hand, an equivalent formulation of $(\equiv)$ is

$$(\equiv') \qquad \langle \overline{u}, \overline{v} \rangle \equiv \langle \overline{x}, \overline{y} \rangle \quad \text{iff} \quad \overline{uv} = \overline{xy} \quad \text{and} \quad \overline{ux} \in D(1, \overline{xy}).$$

$G_{red}(F)$ is called the **reduced special group** of F, where reducibility means that for all $a \in \dot{F}$,

$$\langle a, a \rangle \equiv \langle 1, 1 \rangle \quad \text{iff} \quad \overline{a} = 1.$$

Let $\mathbb{Z}_2 = \{1, -1\}$ be the multiplicative subgroup of $\mathbb{Z}$. A **SG-character** of $G_{red}(F)$ is a group homomorphism, $\sigma : G_{red}(F) \longrightarrow \mathbb{Z}_2$, such that $\sigma(-1) = -1$, and for all $x \in F$,

$$\text{(ch)} \qquad \overline{x} \in \ker \sigma \quad \text{implies} \quad D(1, \overline{x}) \subseteq \ker \sigma.$$

Write X_F for the set of characters of $G_{red}(F)$. It is a *closed* set in the product topology of the power $2^{G_{red}(F)}$; hence, with the induced topology, it is Boolean space, with a basis of clopens given by finite intersections of sets of the type

$$[\overline{x} = 1] = \{\sigma \in X_F : \sigma(\overline{x}) = 1\}.$$

For $\sigma \in X_F$, define

$$\text{(C)} \qquad P_\sigma = \{0\} \cup \{x \in \dot{F} : \sigma(\overline{x}) = 1\},$$

where 0 is the additive neutral of F. Clearly, $\Sigma\dot{F}^2 \subseteq P_\sigma$. Moreover,

- If $x, y \in P_\sigma$, then $xy \in P_\sigma$, because σ is a homomorphism; further, since $\overline{x} = \overline{(1/x)}$, $x \in P_\sigma$ iff $1/x \in P_\sigma$;

- $-1 \notin P_\sigma$, because $\sigma(-1) = -1$;

- For $x \in P_\sigma$, note that $\overline{1+x} \in D(1, \overline{x})$; since σ is a SG-character, we get $(1 + x) \in P_\sigma$. It now follows easily that P_σ is closed under sum.

- If $x \in \dot{F}$, then either $\sigma(\overline{x}) = 1$ or $\sigma(\overline{-x}) = 1$ (but not both); thus, $P_\sigma \cup -P_\sigma = F$ and this union is disjoint.

Hence, P_σ is a *maximal cone* in F, corresponding to the set of positive elements of a *unique* ordering on F (see section 8.3 in [Co], p. 309 ff).

Conversely, if P is the positive cone of an ordering $\leq$ on F, define $\sigma_P : G_{red}(F) \longrightarrow \mathbb{Z}_2$, by

$$\text{For all } x \in \dot{F}, \ \ \sigma_P(\overline{x}) = 1 \quad \text{iff} \quad x \in P.$$

Note that $\sigma_P(1) = 1$ ($1 \in P$) and $\sigma_P(-1) = -1$, because $-1 \notin P$. The above definition is independent of representatives in the same square class, since P is closed under products and contains $\Sigma\dot{F}^2$. The fact that P is closed under multiplication and $P \cup -P = F$, implies that σ_P is a group homomorphism. If $\sigma_P(\overline{x}) = 1$ and that $\overline{y} \in D(1, \overline{x})$, then $y = s + tx$, where s and t are sums of squares in F. Since $\Sigma\dot{F}^2 \subseteq P$ and P is closed under sums, we conclude that $y \in P$ and $D(1, \overline{x}) \subseteq \ker \sigma_P$. Consequently, $\sigma_P \in X_F$.

It is easily verified that for all maximal cones P in F and all $\sigma \in X_F$,

$$\sigma_{P_\sigma} = \sigma \quad \text{and} \quad P_{\sigma_P} = P,$$

establishing a natural bijective correspondence between orders on F and SG-characters of $G_{red}(F)$. Note that if $a_1, \ldots, a_n \in \dot{F}$, then the orders in which all a_j's are positive correspond exactly to the characters that are equal to 1 in all $\overline{a}_j$'s, $1 \leq j \leq n$. Thus, the Harrison set $H(a_1, \ldots, a_n)$, a typical basic clopen in the usual topology on the space of orders of F, is taken to the the basic clopen $(\cap_{j=1}^n [\overline{a}_j = 1])$ in X_F : the natural correspondence between orders and SG-characters, described above, is also topological.

Now suppose that $F \subseteq L$ is a field extension, with L formally real. There is a natural group homomorphism

$$\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L), \text{ given by } \overline{x}_F \mapsto \overline{x}_L.$$

Observe that ι_{FL} takes -1 to -1. Moreover, if $x, y \in \dot{F}$,

$$\overline{x}_F \in D(1, \overline{y}_F) \quad \text{implies} \quad \overline{x}_L \in D(1, \overline{y}_L),$$

that is, ι_{FL} is a *morphism of special groups (SG-morphism)*. In general, ι_{FL} *is not* injective, for an element of $\dot{F}$ may be a sum of squares in $\dot{L}$, but not in F. Note that for every $\sigma \in X_L$, composition with ι_{FL} yields a SG-character of $G_{red}(F)$. Thus, ι_{FL} induces a map

$$\iota^*{}_{FL} : X_L \longrightarrow X_F,$$

that is, in fact, continuous. The next result gives equivalent sufficient conditions for ι_{FL} to be injective (compare with Theorem 5.2 in [DM1]) :

Lemma 1.1 *Notation as above, the following conditions are equivalent :*

(1) *All orders on F can be extended to orders on L.*

(2) *ι_{FL} is injective and all SG-characters of $G_{red}(F)$ can be extended to SG-characters of $G_{red}(L)$.*

(3) *For all forms φ, ψ over $G_{red}(F)$,*

$$\varphi \equiv_{G_{red}(F)} \psi \quad \textit{iff} \quad \iota_{FL} \star \varphi \equiv_{G_{red}(L)} \iota_{FL} \star \psi,$$

where $\iota_{FL} \star \langle a_1, \ldots, a_n \rangle = \langle \iota_{FL}(a_1), \ldots, \iota_{FL}(a_n) \rangle$ (the image of $\langle a_1, \ldots, a_n \rangle$ by ι_{FL}).

Proof. (1) $\Leftrightarrow$ (2) : Given the natural correspondence between SG-characters and orders, to show that (1) implies (2), it is enough to check that ι_{FL} is injective. If $\iota_{FL}(x_F) = 1$, then x is a sum of squares in L. If x was not in $\Sigma\dot{F}^2$, by Artin-Schreier, there would be an order on F in which $x < 0$. Clearly, this order cannot be extended to L. Hence, $\overline{x}_F = 1$ and ι_{FL} is indeed injective. The converse is similar and omitted. The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 5.2 in [DM1]. $\diamond$

Morphisms of special groups that satisfy condition (3) in Lemma 1.1 are called **complete embeddings**; conditions (2) and (3) are equivalent for reduced special groups in general. The reader can consult section 1 of chapter 5 in [DM1] for more details.

