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Hochschild cohomology: some methods for computations [1]

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Abstract: We present some results on computing Hochschild cohomology groups. We describe the lower cohomology groups and provide several examples. In the particular case of hereditary algebras, radical square zero algebras and incidence algebras, we construct convenient projective resolutions that allow us to compute their cohomology groups. Finally, we show an inductive method to compute the Hochschild cohomology groups.

Key words: cohomology, Hochschild, finite-dimensional algebras.

1 Introduction

These notes correspond to a series of three lectures given in the Workshop on Representations of Algebras that took place in São Paulo in July, 1999, before the Conference on Representations of Algebras (CRASP).

The purpose of these lectures was to present some results on computing Hochschild cohomology groups.

Let A be a finite–dimensional k–algebra (associative, with unit) and let M be an A–bimodule. The Hochschild cohomology groups $H^i(A, M)$ were introduced by Hochschild [19] in 1945. He considered the group of i–linear applications $L_k^i(A, M)$ and he defined a coboundary operator $L_k^i(A, M) \to L_k^{i+1}(A, M)$ in analogy with the corresponding in algebraic topology. He proved that A is separable if and only if $H^i(A, M) = 0$ for any A–bimodule M, and that there is a one to one correspondence between $H^2(A, M)$ and the set of equivalence classes of singular extensions of A by M.

The low–dimensional groups ($i \leq 2$) have a very concrete interpretation of classical algebraic structures such as derivations and extensions. Moreover, $H^2(A, A)$ has a close connection to algebraic geometry. It was observed by Gerstenhaber [15] that $H^2(A, A)$ controls the deformation theory of A, and it was shown that the vanishing of $H^2(A, A)$ implies that A is rigid, that is, any 1–parametric deformation is isomorphic to the trivial one [16]. The converse is not true in general, but it holds if we add the condition $H^3(A, A) = 0$.

There exists also a connection between Hochschild cohomology and the representation theory of finite–dimensional algebras. It is known that if A is of finite representation type (this means that there exists a finite number of non-isomorphic indecomposable A–modules) then A is simply connected if and only if A is representation-directed and $H^1(A, A) = 0$, see [18]. The importance of the

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simply connected algebras follows from the fact that usually we may reduce the study of indecomposable modules over an algebra to that for the corresponding simply connected algebras, using Galois coverings.

Despite this very little is known about computations for particular classes of finite–dimensional algebras, since the computations of these groups by definition is rather complicated, and it has been done only in particular situations where explicit formulas have been obtained. The aim of these notes is to show how some computations can be done in particular cases.

In Section 2 we provide an introduction to the subject, that is to say, given any associative k–algebra A with unit, with k a commutative ring, we define the Hochschild (co)–homology groups of A with coefficients in an A–bimodule M.

In Section 3 we consider the lower cohomology groups, that is, $H^i(A, M)$ for $i = 0, 1, 2$. These groups have a concrete interpretation in terms of classical algebraic structures such as derivations and extensions. We provide several examples concerning algebras of the form kQ/I where Q is a quiver, kQ is its path algebra and I is an ideal of kQ. For basic information on this subject we refer the reader to [1], [21].

In Section 4 we consider finite dimensional algebras over an algebraically closed field k. We provide convenient projective resolutions of A over the enveloping algebra A^e, which allow us to compute the Hochschild cohomology groups of hereditary algebras, radical square zero algebras and incidence algebras.

In Section 5 we present an inductive method to compute the Hochschild cohomology groups of triangular algebras. We use a result due to Happel that says that for one point extension algebras $A = B[M]$ there exists a long exact sequence connecting the Hochschild cohomology groups of A and B, see [18].

2 Definition of Hochschild cohomology groups

Let k denote a commutative ring with unit and let A be a k–algebra (associative with an identity). The *enveloping algebra* A^e is the k–algebra whose underlying k–module is $A \otimes_k A^{op}$ with product $(a \otimes b)(a' \otimes b') = aa' \otimes b'b$. The following lemma shows the importance of the enveloping algebra:

Lemma 2.1 *The category of A–bimodules is equivalent to the category of left (right) A^e–modules.*

Proof: If M is an A–bimodule, we define a left (right) A^e–structure in the following way:

$$(a \otimes b)m = amb, \qquad (m(a \otimes b) = bma).$$

On the other hand, if N is an A^e–module, we define

$$am = (a \otimes 1)m, \qquad mb = (1 \otimes b)m.$$

The axioms are verified and this defines an equivalence. □

Example 2.2 *The tensor product $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$ of A n–times over k is an A–bimodule, with $a(a_1 \otimes \cdots \otimes a_n)b = aa_1 \otimes \cdots \otimes a_n b$. Hence $A^{\otimes n}$ is an A^e–module.*
The map $b'_{n-1} : A^{\otimes n+1} \to A^{\otimes n}$, $n \geq 1$, given by

$$b'_{n-1}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

is a morphism of A–bimodules. Hence, it is a map of A^e–modules.

Lemma 2.3

$$(A^{\otimes n}, b') : \quad \cdots \to A^{\otimes n+1} \stackrel{b'_{n-1}}{\to} A^{\otimes n} \to \cdots \to A^{\otimes 3} \stackrel{b'_1}{\to} A^{\otimes 2} \stackrel{b'_0}{\to} A \to 0$$

is a resolution of A over A^e, the so–called Hochschild resolution of A.

Proof: The map $s : A^{\otimes n} \to A^{\otimes n+1}$ given by $s(x) = 1 \otimes x$, for any x in $A^{\otimes n}$, verifies:

$$\begin{cases} b'_n s + s b'_{n-1} = id_{A^{\otimes n+1}}, & \forall n \geq 1, \\ b'_0 s = id_A. \end{cases}$$

Then b'_0 is an epimorphism and $\operatorname{Ker} b'_{n-1} \subset \operatorname{Im} b'_n$.
To prove that $(b')^2 = 0$, we proceed by induction. Since A is associative

$$b'_0 b'_1 (a \otimes b \otimes c) = b'_0 (ab \otimes c - a \otimes bc) = (ab)c - a(bc) = 0.$$

By induction

$$b'_n b'_{n+1} s = b'_n (id - s b'_n) = (id - b'_n s) b'_n = s b'_{n-1} b'_n = 0.$$

Since $\operatorname{Im} s$ generates $A^{\otimes n}$ as an A–module and b' is a morphism of A–modules, then $b'_n b'_{n+1} = 0$. □

Let M be an A–bimodule. If we apply the functor $M \otimes_{A^e}$. (respectively $\operatorname{Hom}_{A^e}(., M)$) to the Hochschild resolution of A, we get a complex whose homology (respectively cohomology) is the Hochschild (co)–homology of A with coefficients in M, $H_i(A, M)$ (respectively $H^i(A, M)$).

Let us see in detail the definition of $H^i(A, M)$. Applying the functor $\operatorname{Hom}_{A^e}(., M)$ to the Hochschild resolution of A and using the isomorphism

$$\operatorname{Hom}_{A^e}(A^{\otimes n}, M) \cong \operatorname{Hom}_k(A^{\otimes n-2}, M)$$

given by $f \to \tilde{f}$, with $\tilde{f}(x) = f(1 \otimes x \otimes 1)$, we get the following isomorphism of complexes:
M)

$$
\begin{array}{ccc}
\cdots \to \operatorname{Hom}_{A^e}(A^{\otimes n+2}, M) & \xrightarrow{(b',M)} & \operatorname{Hom}_{A^e}(A^{\otimes n+3}, M) \to \cdots \\
\cong\big\downarrow & & \cong\big\downarrow \\
\cdots \to \operatorname{Hom}_k(A^{\otimes n}, M) & \xrightarrow{\delta^n} & \operatorname{Hom}_k(A^{\otimes n+1}, M) \to \cdots
\end{array}
$$

where the maps δ^n are defined so as to make all the squares commutative. It can be verified directly that $\delta^n : \operatorname{Hom}_k(A^{\otimes n}, M) \to \operatorname{Hom}_k(A^{\otimes n+1}, M)$ is given by

$$
\begin{aligned}
(\delta^n f)(a_0 \otimes \cdots \otimes a_n) &= a_0 f(a_1 \otimes \cdots \otimes a_n) \\
&+ \sum_{i=0}^{n-1} (-1)^{i+1} f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
&+ (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) a_n.
\end{aligned}
$$

Then $H^i(A, M) \cong \operatorname{Ker} \delta^i / \operatorname{Im} \delta^{i-1}$.

Remark 2.4

i) If A is k–projective, then $A^{\otimes n-2}$ is k–projective. Hence the A^e–module $A^{\otimes n}$ is projective, for $n > 1$, since

$$A^{\otimes n} = A \otimes_k A^{\otimes n-2} \otimes_k A \cong A \otimes_k A^{op} \otimes_k A^{\otimes n-2} = A^e \otimes_k A^{\otimes n-2}.$$

Then $(A^{\otimes n}, b')$ is a projective resolution of A over A^e. So we may define the Hochschild (co)–homology of A with coefficients in M in the following way:

$$H_i(A, M) = \operatorname{Tor}_i^{A^e}(M, A),$$

$$H^i(A, M) = \operatorname{Ext}^i_{A^e}(A, M).$$

It follows that, in this case, the Hochschild (co)–homology of A with coefficients in M does not depend on the projective resolution we consider to compute it.

ii) Let $D = \operatorname{Hom}_k(., k)$. For any A–bimodule M, we can define maps

$$\phi : H^i(A, D(M)) \to D(H_i(A, M)),$$

$$\psi : H_i(A, D(M)) \to D(H^i(A, M)).$$

If k is a field then ϕ is an isomorphism, and if A is a finite dimensional k–algebra then ψ is also an isomorphism, see [7, page 181].

iii) Assume that k is a field. If A is A^e–projective, by i) we have that $H^i(A, M) = 0$ for any $i > 0$ and for any A–bimodule M. By ii), we deduce that $H_i(A, M) = 0$ for any $i > 0$ and for any A–bimodule M.

In fact, the following conditions are equivalent:

a) A is A^e–projective,

b) $H^i(A, M) = 0$, $\forall i > 0$, for any A–bimodule M,

c) $H_i(A, M) = 0$, $\forall i > 0$, for any A–bimodule M,

d) A is separable.