Example 1.2 By Lemma 8.3.6 in [Co] (p. 312), if L is an odd degree extension of F, or a quadratic extension of the form $F(\sqrt{s})$, where $s \in \Sigma\dot{F}^2$, then ι_{FL} is a complete embedding.

For odd degree extensions, a result of T. A. Springer (see Theorem II.5.3, pg.46 of [Sc]), guarantees that the embedding ι_{FL} has an even stronger property, namely, that it reflects isotropy (see Proposition 4.14). $\diamond$

Example 1.3 By Theorem II.3.6 in [Be2], if Ω is a prime-closed extension of F, then every order on F can be extended to F^{Ω}. Thus, $\iota_{FF^{\Omega}}$ is a complete embedding. In particular, all the SG-morphisms that come from the sequence

$$F \subseteq F^{\pi} \subseteq F^{\Omega} \subseteq F^{\Sigma},$$

are complete embeddings. $\diamond$

2 Groups of Automorphisms

If L is an extension of F, write $Aut_F(L)$ for the group of automorphisms of L that leave F pointwise fixed. If L is formally real and $f \in Aut_F(L)$, define

$$\overline{f} : G_{red}(L) \longrightarrow G_{red}(L), \quad \text{by} \quad \overline{f}(\overline{a}) = \overline{f^{-1}(a)}.$$

Note that if $\overline{a} = \overline{b}$, then $ab \in \Sigma\dot{L}^2$, and so $f^{-1}(ab) = f^{-1}(a)f^{-1}(b) \in \Sigma\dot{L}^2$. Thus, $\overline{f}$ is well defined. It is straightforward to verify that

1) $\overline{f}$ is a SG-automorphism of $G_{red}(L)$, i.e., a bijective SG-morphism such that for all x, $y \in \dot{L}$,

$$\overline{x} \in D(1, \overline{y}) \quad \textbf{iff} \quad \overline{f}(\overline{x}) \in D(1, \overline{f}(\overline{y})).$$

Moreover, the inverse of $\overline{f}$ is $\overline{f^{-1}}$.

2) $\overline{f \circ g} = \overline{g} \circ \overline{f}$ and $\overline{Id_L} = Id_{G_{red}(L)}$.

3) $\iota_{FL} = \overline{f} \circ \iota_{FL}$, that is, the following diagram is commutative :

$$\begin{tikzcd}
G_{red}(L) \arrow[rr, "\overline{f}"] & & G_{red}(L) \\
& G_{red}(F) \arrow[ul, "\iota_{FL}"] \arrow[ur, "\iota_{FL}"'] &
\end{tikzcd}$$

Although ι_{FL} might not be injective, an automorphism of $G_{red}(L)$ that satisfies (3), is called an automorphism over F or an F-automorphism. Write $Aut_F(G_{red}(L))$ for the group of F-automorphisms of $G_{red}(L)$.

By Item (2) above, $f \mapsto \overline{f}$ is a group anti-homomorphism, from $Aut_F(L)$ to $Aut_F(G_{red}(L))$. The reason for the defining $\overline{f}$ as above will become apparent in section 3.

We shall use the following extension result, that can be extracted from the proof of Theorem 1.3.2 in [BCR], particularly the paragraphs before and after Lemma 1.3.3 :

Proposition 2.4 *Let L be an ordered field, with real closure $\widetilde{L}$ and let R be a real closed field. Then, any order embedding $L \xrightarrow{h} R$* [1] *has a unique extension to an order embedding of $\widetilde{L}$ into R.* $\diamond$

Proposition 2.5 *With notation as above, let L be a formally real algebraic extension of F.*

a) The following are equivalent, for $f \in Aut_F(L)$:

(1) *For some order on L, f is an order automorphism of L* [2]*;*

(2) *For all $a \in \dot{L}$, $\overline{f(a)}_L = \overline{a}_L$;*

(3) $f = Id_L$.

b) The anti-homomorphism $f \in Aut_F(L) \mapsto \overline{f} \in Aut_F(G_{red}(L))$ is injective.

Proof. a) Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). To show that (1) implies (3), let $\leq$ be an ordering on L, which is preserved by f. Write $\widetilde{L}$ for the real closure of $\langle L, \leq \rangle$ in F^a. L being algebraic over F, $\widetilde{L}$ is also the real closure of $\langle F, \leq_{|F} \rangle$. By Proposition 2.4, f has a unique extension to an automorphism $\widetilde{f}$ of $\widetilde{L}$. On the

[1] I.e., $a \geq 0 \Rightarrow h(a) \geq 0$.

[2] $a \geq 0 \Rightarrow f(a) \geq 0$, $\forall\, a \in \dot{L}$.

other hand, because $f_{|F} = Id_F$ and $\widetilde{L}$ is the real closure of F with the induced order, we must have $\widetilde{f} = Id_{\widetilde{L}}$. It follows that $\widetilde{f}_{|L} = f = Id_L$.

b) Suppose that $f, g \in Aut_F(L)$ are such that $\overline{f} = \overline{g}$. Then,

$$\overline{g} \circ \overline{f}^{-1} = \overline{g} \circ \overline{f^{-1}} = \overline{f^{-1} \circ g} = Id_{G_{red}(L)}.$$

The preceding equation means that for all $a \in \dot{L}$, $\overline{[g^{-1} \circ f](a)} = \overline{a}$. By (a), $g^{-1} \circ f = Id_L$, showing that $f = g$, as desired. ◇

Remark 2.6 If we drop the assumption that our fields are formally real, there are perturbations of the identity by sums of squares, distinct from the identity. One example that comes to mind is the Frobënius automorphism, $x \mapsto x^p$, p an odd prime. ◇

As observed in section 1, the natural map inclusion of F into a formally real algebraic extension L, induces a SG-morphism $\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L)$, which in turn yields a continuous map $\iota^*_{FL} : X_L \longrightarrow X_F$. We may ask if there is a geometrical meaning to the fiber of $\iota^*{}_{FL}$ over the points of X_F. Or equivalently, what is the geometrical meaning, for $\sigma, \tau \in X_L$, of the equation $\sigma \circ \iota_{FL} = \tau \circ \iota_{FL}$. The answer is the content of the next Proposition. Before its statement, recall that an algebraic extension L/F is **normal** if every F-embedding of L into F^a is an automorphism of L. Or equivalently, if every F-automorphism of any field containing L, restricts to an F-automorphism of L (see Corollary V.3.9 in [Co]).