So we are interested in determining when A is A^e–projective. Let $\mu : A^e \to A$ be the map defined by $\mu(a \otimes b) = ab$. Then μ is a morphism of A–bimodules.

Lemma 2.5 *The A^e-module A is projective if and only if there exists an element $e \in A^e$ such that $\mu(e) = 1$ and $ae = ea$, for any a in A.*

Proof: Assume that A is A^e–projective. Then the A^e–epimorphism

$$A^e \xrightarrow{\mu} A \to 0$$

splits. Hence there exists an A^e–morphism $\sigma : A \to A^e$ such that $\mu\sigma = id_A$. Let $e = \sigma(1)$. Then $\mu(e) = \mu\sigma(1) = 1$ and $ae = a\sigma(1) = \sigma(a) = \sigma(1)a = ea$.
The converse is immediate if we define the map σ by $\sigma(a) = ae$. □

Example 2.6

i) Let $A = M_n(k)$. The element

$$e = \sum_{i=1}^{n} e_{i1} \otimes e_{1i}$$

verifies the conditions of the previous lemma. Then $H^i(M_n(k), M) = 0 = H_i(M_n(k), M)$ for $i > 0$ and for any $M_n(k)$–bimodule M.

ii) Let $A = k[G]$, G a group with $o(G) = n$, such that $n^{-1} \in k$. Then the element

$$e = \frac{1}{n} \sum_{x \in G} x^{-1} \otimes x$$

verifies the conditions of the previous lemma. Hence $H^i(k[G], M) = 0 = H_i(k[G], M)$ for $i > 0$ and for any $k[G]$–bimodule M.

iii) Let $A = k[x]/ < x^n >$. We want to compute the Hochschild (co)–homology of A with coefficients in the A–bimodule A, $H^i(A) = H^i(A, A)$ and $H_i(A) = H_i(A, A)$. Since A is k–free, we may consider any projective resolution. Now,

$$\cdots \to A^e \xrightarrow{d_n} A^e \to \cdots \to A^e \xrightarrow{d_1} A^e \xrightarrow{\mu} A \to 0$$

is a projective resolution of A over A^e, with d_{2i} the multiplication by $\sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}$ and d_{2i+1} the multiplication by $1 \otimes x - x \otimes 1$, see [17, page 54].

If we apply the functors $A \otimes_{A^e} .$ and $\mathrm{Hom}_{A^e}(., A)$, using that A is commutative, we get complexes isomorphic to the following ones:

$$\cdots \to A \xrightarrow{b_n} A \to \cdots \to A \xrightarrow{b_1} A \to 0,$$

$$0 \to A \xrightarrow{b^1} A \to \cdots \to A \xrightarrow{b^n} A \to \ldots$$

where $b_{2i} = b^{2i}$ is the multiplication by $\mu(\sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}) = nx^{n-1}$ and $b_{2i+1} = b^{2i+1}$ is the multiplication by $\mu(1 \otimes x - x \otimes 1) = 0$. Then

$$H_i(A) = \begin{cases} A, & \text{if } i = 0, \\ A/nx^{n-1}A, & \text{if } i \text{ is odd}, \\ \mathrm{Ann}(nx^{n-1}), & \text{if } i \text{ is even and } i > 0. \end{cases}$$

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In particular, if $\frac{1}{n} \in k$, then

$$H^i(A) = H_i(A) = \begin{cases} A, & \text{if } i = 0, \\ A/nx^{n-1}A, & \text{if } i > 0. \end{cases}$$

If $n = 0$ in k then $H^i(A) = H_i(A) = A$ for any $i \geq 0$.
In fact these computations may be generalized for any monic polynomial $f \in k[x]$, see [17, page 54], and we get

$$H_i(A) = \begin{cases} A, & \text{if } i = 0, \\ A/f'A, & \text{if } i \text{ is odd}, \\ \mathrm{Ann}(f'), & \text{if } i \text{ is even and } i > 0. \end{cases}$$

$$H^i(A) = \begin{cases} A, & \text{if } i = 0, \\ A/f'A, & \text{if } i \text{ is even and } i > 0, \\ \mathrm{Ann}(f'), & \text{if } i \text{ is odd}. \end{cases}$$

Remark 2.7

i) Let A, B be k–algebras, M any A–bimodule and N any B–bimodule. Then, for any $i \geq 0$,

$$H_i(A \times B, M \times N) = H_i(A, M) \oplus H_i(B, N),$$

$$H^i(A \times B, M \times N) = H^i(A, M) \oplus H^i(B, N),$$

see [22, page 305]. Hence, we may just consider indecomposable algebras.

ii) The Hochschild (co)–homology is invariant under Morita equivalence: given k–algebras A and B such that $\operatorname{mod} A$ is equivalent to $\operatorname{mod} B$, then $H_i(A) \cong H_i(B)$ and $H^i(A) \cong H^i(B)$, for any $i \geq 0$, see [22, page 328]. Hence, we may just consider basic algebras.

3 Interpretation of the lower cohomology groups

Recall that the Hochschild cohomology of A with coefficients in M is the cohomology of the following complex:

$$0 \to M \xrightarrow{\delta^0} \operatorname{Hom}_k(A, M) \xrightarrow{\delta^1} \operatorname{Hom}_k(A^{\otimes 2}, M) \xrightarrow{\delta^2} \operatorname{Hom}_k(A^{\otimes 3}, M) \to \dots$$

3.1 The 0–Hochschild cohomology group

We have

$$\begin{aligned} H^0(A, M) &= \operatorname{Ker}(\delta^0) \\ &= \{m \in M : \delta^0(m) = 0\} \\ &= \{m \in M : \delta^0(m)(a) = am - ma = 0, \forall a \in A\}. \end{aligned}$$

In particular, $H^0(A) = Z(A)$ the center of A.

3.2 The first Hochschild cohomology group

We have $H^1(A, M) = \operatorname{Ker}(\delta^1)/\operatorname{Im}(\delta^0)$. Now,

$$\begin{aligned} \operatorname{Ker}(\delta^1) &= \{f \in \operatorname{Hom}_k(A, M) : \delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b = 0, \\ &\quad \forall a, b \in A\} \\ &= \operatorname{Der}_k(A, M) \end{aligned}$$

is the space of derivations of A in M, and

$$\begin{aligned} \operatorname{Im}(\delta^0) &= \{f \in \operatorname{Hom}_k(A, M) : f = \delta^0(m), m \in M\} \\ &= \{f_m \in \operatorname{Hom}_k(A, M), m \in M : f(a) = am - ma\} \\ &= \operatorname{Der}_k^0(A, M) \end{aligned}$$

is the space of inner derivations of A in M.

Then $H^1(A, M) \cong \operatorname{Der}_k(A, M)/\operatorname{Der}_k^0(A, M)$.

Example 3.1 *Let $A = k[x]/ < x^2 >$. The k–linear map $\delta : A \to A$ given by $\delta(a + b\overline{x}) = b\overline{x}$ is a derivation. Since A is commutative, $\operatorname{Der}_k^0(A, A) = 0$. Hence $H^1(A) \cong \operatorname{Der}_k(A, A) \neq 0$.*

Example 3.2 *Let $A = kQ/J^2$, with J the ideal generated by the arrows. We want to show that if $H^1(A) = 0$ then Q is a tree (the underlying graph has no cycles). Assume $Q_0 = \{1, \ldots, n\}$ and $Q_1 = \{\alpha_1, \ldots, \alpha_r\}$. If Q is not a tree, there exists an arrow $\alpha \in Q_1$ such that $Q \setminus \{\alpha\}$ is connected. Suppose that $\alpha = \alpha_1$. We may define a derivation $\delta : A \to A$ by*

$\delta(e_i) = 0$ for $i = 1, \ldots, n$,

$\delta(\alpha_1) = \alpha_1$

$\delta(\alpha_i) = 0$ for $i = 2, \ldots, r$, and we extend by linearity.

Let us see that δ is not an inner derivation. If it were, there would exist $x \in A$ such that $\delta = \delta_x$ and $\delta(a) = ax - xa$ for any $a \in A$. Let $x = \sum_{i=1}^{n} \lambda_i e_i + \sum_{j=1}^{r} \mu_j \alpha_j$, $\lambda_i, \mu_j \in k$. Then

$$\alpha_1 = \delta(\alpha_1) = \alpha_1 x - x\alpha_1 = (\lambda_{s(\alpha_1)} - \lambda_{e(\alpha_1)})\alpha_1$$
$$0 = \delta(\alpha_i) = \alpha_i x - x\alpha_i = (\lambda_{s(\alpha_i)} - \lambda_{e(\alpha_i)})\alpha_i, \quad i = 2 \ldots, r.$$

So $\lambda_{s(\alpha_1)} - \lambda_{e(\alpha_1)} = 1$ and $\lambda_{s(\alpha_i)} - \lambda_{e(\alpha_i)} = 0$ for $i = 2, \ldots, r$. But this is a contradiction since $Q \setminus \{\alpha_1\}$ connected implies that $\lambda_i = \lambda_j$, $\forall i, j \in Q_0$.
Hence δ is a derivation which is not inner, so $H^1(A) \neq 0$.
In fact, the following general result holds (see [18, page 114]): if $A = kQ/J^2$, the following conditions are equivalent

a) $H^i(A) = 0$, $\forall i \geq 1$;

b) $H^1(A) = 0$;

c) Q is a tree.