Proposition 2.7 *With notation as above, let L be a formally real algebraic extension of F. For $\sigma, \tau \in G_{red}(L)$, consider the following conditions :*

(1) $\sigma \circ \iota_{FL} = \tau \circ \iota_{FL}$;

(2) *The orders on L, associated to σ and τ, respectively, coincide in F.*

(3) *The orders on L, associated to σ and τ, respectively, are conjugate by a F-automorphism of L.*

(4) *There is $f \in Aut_F(L)$ such that $\tau = \sigma \circ \overline{f}$.*

Then, (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). *If L is a normal extension of F, then all four conditions are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) : It suffices to check that if $\sigma \in X_L$ and $\mu = \sigma \circ \iota_{FL}$ then $P_\sigma \cap F = P_\mu$, where $P_{(\cdot)}$ is the maximal cone associated to SG-characters via relation (C) in section 1. But for all $x \in \dot{F}$, we have

$$[\sigma \circ \iota_{FL}](\overline{x}_F) = \sigma(\overline{x}_L),$$

and the desired conclusion follows immediately.

It is clear that (3) is equivalent to (4), as well as that these conditions imply (2) (or (1)). It remains to show, for instance, that (2) implies (3). Suppose that σ and τ are orders on L, that coincide in F. Let

— $\widetilde{F}$ be the real closure of F inside F^a;

— L_σ be the real closure of $\langle L, \sigma \rangle$ inside F^a;

— L_τ be the real closure of $\langle L, \tau \rangle$ inside F^a.

Since L is algebraic over F, it follows from Theorem 1.3.2 in [BCR] — or Proposition 2.4 and Zorn's Lemma —, that there are F-order isomorphisms $\lambda_\sigma : \widetilde{F} \longrightarrow L_\sigma$ and $\lambda_\tau : \widetilde{F} \longrightarrow L_\tau$; clearly, $\lambda = \lambda_\tau \circ \lambda_\sigma^{-1}$ is an F-order isomorphism from L_σ onto L_τ. Let $\mu : L_\sigma(\sqrt{-1}) \longrightarrow L_\tau(\sqrt{-1})$ be the unique extension of λ to an F-isomorphism, taking $\sqrt{-1}$ to itself. By Theorem VIII.3.7 in [Co], we have $F^a = L_\sigma(\sqrt{-1}) = L_\tau(\sqrt{-1})$ and so μ is an F-automorphism of F^a. Since L is normal, we conclude that $\lambda_{|L}$ is an F-automorphism of L; by construction, $\lambda_{|L}$ takes the order σ to the order τ, as desired. ◇

With notation as in Example 1.3, we have

Corollary 2.8 *Let F be a formally real field and Ω a prime-closed extension of F. For all σ, $\tau \in X_{F^\Omega}$, the following are equivalent :*

(1) $\sigma \circ \iota_{FF^\Omega} = \tau \circ \iota_{FF^\Omega}$;

(2) *There is $f \in Aut_F(F^\Omega)$ such that $\tau = \sigma \circ \overline{f}$.* ◇

3 Galois Groups and Orders

Let L be a formally real algebraic extension of F. We start by constructing a group homomorphism from $Aut_F(L)$ into $Homeo(X_L)$, the group of homeomorphisms of the space of orders of L. For $f \in Aut_F(L)$, define $f_* : X_L \longrightarrow X_L$, by $\sigma \mapsto \sigma \circ \overline{f}$.

$$\begin{tikzcd} G_{red}(L) \arrow[rr, "\overline{f}"] \arrow[dr, "f_*(\sigma)"'] & & G_{red}(L) \arrow[dl, "\sigma"] \\ & \{\pm 1\} & \end{tikzcd}$$

Note that f_* is bijective, because the same is true of $\overline{f}$; $\sigma \circ \overline{f}$ is a SG-character of $G_{red}(L)$ because $\overline{f}$ is a SG-morphism; continuity of f_* comes from the identity

$$f_*^{-1}([\overline{a}=1]) = [\overline{f^{-1}(a)}=1],$$

which holds for all $a \in L$. Since X_L is a compact Hausdorff space and f_* is a continuous bijective map, we have $f_* \in Homeo(X_L)$. Further, $Id_* = Id_{X_L}$ and

$$(f \circ g)_*(\sigma) = \sigma \circ (\overline{g \circ f}) = \sigma \circ (\overline{g} \circ \overline{f}) = (\sigma \circ \overline{g}) \circ \overline{f} = f_*(g_*(\sigma)),$$

for all $f, g \in Aut_F(L)$ and all $\sigma \in X_L$. Hence, $f \mapsto f_*$ is a group homomorphism from $Aut_F(L)$ into $Homeo(X_L)$.

Remark 3.9 a) Since we wished $f \mapsto f_*$ to be a group homomorphism, it was necessary to define $\overline{f}$ as in section 2.

b) For all $f \in Aut_F(L)$, f_* is, in fact, an isomorphism of abstract order spaces, because it is the dual of the SG-automorphism $\overline{f}$. $\diamond$

Theorem 3.10 *Let L be a formally real Galois extension of F. Let $T =_{def} Aut_F(L)$ be the Galois group of L over F. Then,*

a) The map

$$T \times X_L \longrightarrow X_L, \quad \langle f, \sigma \rangle \mapsto f_*(\sigma) =_{def} f \cdot \sigma,$$

defines a free and continuous T-action on X_L.

b) If X_L/T is the space of orbits of this T-action on X_L, with the quotient topology, there is a continuous injection $\omega : X_L/T \longrightarrow X_F$, such that the following diagram is commutative :

$$\begin{tikzcd}
X_L \arrow[rr, "p_T"] \arrow[dr, "\iota_*"'] & & X_L/T \arrow[dl, "\omega"] \\
& X_F &
\end{tikzcd}$$

with ι_* the continuous dual of the SG-morphism $\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L)$ and p_T the canonical quotient map.

c) The continuous injection ω is a homeomorphism iff ι_{FL} is a complete embedding.

d) If $[L : F]$ is infinite, every orbit of the action of T in X_L, as well as X_L itself, are perfect compacts.

Proof. Our basic reference for actions of topological groups on topological spaces is [Br], particularly Chapter 1.

a) The displayed rule will be a T-action on X_L if for all $f, g \in T$

(A) For all $\sigma \in X_L$, $(f \circ g) \cdot \sigma = f \cdot (g \cdot \sigma)$;

(B) For all $\sigma \in X_L$, $Id_L \cdot \sigma = \sigma$;

(C) (Freeness) If $f \cdot \sigma = \sigma$, for some $\sigma \in X_L$, then $f = Id$.

(A) and (B) are clear; for (C), suppose that $f \cdot \sigma = \sigma$, for some $\sigma \in X_L$. If $\leq$ is the order on L associated to σ, then f is an automorphism of the ordered field $\langle L, \leq \rangle$. By Proposition 2.5.(a), $f = Id_L$, as required.