Example 3.3 *Assume k has characteristic zero, $A = kQ/I$, I an homogeneous ideal (this means that I is generated by linear combinations of paths that have the same length). We want to show that if $H^1(A) = 0$ then Q has no oriented cycles. We may define $\delta : kQ \to kQ$ by $\delta(w) = l(w).w$, where $l(w)$ is the length of the path w, and extend by linearity. A direct computation shows that δ is a derivation of kQ. Since I is homogeneous, δ induce a derivation in A,*
$\delta : A \to A$.
Since $H^1(A) = 0$, δ must be inner. Hence there exist $a \in A$ such that $\delta(x) = xa - ax$ for all x in A. Now $\delta(e_i) = l(e_i)e_i = 0$ for all i, so $ae_i = e_i a$ for all i. Hence $a = \sum a_i e_i + y$, with $y \in \oplus e_i(\mathrm{rad}\, A)e_i$.
Take $\alpha \in Q_1$. Then

$$\alpha = \delta(\alpha) = \alpha a - a\alpha = (a_{s(\alpha)} - a_{e(\alpha)})\alpha + y\alpha - \alpha y.$$

Since $y\alpha - \alpha y \in \mathrm{rad}^2 A$, we have that $a_{s(\alpha)} - a_{e(\alpha)} = 1$ for any arrow $\alpha \in Q_1$. Assume that Q has an oriented cycle $1 \to 2 \to \cdots \to n \to 1$. Then

$$\left\{ \begin{array}{lcl} a_2 - a_1 & = & 1 \\ a_3 - a_2 & = & 1 \\ \ldots & & \\ a_1 - a_n & = & 1 \end{array} \right.$$

is a linear system that has no solution if k has characteristic 0.
Using this result we get that for algebras A with $\operatorname{rad}^3 A = 0$, the vanishing of the first Hochschild cohomology group implies that its associated quiver Q has no oriented cylces.

For some time it was suspected that the vanishing of the first Hochschild cohomology group implies that the corresponding quiver has no oriented cycles. Now it is known that this is not true (see [4]).

3.3 The second Hochschild cohomology group

Recall that $H^2(A, M) = \operatorname{Ker}\delta^2 / \operatorname{Im}\delta^1$. Now,

$$\begin{aligned}\operatorname{Ker}\delta^2 &= \{f : A \otimes A \to M : \delta^2(f) = 0\} \\ &= \{f : A \otimes A \to M : af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\delta^1 &= \{f : A \otimes A \to M : f = \delta^1(g), g \in \operatorname{Hom}_k(A, M)\} \\ &= \{f : A \otimes A \to M : f(a \otimes b) = ag(b) - g(ab) + g(a)b, g \in \operatorname{Hom}_k(A, M)\}.\end{aligned}$$

Definition 3.4 *An extension of A is a k-algebra epimorphism $\phi\colon B \to A$ that is k-split.*

Let M be the kernel of ϕ. Since M is a two-sided ideal of B, then M has an structure of B-module. The product in B induces a product in M. If this product is such that $M^2 = 0$ this allows us to consider M as an A-bimodule in the following way:

$$\begin{aligned} a.m &= b.m, \quad \text{if } \phi(b) = a, \\ m.a &= m.b, \quad \text{if } \phi(b) = a. \end{aligned} \tag{1}$$

Observe that this is well defined since $\phi(b) = \phi(b')$ implies that $b - b' \in M$, so $(b - b')m = 0$ since $M^2 = 0$.
On the other hand, if M has an structure of A-bimodule satisfying (1), then the product in M induced by the product in B is zero.

Definition 3.5 *Let A be a k-algebra, M an A-bimodule. An extension of A by M is a short exact sequence*

$$0 \to M \xrightarrow{i} B \xrightarrow{\phi} A \to 0$$

with ϕ an epimorphism of algebras that is k-split, i a monomorphism of k-modules such that

$$\begin{aligned} i(\phi(b).m) &= b.i(m), \\ i(m.\phi(b)) &= i(m).b, \qquad \forall b \in B, m \in M. \end{aligned} \tag{2}$$

Two extensions of A by M are said to be equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{\phi} & A & \longrightarrow & 0 \\ & & \| & & {\scriptstyle F}\downarrow & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & B' & \xrightarrow{\phi} & A & \longrightarrow & 0 \end{array}$$

with F a morphism of algebras (necessarily isomorphism).

Remark 3.6 *The conditions (2) are simply a translation of (1) when i is the inclusion $M \hookrightarrow B$.*

Proposition 3.7 *The set* $\mathrm{Ext}(A, M)$ *of isomorphic classes of extensions of A by M is in natural bijection with $H^2(A, M)$.*

Proof: Let

$$0 \to M \xrightarrow{i} B \xrightarrow{\phi} A \to 0$$

be an extension of A by M, and let $\gamma : A \to B$ be the k–linear map such that $\phi\gamma = id_A$. Then $B \cong A \oplus M$ as k–modules.
If γ is an algebra morphism, $B \cong A \ltimes M$ as k–algebras, with $(a, m).(b, n) = (ab, an + mb)$. In this case, B is said to be the trivial extension of A by M.
In general, γ is not an algebra morphism. The failure of γ to be a morphism is measured by

$$f(a \otimes b) = \gamma(a)\gamma(b) - \gamma(ab).$$

Since ϕ is a morphism of algebras, we have

$$\phi(f(a \otimes b)) = ab - ab = 0$$

and γ is k–linear, so $f\colon A \otimes A \to M$. Now, B is completely determined by A, M and f as the k–module $A \oplus M$ with multiplication $(a, m).(b, n) = (ab, an + mb + f(a \otimes b))$. We write $B \cong A \ltimes_f M$.
Derived from the associative law, we have that f satisfies