We now verify that $\langle f, \sigma \rangle \mapsto f \cdot \sigma$ is continuous, where $T \times X_L$ has the product topology. Recall that T is the projective limit of the finite groups $Aut_F(K)$, as K ranges over the finite Galois extensions of F, contained in L (see Theorem V.6.6 in [Co]). For $f \in Aut_F(L)$, a basic clopen neighborhood of f is given by

$$\mathcal{V}(f, K) = \{g \in Aut_F(L) : f_{|K} = g_{|K}\},$$

where K is a finite normal extension of F contained in L. With these preliminaries, let $a_1, \ldots, a_n$ be elements of L and suppose that $(f \cdot \sigma) \in V =_{def} \bigcap_{j=1}^{n} [\overline{a}_j = 1]$; we must find a clopen U containing f in T and a clopen W containing σ in X_L, such that $\langle g, \tau \rangle \in U \times W$ implies $g \cdot \tau \in V$. Let N be a finite normal extension of F, containing $\{a_1, \ldots, a_n\}$; it suffices, for example, to take N as the splitting field of the minimal polynomials of the a_j's over F. Set

$$U = \mathcal{V}(f, N) \quad \text{and} \quad W = \bigcap_{j=1}^{n} [\overline{f^{-1}(a_j)} = 1].$$

Since $f \cdot \sigma(a_j) = \sigma(\overline{f^{-1}(a_j)}) = 1$, it is clear that $\sigma \in W$. If $\langle g, \tau \rangle \in U \times W$, we have

$$\text{For } 1 \leq j \leq n, \ \ [g \cdot \tau](\overline{a}_j) = \tau(\overline{g}(\overline{a}_j)).$$

Hence, to show that $g \cdot \tau \in V$, it is enough to check that $g^{-1}(a_j) = f^{-1}(a_j)$. But this is immediate from the fact that, for $g \in \mathcal{V}(f, N)$, $f(a_j) = g(a_j)$, for all $1 \leq j \leq n$.

As is the case for general actions, if we fix $f \in T$, the above action induces the homeomorphism f_* from X_L to X_L.

b) Let X_L/T be the space of orbits of the T-action on X_L (see section 3, Chapter 1 in [Br]). Let

$$p_T : X_L \longrightarrow X_L/T$$

be the natural projection. Recall that if $\sigma \in X_L$, the <u>orbit of σ</u> is the set

$$T(\sigma) = \{f \cdot \sigma : f \in T\};$$

$T(\sigma)$ is the equivalence class of σ under the equivalence relation

$$\sigma \sim \tau \quad \text{iff} \quad \exists\, f \in T(f \cdot \sigma = \tau).$$

As a set, X_L/T is the set of equivalence classes (or orbits) of this equivalence relation. We endow X_L/T with the quotient topology induced by p_T :

A subset of X_L/T is open iff its inverse image by p_T is open in X_L.

A subset $A \subseteq X_L$ is invariant if $f_*(A) \subseteq A$, for all $f \in T$. For $A \subseteq X_L$, the saturation of A is

$$T(A) = \bigcup_{f\in T} f_*(A),$$

being the least invariant subset of X_L containing A. Since the f_* are all homeomorphisms, it is clear that if U is open in X_L, the same is true of $T(U)$. In particular, p_T is continuous and *open*. By Theorem I.3.1 in [Br], X_L/T is Hausdorff and compact. But we need a bit more, namely

Fact 1. X_L/T is a Boolean space and the diagram $X_L \xrightarrow{p_T} X_L/T$ has the following universal property :

If $X_L \xrightarrow{\eta} Y$ is a continuous map such that for all $y \in Y$, $f^{-1}(y)$ is invariant in X_L, then, there is a unique continuous map $\eta_T : X_L/T \longrightarrow Y$, making the following diagram commutative :

(*)
$$\begin{tikzcd} X_L \arrow[rr, "p_T"] \arrow[dr, "\eta"'] & & X_L/T \arrow[dl, "\eta_T"] \\ & Y & \end{tikzcd}$$

Proof. Another way to view the saturation of a subset A in X_L is as the image, by the operation $T \times X_L \longrightarrow X_L$, of the product $T \times A$. Since X_L is Boolean and T is compact, we get that the saturation of any clopen set is again clopen. To see this, let V be a clopen set in X_L. As already observed, $T(V)$ is open; on the other hand, $T(V)$ is the image by a continuous map of the compact set $T \times V$, thus being compact in X_L. Since a compact subset of a Hausdorff space is always closed, $T(V)$ is closed.

To show that X_L/T is Boolean, it is enough to verify that all of its opens can be written as unions of clopens. For U open in X_L/T, write

$$p_T^{-1}(U) = \bigcup_{i\in I} V_i,$$

where each V_i is clopen in X_L. Since $p_T^{-1}(U)$ is invariant, we have

$$p_T^{-1}(U) = \bigcup_{i\in I} T(V_i),$$

with each $T(V_i)$ clopen in X_L. It follows easily from the definition of quotient topology, that the image by p_T of an invariant clopen is a clopen in X_L/T. Thus,

$$U = \bigcup_{i \in I} p_T(T(V_i)),$$

is a rendering of U as a union of clopens. To verify (*), for $T(\sigma) \in X_L/T$, set $\eta_T(T(\sigma)) = \eta(\sigma)$; then η_T is the unique continuous map making the displayed diagram commutative. Details are left to the reader.

As in section 1, the inclusion $F \subseteq L$ yields a SG-morphism

$$\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L),$$

which in turn originates, by composition, a continuous map $\iota_* : X_L \longrightarrow X_F$. By Proposition 2.7, for all $\sigma, \tau \in X_L$

(1) $\qquad \iota_*(\sigma) = \iota_*(\tau) \quad \text{iff} \quad \sigma \circ \iota_{FL} = \tau \circ \iota_{FL} \quad \text{iff} \quad T(\sigma) = T(\tau),$

i.e., $\iota_*^{-1}(\mu)$ is a T-invariant (possibly empty) subset of X_L, for all $\mu \in X_F$. By (*) in Fact 1, there is a unique continuous map $\omega : X_L/T \longrightarrow X_F$, such that $\omega \circ p_T = \iota_*$.

$$\begin{tikzcd} X_L \arrow[rr, "p_T"] \arrow[dr, "\iota_*"'] & & X_L/T \arrow[dl, "\omega"] \\ & X_F & \end{tikzcd}$$

Now observe that if $\omega(T(\sigma)) = \omega(T(\tau))$, then $\iota_*(\sigma) = \iota_*(\tau)$, which by (1), implies $T(\sigma) = T(\tau)$, showing that ω is indeed injective.

c) Since we are dealing with compact Hausdorff spaces and ω is continuous and injective, it is enough to prove that ω is surjective iff ι_{FL} is a complete embedding. But we have

ω is surjective iff ι_* is surjective

iff every ordering on F extends to an ordering in L,

and so, ω is onto iff ι_{FL} is a complete embedding (see Lemma 1.1).

d) It is enough to prove that every orbit in X_L is a perfect compact. Since the action of T is free, the isotropy subgroup of σ in T,

$$T_\sigma = \{f \in T : f \cdot \sigma = \sigma\},$$

is equal to $\{Id_L\}$. Because T is compact, Proposition I.4.1 in [Br] guarantees that the map

$$T \longrightarrow T(\sigma), \quad f \mapsto f \cdot \sigma,$$

is a homeomorphism, where $T(\sigma)$ has the topology induced by X_L. When $[L : F]$ is infinite, T is an infinite compact topological group, and the underlying space of any such group must be perfect. But then, $T(\sigma)$ is also a perfect compact, as desired. ◇

When F is a countable field, we have

Theorem 3.11 *Let F be a countable formally real field. Let L be a Galois extension of F, such that every order on F has an extension to L. Let T be the Galois group of L over F. Then, there is an equivariant homeomorphism, $h : T \times X_F \longrightarrow X_L$, such that the following diagram is commutative*

$$\begin{tikzcd} T \times X_F \arrow[rr, "h"] \arrow[dr, "\pi"'] & & X_L \arrow[dl, "\iota_*"] \\ & X_F & \end{tikzcd}$$

where π is the canonical projection and ι_* is the dual of the SG-morphism ι_{FL}.