$$f(a \otimes b)c + f(ab \otimes c) = af(b \otimes c) + f(a \otimes bc).$$

This shows f to be in $\mathrm{Ker}\, \delta^2$.
Hence we have a surjective map

$$\mathrm{Ker}\, \delta^2 \to \mathrm{Ext}(A, M).$$

Two extensions $A \ltimes_{f_1} M$, $A \ltimes_{f_2} M$ are equivalent if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & A \ltimes_{f_1} M & \xrightarrow{\phi} & A & \longrightarrow & 0 \\ & & \| & & {\scriptstyle F}\downarrow & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & A \ltimes_{f_2} M & \xrightarrow{\phi} & A & \longrightarrow & 0 \end{array}$$

with F a morphism of algebras.
The commutativity of this diagram implies that $F(a,m) = (a, m + g(a))$, for $g \in \operatorname{Hom}_k(A, M)$. Now, F is a morphism of algebras if and only if

$$f_1(a \otimes b) - f_2(a \otimes b) = ag(b) - g(ab) + g(a)b, \quad \forall a, b \in A.$$

This is just the condition for $f_1 - f_2$ to be in $\operatorname{Im} \delta^1$. □

Remark 3.8

1) The trivial extension of A by M corresponds to the zero element in $H^2(A, M)$.

2) If $A = kQ/J^2$, then $H^2(A) = 0$ if and only if Q does not contain loops, does not contain non-oriented triangles, and Q is not $1 \overset{\longleftarrow}{\longrightarrow} 2$, see [9, page 213]. This says that if Q satisfies the hypothesis we have just mentioned, any extension of A by A splits.

4 Convenient projective resolutions of A over A^e

From now on A will denote a finite dimensional algebra over an algebraically closed field k. Moreover, we will assume that A is basic and connected. For information on this subject see [1].

4.1 Minimal projective resolution

Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Then $\{e_i \otimes e_j\}_{1 \leq i,j \leq n}$ is a complete set of primitive orthogonal idempotents of A^e.
So $\{P(i,j) = A^e(e_i \otimes e_j) \simeq Ae_i \otimes_k e_jA\}$ is a complete set of representatives from the isomorphism classes of indecomposable projective A^e–modules.

Lemma 4.1 *[18, page 110] Let*

$$\cdots \to R_m \to R_{m-1} \to \cdots \to R_1 \to R_0 \to A \to 0$$

be a minimal projective resolution of A over A^e. Then

$$R_m = \bigoplus_{i,j} P(i,j)^{\dim_k \operatorname{Ext}^m_A(S_j, S_i)}.$$

Proof: Let $R_m = \bigoplus_{i,j} P(i,j)^{r_{ij}}$. Denote $S(i,j) = \operatorname{top} P(i,j)$ the corresponding simple A^e–module. Observe that $S(i,j) \simeq \operatorname{Hom}_k(S_j, S_i)$. Then by definition we have that

$$\begin{aligned} r_{ij} &= \dim_k \operatorname{Ext}^m_{A^e}(A, S(i,j)) \\ &= \dim_k \operatorname{Ext}^m_{A^e}(A, \operatorname{Hom}_k(S_j, S_i)) \\ &= \dim_k \operatorname{Ext}^m_A(S_j, S_i) \end{aligned}$$

The last equality follows from [7, Corollary 4.4, page 170]. □

The projective resolution constructed above allows us to get the immediate following consequences:

Proposition 4.2 $\mathrm{pd}_{A^e} A = \mathrm{gl.dim}\, A$.

Proposition 4.3 *[8] Let $A = kQ/I$, Q with no oriented cycles. Then*

$$H_i(A) = \begin{cases} k^{|Q_0|} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Proof: This follows from the fact that applying the functor $A \otimes_{A^e} .$ to the minimal projective resolution given in the previous lemma, we may identify

$$A \otimes_{A^e} P(i,j) = A \otimes_{A^e} A^e(e_i \otimes e_j) \simeq e_j A e_i.$$

But $\mathrm{Ext}_A^m(S_j, S_i) \neq 0$ for some $m \geq 1$ implies that there is a path in Q from j to i. Since Q has no oriented cycles, then $e_j A e_i = 0$. Hence $A \otimes_{A^e} R_m = 0$ for all $m \geq 1$. □

Proposition 4.4 *[18, page 111] Let A be a basic indecomposable finite dimensional hereditary algebra, this means, $A = kQ$, Q connected without oriented cycles. Then*

$$\dim_k H_i(A) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i > 1, \\ 1 - n + \sum_{\alpha \in Q_1} \dim_k e_{e(\alpha)} A e_{s(\alpha)} & \text{if } i = 1. \end{cases}$$

where $n = |Q_0|$ and $e_{e(\alpha)} A e_{s(\alpha)}$ is the subspace of A generated by all the paths from $s(\alpha)$ to $e(\alpha)$.