Proof. If F is a countable field, then so is F^a. Consequently, the normal algebraic extension L is also countable. Hence,

– X_L and and X_F are Boolean metric spaces, since they are compact and have a countable basis of clopens, consisting of finite intersections of sets of the type $[a = 1]$. In particular, X_L and X_F are complete metric spaces.

– The Galois group of L over F, T, is the projective limit of a countable family of finite groups, because there are only countably many normal fields between F and L. Therefore, T is a compact metrizable topological group.

The stage is set for an application of

Theorem A *If X is a paracompact and zero-dimensional Hausdorff space and Y is a complete metric space, then every lower semi-continuous function Φ, from X to the closed, non-empty subsets of Y, admits a continuous selection.*

This result is due to E. Michael ([Mi1]); Chapter 1 of [Pa] contains a nice exposition of this and related results. Write 2^Y for the set of closed, non-empty subsets of a topological space Y.

Consider the continuous map $\iota_* : X_L \longrightarrow X_F$; since every order on F can be extended to an order in L, ι_{FL} is a complete embedding (Lemma 1.1). Thus, ι_* is a surjection. By item (c) in Theorem 3.10, there is a homeomorphism $\omega : X_L/T \longrightarrow X_F$, such that $\omega \circ p_T = \iota_*$. Since p_T is an open map, we conclude that the same is true of ι_*. Recall that by Proposition 2.7,

$$\text{(1)} \qquad \text{For all } \tau \in X_F, \quad {\iota_*}^{-1}(\tau) = T(\sigma),$$

where σ is any extension of τ to an order on L. Now define

$$\Phi : X_F \longrightarrow 2^{X_L}, \text{ by } \Phi(\tau) = {\iota_*}^{-1}(\tau).$$

Since ι_* is a continuous *open* map, it follows from Examples 1.1 and 1.1* in Chapter 1 of [Pa] (p. 3 and 4), that Φ is lower semi-continuous. Since X_F is Boolean and metric, it is paracompact and zero-dimensional; hence Theorem A applies, to yield a continuous $s : X_F \longrightarrow X_L$, such that $s(\tau) \in {\iota_*}^{-1}(\tau)$, for all $\tau \in X_F$. Define

$$h : T \times X_F \longrightarrow X_L, \quad h(f, \tau) = f \cdot s(\tau).$$

If $\langle f, \tau \rangle, \langle g, \mu \rangle \in T \times X_F$ are such that $f \cdot s(\tau) = g \cdot s(\mu)$, then, from (1) comes

$$g^{-1} \cdot (f \cdot s(\tau)) = s(\mu) \;\Rightarrow\; \tau = \iota_*(g^{-1} \cdot (f \cdot s(\tau))) = \iota_*(s(\mu)) = \mu.$$

Hence, since the action is free, $(g^{-1} \circ f) \cdot s(\tau) = s(\tau)$ implies that $f = g$. We have just verified that h is injective. Since it is clearly surjective, we conclude that h is bijective. To show that h is a homeomorphism it is, once again, enough to check continuity. Let V be an open neighborhood of $f \cdot s(\tau)$ in X_L. Since the T-action is continuous, there is an open U in T, containing f, and an open W in X_L, containing $s(\tau)$, such that

$$\text{(2)} \qquad \langle g, \sigma \rangle \in U \times W \text{ implies } g \cdot \sigma \in V.$$

Let $W' = s^{-1}(W)$; by the continuity of s, W' is an open set in X_F. Consider $V' = U \times W'$; clearly, V' is open in $T \times X_F$. But then,

$$\langle g, \mu \rangle \in V' \implies \langle g, s(\mu) \rangle \in U \times W \overset{(2)}{\implies} g \cdot s(\mu) \in V,$$

completing the proof that h is continuous. It is straightforward that the diagram displayed in the statement is commutative. Note that for all $f \in T$ and all $\langle g, \tau \rangle \in T \times X_F$, we have

$$\text{(3)} \qquad f \cdot h(g, \tau) = f \cdot (g \cdot s(\tau)) = (f \circ g) \cdot s(\tau) = h(f \circ g, \tau).$$

There is a natural T-action on the topological product $T \times X_F$, given by

$$f \cdot \langle g, \tau \rangle = \langle f \circ g, \tau \rangle.$$

Clearly, this action is free. Then, equation (3) guarantees that h is an equivariant map from $T \times X_F$ to X_L, completing the proof. ◇

Since for all prime-closed Ω containing a formally real field F, F^{Ω} is a Galois extension of F, to which every order on F can be extended, Theorems 3.10 and 3.11 apply, *ipsis literis*, to this situation.

We end this section with another proof that the orbits of the action of the

Galois of an infinite extension are perfect. Note that, since the Galois group operates by homeomorphisms, if an orbit of this action had an isolated point, all of its points would be isolated (in the induced topology). Since the orbit is compact, that would imply that it must be finite. Thus, to show that each orbit is perfect, it is enough to prove

Proposition 3.12 *Let F be a formally real, non-Pythagorean field, and let Ω be a prime-closed extension of F. Then, each order on F has, at least, continuum many distinct extensions to F^{Ω}.*

Proof. Fix an order τ in F. One should keep in mind the fact, due to Diller and Dress, that a finite extension of a non-Pythagorean field is also non-Pythagorean.

Write 2^{n-1} for the set of maps from $\{1, 2, \ldots, n\}$ into $\{0, 1\}$, that take 1 to 1. By induction on $n \geq 1$, we construct a sequence of fields

$$F = F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \subseteq F_{n+1} \subseteq \ldots \subseteq F^{\Omega},$$

and a sequence of sets of 2^{n-1} distinct orders on F_n,

$$\mathcal{O}_n = \{\sigma_s : s \in 2^{n-1}\},$$

such that $\mathcal{O}_1 = \{\tau\}$ and for all $n \geq 1$,

i) $F_{n+1} = F_n(\sqrt{1+a^2})$, is a proper quadratic extension of F_n, with $a \in \dot{F}_n$;

ii) For all $s \in 2^n$, σ_s is an extension to F_{n+1} of σ_t, where $t = s_{|\{1,2,\ldots,n\}}$.

Suppose that F_n and $\mathcal{O}_n$ have already been constructed and satisfy properties (i) and (ii), for all $1 \leq k \leq n$. Since F_n is a finite extension of F, F_n cannot be Pythagorean. Thus, there is $a \in \dot{F}_n$, such that $1 + a^2$ is not a square in F_n. Set $F_{n+1} = F_n(\sqrt{1+a^2}) \subseteq F^{\Omega}$. For $t \in 2^{n-1}$, define the extensions of t by 0 and 1, $t\frown 0$, $t\frown 1 \in 2^n$ by

$$t\frown 0\,(k) = \begin{cases} t(k) & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k = n+1 \end{cases} \qquad t\frown 1\,(k) = \begin{cases} t(k) & \text{if } 1 \leq k \leq n \\ 1 & \text{if } k = n+1 \end{cases}$$

Clearly, every $s \in 2^n$ is of the form $t\frown i$, $(i = 0, 1)$, for some $t \in 2^{n-1}$. Now define

$$\sigma_{t\frown i} = \begin{cases} \text{the extension of } \sigma_t \text{ to } F_{n+1} \text{ such that } \sqrt{1+a^2} < 0 & \text{if } i = 0 \\ \text{the extension of } \sigma_t \text{ to } F_{n+1} \text{ such that } \sqrt{1+a^2} > 0 & \text{if } i = 1 \end{cases}$$

and set $\mathcal{O}_{n+1} = \{\sigma_{t\frown i} : t \in 2^n \text{ and } i = 0, 1\}$. It is clear that $\langle F_{n+1}, \mathcal{O}_{n+1} \rangle$ satisfies the required properties.

Let $L = \bigcup_{n\geq 1} F_n$; any map $h : \mathbb{N} \longrightarrow \{0, 1\}$, such that $h(1) = 1$, determines a family of compatible orderings

$$\langle F_n, \sigma_{h_{|\{1,\ldots,n\}}} \rangle$$

in the tower $\{F_n : n \geq 1\}$, which are all extensions of τ on F; this family of compatible orderings defines an ordering σ_h on L, whose restriction to each F_n is $\sigma_{h_{|\{1,\ldots,n\}}}$. Thus, σ_h is an extension of τ. Note that $h \neq k$ implies $\sigma_h \neq \sigma_k$, because if $n = min\ \{j \geq 2 : h(j) \neq k(j)\}$, then, by construction, σ_h and σ_k are distinct on F_n. We have just shown that L has continuum many distinct orders, all of which are extensions of τ. Since $F \subseteq L \subseteq F^\Omega$, we have $L^\Omega = F^\Omega$; hence, each order on L can be extended to F^Ω. Consequently, τ has continuum many distinct extensions to F^Ω, ending the proof. $\diamond$

4 An Isotropy Reflection Principle

Definition 4.13 *Let $G \xrightarrow{f} H$ be a morphism of special groups. We say that f* **reflects isotropy** *if for all forms $\varphi = \langle a_1, \ldots, a_n \rangle$ over G,*

$$f \star \varphi \text{ isotropic in } H \quad \text{implies} \quad \varphi \text{ isotropic in } G,$$

where $f \star \varphi = \langle f(a_1), \ldots, f(a_n) \rangle$ is the image form in H.

The following result appears as Proposition 5.32 in [DM1] :

Proposition 4.14 *If $G \xrightarrow{f} H$ reflects isotropy, then f is a complete embedding, i.e., for all forms φ, ψ, of dimension $n \geq 1$ over G,*

$$\varphi \equiv_G \psi \quad \textit{iff} \quad f \star \varphi \equiv_H f \star \psi,$$

where $\equiv_G$, $\equiv_H$ denote the isometry relation in G and H, respectively.

If G is a special group, write $Sat(G)$ for the least saturated subgroup of G, that is,

$$Sat(G) = \textstyle\bigcup_{k \geq 1} D_G(2^k),$$

where $2^k = \begin{cases} \bigotimes_{i=1}^k \langle 1, 1 \rangle & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \end{cases}$.

A special group is **formally real** iff $-1 \notin Sat(G)$. In this case, the quotient $G/Sat(G)$ is a *reduced* special group, written G_{red}. There is a natural SG-morphism

$$\pi : G \longrightarrow G_{red}, \ \ \pi(a) = a/Sat(G) \ \ \text{(the class of } a \text{ modulo } Sat(G)).$$

For n-forms φ, ψ over G, Proposition 2.21.(a) of [DM1] yields

$$\pi \star \varphi \equiv_{G_{red}} \pi \star \psi \quad \text{iff} \quad \text{For some integer } k \geq 0,\ \ 2^k \otimes \varphi \equiv_G 2^k \otimes \psi. \qquad (*)$$

If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over G_{red}, since π is surjective, we can always find a n-form θ in G, such that $\pi \star \theta = \varphi$; θ is called a **lifting** of φ to G.

Example 4.15 Let F be a field of characteristic $\neq 2$. As in section 3 of chapter 1 of [DM1], we can associate to F two special groups, $G(F)$ and $G_{red}(F)$; the latter, we have already presented in section 1. To describe the former, set $G(F) = \dot{F}/\dot{F}^2$, and write $\widehat{a}_F$ for the square class of $a \in \dot{F}$. We select $\widehat{-1}_F$ as the distinguished element of $G(F)$ and write it as -1.

$$\langle \widehat{a}_F, \widehat{b}_F \rangle \equiv_{G(F)} \langle \widehat{c}_F, \widehat{d}_F \rangle \quad \text{iff} \quad \begin{cases} \widehat{ab}_F = \widehat{cd}_F \text{ and } \exists\, x, y \in F \\ \text{such that } ac = x^2 + y^2(cd). \end{cases}$$

Then $\langle G(F), \equiv_{G(F)}, -1 \rangle$ is a special group. Moreover, $G(F)$ is formally real iff F is formally real. In this case, the SG-morphism π is given by $\widehat{a}_F \mapsto \overline{a}_F$, where $\overline{a}_F$ is the class of $a \in \dot{F}$ modulo sums of squares; relation (*) above is a well-known connection, due to Pfister, between the reduced and non-reduced theory of quadratic forms over F. When the field F is clear from context, we drop its mention from the notation.

A field extension $F \subseteq L$ induces a SG-morphism, $\eta_{FL} : G(F) \longrightarrow G(L)$, in a natural way : $\widehat{a}_F \mapsto \widehat{a}_L$. When L is formally real, we have a commutative diagram

$$\text{(D)} \qquad \begin{array}{ccc} G(F) & \xrightarrow{\eta_{FL}} & G(L) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G_{red}(F) & \xrightarrow[\iota_{FL}]{} & G_{red}(L) \end{array}$$

As examples, we state some of the results we shall need, in the language of special groups.

(I) A regular quadratic form over F of dimension $n \geq 1$, is isotropic iff it has a non-zero isotropic vector in F^n. In the language of special groups, let $\varphi = \langle \widehat{a}_1, \ldots, \widehat{a}_n \rangle$ be a n-form over $G(F)$. Then,

$$\varphi \equiv_G \langle 1, -1 \rangle \oplus \lambda \quad \text{iff} \quad \begin{cases} \exists\, (t_1, \ldots, t_n) \in F^n, \text{ not all zero,} \\ \quad \text{such that } \sum_{i=1}^n a_i t_i^2 = 0, \end{cases}$$

where λ is a $(n-2)$-form over $G(F)$. Since a SG-morphism preserves isometry in all dimensions, it is clear that the isotropy of φ in $G(F)$ implies that of $\pi \star \varphi$ in $G_{red}(F)$.