Proof: Clearly $H^0(A) = Z(A) = k$ since Q is connected and has no oriented cycles.
Since A is hereditary, we have that $\mathrm{gl.dim}\, A \leq 1$, so $R_m = 0$ for all $m \geq 2$. Hence $H^i(A) = 0$ for $i \geq 2$ and

$$0 \to R_1 \to R_0 \to A \to 0$$

is the minimal projective resolution of A over A^e, with $R_0 = \bigoplus_{i \in Q_0} P(i,i)$ and $R_1 = \bigoplus_{\alpha \in Q_1} P(e(\alpha), s(\alpha))$, because $\dim_k \mathrm{Ext}_A^1(S_i, S_j)$ coincides with the number of arrows from i to j. Applying $\mathrm{Hom}_{A^e}(., A)$ to the previous exact sequence, we get

$$0 \to \mathrm{Hom}_{A^e}(A, A) \to \mathrm{Hom}_{A^e}(R_0, A) \to \mathrm{Hom}_{A^e}(R_1, A) \to 0.$$

But

$$\mathrm{Hom}_{A^e}(A, A) \simeq k,$$

$$\mathrm{Hom}_{A^e}(R_0, A) \simeq \bigoplus_{i \in Q_0} e_i A e_i \simeq k^n$$

and

$$\mathrm{Hom}_{A^e}(R_1, A) \simeq \bigoplus_{\alpha \in Q_1} e_{e(\alpha)} A e_{s(\alpha)}.$$

Thus $\dim_k H^1(A) = 1 - n + \sum_{\alpha \in Q_1} \dim_k e_{e(\alpha)} A e_{s(\alpha)}$. □

Corollary 4.5 *Let $A = kQ$, Q without oriented cycles. Then $H^1(A) = 0$ if and only if Q is a tree.*

Remark 4.6

1) *Locateli describes the minimal resolution considered in Lemma 4.1 in the particular case of truncated algebras $A = kQ/J^m$, and she computes the corresponding Hochschild cohomology groups [20].*

2) *Butler and King [6] and Bardzell [2] describe the morphisms of this minimal resolution in particular cases (monomial algebras, truncated algebras, Koszul algebras).*

3) *The equation for the dimension of the first Hochschild cohomology group given in Proposition 4.4 holds in a more general context. In fact,*

$$\dim_k H^1(A) = \dim_k Z(A) - \sum_{i \in Q_0} \dim_k e_i A e_i + \sum_{\alpha \in Q_1} \dim_k e_{e(\alpha)} A e_{s(\alpha)}$$

if $A = kQ/I$ and

a) *$I = J^m$, see [3, 20]; or*

b) *the ideal I is pregenerated, that is, $e_i I e_j = e_i kQ e_j$ or $e_i(IJ + JI)e_j$ for any $i, j \in Q_0$, see [10, page 647] and [13]; or*

c) *A is schurian and semi-commutative, that is, $\dim_k \mathrm{Hom}_A(P, P') \leq 1$ for any indecomposable projective modules P, P' and if w, w' are two paths in Q sharing starting and ending points, $w \in I$ implies $w' \in I$, see [18, page 113].*

Example 4.7

1) *Let $A = T_n(k)$ be the $n \times n$-upper triangular matrices over k. Then A is an hereditary algebra and the ordinary quiver associated*

$$Q: \quad 1 \to 2 \to \cdots \to n$$

is a tree. So

$$H^i(A) = \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

2) Let $A = kQ$, with $Q_0 = \{1,2\}$ and $Q_1 = \{\alpha_i : 1 \to 2\}_{1 \leq i \leq m}$. So

$$\dim_k H^i(A) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i > 1, \\ 1 - 2 + \sum_{i=1}^m m = m^2 - 1 & \text{if } i = 1. \end{cases}$$

3) Let $A = kQ/J^m$, with Q the oriented cycle

$$1 \to 2 \to \cdots \to n \to 1$$

Then $\dim_k H^1(A) = 1 - n + n = 1$.

Recall that a left A–module T is called a tilting module if $\operatorname{pd} T < \infty$, $\operatorname{Ext}^i_A(T,T) = 0$ for all $i > 0$ and there exists an exact sequence $0 \to A \to T^0 \to \cdots \to T^d \to 0$ with $T^i \in \operatorname{add} T$.

Theorem 4.8 *Let A be a finite dimensional k–algebra, T a tilting left A–module. Let $B = \operatorname{End}_A(T)$. Then $H^i(A) \simeq H^i(B)$.*

Proof: It is known that if $B = \operatorname{End}_A(T)$, T a tilting A–module, then there is an isomorphism between the corresponding derived categories, $\phi_{B,T,A} : D^b(A) \simeq D^b(B)$. Using this isomorphism, we may construct an isomorphism between the derived categories of the enveloping algebras A^e and B^e.
In fact,

i) $A \otimes_k T$ is a tilting $A \otimes_k B^{op}$–module and $A^e \simeq \operatorname{End}_{A \otimes_k B^{op}}(A \otimes_k T)$

ii) $T \otimes_k B^{op}$ is a tilting $A \otimes_k B^{op}$–module and $B^e \simeq \operatorname{End}_{A \otimes_k B^{op}}(T \otimes B^{op})$