(II) If L is an odd-dimensional extension of F, Springer's Theorem (Theorem II.5.3, p.46 of [Sc]) may be stated as

(S) $\qquad \eta_{FL}$ reflects isotropy.

(III) Lemma VII.3.1 in p. 200 of [L1], can be phrased as follows : Let $L = F(\sqrt{d})$ be a quadratic extension of F and let φ be an anisotropic n-form over $G(F)$. If $\eta_{FL} \star \varphi$ is isotropic in $G(L)$, then there is $a \in \dot{F}$, such that

$$\varphi \equiv_{G(F)} \widehat{a}\langle 1, -\widehat{d}\rangle \oplus \lambda, \tag{L}$$

where λ is a $(n-2)$-form over $G(F)$. ◇

Proposition 4.16 *Let F be a formally real field. If L is an odd-dimensional extension of F, or an extension of the form $F(\sqrt{\beta})$, where $\beta \in \Sigma\dot{F}^2$, then the SG-morphism*

$$\iota_{FL} : G_{red}(F) \longrightarrow G_{red}(L),$$

reflects isotropy.

Proof. The method of proof for both cases is similar, and so we treat them in parallel. Let L be a field extension of F, as in the statement. Let φ be a form over $G_{red}(F)$ such that $\iota_{FL} \star \varphi$ is isotropic in $G_{red}(L)$. Let θ be a lifting of φ to $G(F)$. By the commutativity of the diagram (D) in Example 4.15, we have

$$\pi \star (\eta_{FL} \star \theta) = \iota_{FL} \star (\pi \star \theta) = \iota_{FL} \star \varphi.$$

It follows that $\pi \star (\eta_{FL} \star \theta)$ is isotropic in $G_{red}(L)$. By (*) above, there is a integer $k \geq 0$ such that $2^k \otimes (\eta_{FL} \star \theta) = \eta_{FL} \star (2^k \otimes \theta)$ is isotropic in $G(L)$. Now,

a) If $[L : F]$ is odd, then the formulation (S) of Springer's Theorem described in (II) of Example 4.15, guarantees that $2^k \otimes \theta$ is isotropic in $G(F)$, that is, there is a $(n-2)$-form ξ over $G(F)$, such that $2^k \otimes \theta \equiv_{G(F)} \langle 1, -1\rangle \oplus \xi$; then, since π is a SG-morphism, we get

$$\pi \star (2^k \otimes \theta) = 2^k \otimes (\pi \star \theta) = 2^k \otimes \varphi \equiv_{G_{red}(F)} \langle 1, -1\rangle \oplus (\pi \star \xi),$$

showing that $2^k \otimes \varphi$ is isotropic in $G_{red}(F)$. Since $G_{red}(F)$ is a reduced special group, Proposition 1.6.(e) in [DM1] yields the isotropy of φ in $G_{red}(F)$, as desired.

b) Suppose that $L = F(\sqrt{\beta})$, with $\beta \in \Sigma\dot{F}^2$. If $2^k \otimes \theta$ is isotropic in $G(F)$, the same reasoning as in (a) will show that φ is isotropic in $G_{red}(F)$. If $2^k \otimes \theta$ is anisotropic in $G(F)$, formula (L) in item (III) of Example 4.15 applies, to yield an element $a \in \dot{F}$, such that

$$2^k \otimes \theta \equiv_{G(F)} \widehat{a}\langle 1, -\widehat{\beta}\rangle \oplus \lambda,$$

where λ is a $(n-2)$-form over $G(F)$. Since $\beta \in \Sigma\dot{F}^2$, $\pi(-\widehat{\beta}) = \overline{-\beta} = -1$ in $G_{red}(F)$. But then, taking the image of this isometry by π, we get

$$2^k \otimes \varphi \equiv_{G_{red}(F)} \overline{a}\langle 1, -1\rangle \oplus (\pi \star \lambda) \equiv_{G_{red}(F)} \langle 1, -1\rangle \oplus (\pi \star \lambda),$$

and, as above, φ must be isotropic in $G_{red}(F)$, completing the proof. ◇

We now have

Theorem 4.17 *Let F be a formally real field and Ω a prime-closed extension of F. Then, the natural map* $\iota_{FF^\Omega} : G_{red}(F) \longrightarrow G_{red}(F^\Omega)$ *reflects isotropy.*

Proof. We start with the following

Fact 1. *Let $F \subseteq L \subseteq N \subseteq F^\Omega$ be fields, such that N is Galois over L, with index 2^n, $n \geq 1$. Then, there is a tower of fields*

$$L = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n = N,$$

such that $L_{i+1} = L_i(\sqrt{\beta_i})$, *where* $\beta_i \in \Sigma \dot{L_i}^2$.

Proof. First suppose that $n = 1$ and write $N = L(\alpha)$, $\alpha \in N$. Let $p(X) = X^2 + aX + b$ be the monic irreducible polynomial of α over L. It is clear that $p(X)$ splits into linear factors in $N \subseteq F^\Omega$. Since $L^\Omega = F^\Omega$, this means that the discriminant of $p(X)$, $a^2 - 4b$, must be positive in all Ω-real closures of L. By Theorem II.2.4 (p. 78) of [Be2], all orders on L are induced by a Ω-real closure of L. Thus, we must have $a^2 - 4b = \beta$ where β is a sum of squares in $\dot{L}$. From $b = \dfrac{a^2 - \beta}{4}$ comes

$$p(X) = X^2 + aX + b = (X + a/2)^2 - \beta/4,$$

and so $N = L(\sqrt{\beta})$, as desired. Now, proceed by induction. If the result is true for $(n-1) \geq 1$, since the Galois group $\mathcal{G}$ of N over L is a 2-group, it follows from the Corollary to Theorem I.6.3 (p. 25) in [La], that $\mathcal{G}$ has a normal subgroup, $\mathcal{G}_1$, of index two. Let K be the fixed field of $\mathcal{G}_1$; then K is normal over L of degree 2^{n-1}, while N is of degree 2 over K. Thus, the induction hypothesis implies the desired tower of quadratic extensions.

Next, we show

Fact 2. *Suppose N is a finite Galois extension of F, inside F^Ω. Write $[L : N] = 2^n d$, where d is odd. Then, there is tower of extensions*

$$F \subseteq K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = N,$$

such that $[K : F] = d$ and for $1 \leq i \leq n$, $K_i = K_{i-1}(\sqrt{\beta_i})$, *with* $\beta_i \in \Sigma \dot{K}_{i-1}^2$.

Proof. If $[N : L]$ is odd, there is nothing to prove. If not, let S be a 2-Sylow subgroup of $\mathcal{G}$, the Galois group of N over F. Let K be the fixed field of S. By Theorem VIII.1.2 (p.194) in [La], N is a Galois extension of K, with Galois group S. Thus, $[N : K] = 2^n$, $[K : N] = d$ and the existence of required tower follows directly from Fact 1.

To finish the proof, let φ be a n-form over $G_{red}(F)$, such that $\iota_{FF^\Omega} \star \varphi$ is isotropic in $G_{red}(F^\Omega)$. Thus, there is a $(n-2)$-tuple of non-zero elements in F^Ω, $\langle z_3, \ldots, z_n \rangle$, such that

$$\varphi \equiv_{G_{red}(F^\Omega)} \langle 1, -1 \rangle \oplus \langle (\overline{z}_3)_{F^\Omega}, \ldots, (\overline{z}_n)_{F^\Omega} \rangle. \tag{1}$$

Let N be a *finite* normal extension of F, such that $\{z_3, \ldots, z_n\} \subseteq N$. Since $N^\Omega = F^\Omega$, any order on N can be extended to F^Ω. Thus, ι_{NF^Ω} is a complete embedding (Lemma 1.1). Moreover, $\iota_{FF^\Omega} = \iota_{NF^\Omega} \circ \iota_{FN}$, that is, the following diagram is commutative :

$$\begin{tikzcd} G_{red}(F) \arrow[rr, "\iota_{FN}"] \arrow[dr, "\iota_{FF^\Omega}"'] & & G_{red}(N) \arrow[dl, "\iota_{NF^\Omega}"] \\ & G_{red}(F^\Omega) & \end{tikzcd}$$

Thus, we may rewrite (1) as

$$\iota_{NF^\Omega} \star (\iota_{FN} \star \varphi) \equiv_{G_{red}(F^\Omega)} \iota_{NF^\Omega} \star (\langle 1, -1 \rangle \oplus \langle (\overline{z}_3)_N, \ldots, (\overline{z}_n)_N \rangle).$$

Since ι_{NF^Ω} is a complete embedding, we conclude that

$$\iota_{FN} \star \varphi \equiv_{G_{red}(N)} \langle 1, -1 \rangle \oplus \langle (\overline{z}_3)_N, \ldots, (\overline{z}_n)_N \rangle,$$

i.e., $\iota_{FN} \star \varphi$ is isotropic in $G_{red}(N)$. Let $F \subseteq K \subseteq K_1 \subseteq \ldots \subseteq K_n = N$, be the tower of fields of Fact 2. This tower originates a sequence of SG-morphisms

$$G_{red}(F) \xrightarrow{\iota_{FK}} G_{red}(K) \xrightarrow{\iota_1} G_{red}(K_1) \ldots G_{red}(K_{n-1}) \xrightarrow{\iota_n} G_{red}(N),$$

where $\iota_j = \iota_{K_{j-1}K_j}$, $1 \leq j \leq n$; furthermore, $\iota_{FN} = \iota_n \circ \iota_{n-1} \circ \ldots \circ \iota_{FK}$. But by Proposition 4.16, ι_{FK} and each of the ι_j's reflect isotropy. It follows that the same must be true of ι_{FN}, and so φ is isotropic in $G_{red}(F)$, ending the proof. $\diamond$

References

[Be1] E. Becker, *Euklidische Körper und eucklidische Hüllen von Körpern*, J. reine angew. Math., **268-269**, 41-52, 1974.

[Be2] E. Becker, **Hereditarily-Pythagorean Fields and Orderings of Higher Level**, IMPA Lecture Notes, **29**, Rio de Janeiro, 1978.

[BCR] J. Bochnak, M. Coste, M-F. Roy, **Géométrie Algébrique Réelle**, Ergeb. der Math. Grenz. **12**, Springer Verlag, Berlin, 1980.

[Br] G. E. Bredon, **Introduction to Compact Transformation Groups**, Academic Press, N. York, 1972.

[BS] J. L. Bell, A. B. Slomson, **Models and Ultraproducts : An Introduction**, North-Holland Publ. Co., 1969.

[Co] P. S. Cohn, **Algebra**, Vol. 3 (second edition), John Wiley and Sons, New York, 1991.

[CK] C. C. Chang, H. J. Kiesler, **Model Theory**, North-Holland Publ. Co., 1991.

[Cr] T. Craven, *The Boolean Space of Orderings of a Field*, Trans. Amer. Math. Soc. **209** (1975), 225-235.

[DM1] M. Dickmann, F. Miraglia, *Special Groups : applications of Boolean algebras to the algebraic theory of quadratic forms*, Memoirs of The Amer. Math. Soc. **689**, Providence, R.I., 2000.

[DM2] M. Dickmann, F. Miraglia, *On quadratic forms whose total signature is zero mod pon2. Solution to a problem of M. Marshall*, Invent. Math., **133**, 243-278 (1998)

[Fl] W. M. Fleischman (ed.), **Set Valued Mappings, Selections and Topological Properties of 2^X**, Springer Lecture Notes in Math. **171**, Springer Verlag, N. York, 1970.

[HJ] D. Haran, M. Jarden, *The Absolute Galois Group of a Pseudo Real Closed Field*, Ann. Scuola Norm. Sup. Pisa **22** (1985), 449 - 489.

[Hu] D. Husemoller, **Fiber Bundles**, third edition, MacGraw-Hill, New York, 1990.

[KS] M. Knebusch, W. Scharlau, **Algebraic Theory of Quadratic Forms**, DMV Seminar Notes **1**, 1980, Birkhäuser.

[K] K. Kuratowski, **Topology, I**, Academic Press, PWN, 1966.

[KR] K. Kuratowski, C. Ryll-Nardzewski, *A General Theorem on Selectors*, Bull. Acad. Polon. des Sciences, (série Mat. astr. et phys.), **13**, 397-403, 1965.

[L1] T. Y. Lam, **The Algebraic Theory of Quadratic Forms**, W. A. Benjamin, 1973.

[La] S. Lang, **Algebra**, Addison-Wesley Publ. Co., New York, 1978.

[P] A. Prestel, **Lectures on Formally Real Fields**, Lecture Notes Math. **1093**, Springer Verlag, 1984.

[Mi1] E. Michael, *Continuous Selections, I*, Ann. of Math, **63**, 361-382, 1956.

[Mi2] E. Michael, *Continuous Selections, II*, Ann. of Math, **64**, 562-580, 1956.

[Mi3] E. Michael, *Continuous Selections, III*, Ann. of Math, **65**, 375-390, 1957.

[Pa] T. Parthasarathy, **Selections Theorems and their Applications**, Springer Lecture Notes in Math. **263**, Springer Verlag, N. York, 1972.

[Sc] W. Scharlau, **Quadratic and Hermitian Forms**, Grund. der Math. Wiss. **270**, Springer Verlag, Berlin, 1985.

[St] N. Steenrod, **The Topology of Fiber Bundles**, Princeton Univ. Press, Princeton, New Jersey, 1951.

M. Dickmann
Equipe de Logique Mathématique
Université de Paris VII
dickmann@logique.jussieu.fr
FRANCE

F. Miraglia
Departamento de Matemática
Universidade de São Paulo
miraglia@ime.usp.br
BRAZIL