So the map $F : D^b(A^e) \to D^b(B^e)$,

$$F = \phi^{-1}_{A^e, A \otimes_k T, A \otimes_k B^{op}} \; \phi_{B^e, T \otimes_k B^{op}, A \otimes_k B^{op}}$$

is the desired isomorphism. Moreover, $F(A) = B$ and F commutes with the shift. Hence

$$H^i(A) = \operatorname{Ext}^i_{A^e}(A,A) = \operatorname{Hom}_{D^b(A^e)}(A, A[i]),$$

$$\operatorname{Hom}_{D^b(B^e)}(F(A), F(A[i])) = \operatorname{Hom}_{D^b(B^e)}(F(A), F(A)[i]) = H^i(B)$$

and

$$\operatorname{Hom}_{D^b(A^e)}(A, A[i]) \overset{F}{\simeq} \operatorname{Hom}_{D^b(B^e)}(F(A), F(A[i])).$$

□

Corollary 4.9 *Let A be a finite dimensional k–algebra, $A = A_0, A_1, \ldots, A_m = kQ$, T_i tilting A_i–modules, $A_{i+1} = \operatorname{End}_{A_i}(T_i)$, Q without oriented cycles. Then $H^i(A) = 0$ for all $i \geq 2$, $H^0(A) = k$, and $H^1(A) = 0$ if and only if Q is a tree.*

4.2 The resolution of the radical

The resolution we are going to construct in this section may be used to connect Hochschild cohomology with simplicial cohomology.

Let $A = kQ/I$, E the subalgebra of A generated by the set of vertices Q_0. Then E is semisimple, commutative and $A = E \oplus \operatorname{rad} A$ in the category of E–bimodules.

Lemma 4.10 *Let $A = kQ/I$, $A = E \oplus \operatorname{rad} A$. Then*

$$\cdots \to A \otimes_E (\operatorname{rad} A)^{\otimes_E n} \otimes_E A \xrightarrow{b'} A \otimes_E (\operatorname{rad} A)^{\otimes_E n-1} \otimes_E A$$
$$\to \cdots \to A \otimes_E \operatorname{rad} A \otimes_E A$$
$$\xrightarrow{b'} A \otimes_E A \to A \to 0$$

is a projective resolution of A over A^e, with b' the Hochschild boundary.

Proof: The boundary b' is well defined and $(b')^2 = 0$. The sequence is exact since the map $s : A \otimes_E (\operatorname{rad} A)^{\otimes_E n} \otimes_E A \to A \otimes_E (\operatorname{rad} A)^{\otimes_E n+1} \otimes_E A$ given by $s(a \otimes x) = 1 \otimes \overline{a} \otimes x$, for $x \in (\operatorname{rad} A)^{\otimes_E n} \otimes_E A$, $a = e + \overline{a} \in E \oplus \operatorname{rad} A$, satisfies the equation $b's + sb' = 1$.
On the other hand, $A \otimes_E (\operatorname{rad} A)^{\otimes_E n} \otimes_E A \simeq A \otimes_E A^{op} \otimes_E (\operatorname{rad} A)^{\otimes_E n}$, $(\operatorname{rad} A)^{\otimes_E n}$ is E–projective and $A \otimes_E A^{op}$ is $A \otimes_k A^{op}$ –projective, hence $A \otimes_E (\operatorname{rad} A)^{\otimes_E n} \otimes_E A$ is A^e–projective. □

4.3 Radical square zero algebras

The resolution above allows us to compute completely the Hochschild cohomology of radical square zero algebras, that is, algebras of the form kQ/J^2.

In fact, since $\operatorname{rad}^2 A = 0$, all the middle–sum terms of the boundary b' vanish, so

$$b'(a \otimes r_1 \otimes \cdots \otimes r_n \otimes b) = ar_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes b + (-1)^n a \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n b.$$

Theorem 4.11 *[12, page 96] Let Q be a connected quiver, Q is not an oriented cycle. Then*

$$\dim_k H^n(kQ/J^2) = \begin{cases} 1 + |Q_1||Q_0| & \text{if } n = 0, \\ |Q_1||Q_1| - |Q_0||Q_0| + 1 & \text{if } n = 1, \\ |Q_n||Q_1| - |Q_{n-1}||Q_0| & \text{if } n > 1, \end{cases}$$

where Q_m is the set of paths in Q of length m and $Q_i||Q_j = \{(\gamma, \gamma') \in Q_i \times Q_j : \gamma, \gamma' \text{ parallel paths}\}$.

Corollary 4.12 *[12, page 98] Let Q be a connected quiver, Q is not an oriented cycle. Then $\oplus_{n \geq 0} H^n(kQ/J^2)$ is a finite dimensional vector space if and only if the quiver Q has no oriented cycles.*
If Q has an oriented cycle of length c, then $H^{cn+1}(kQ/J^2) \neq 0$, for any $n > 0$.

Remark 4.13 *The last result has been generalized by Locateli [20, page 660] for truncated algebras, that is, algebras of the form* $A = kQ/J^m$.

Theorem 4.14 *[12, page 98] If* Q *is the oriented cycle* $1 \to 2 \to \cdots \to m \to 1$, *and* $\mathrm{char} k \neq 2$ *then*

i) if $m = 1$, $A = k[x]/ < x^2 >$ *and*

$$H^n(kQ/J^2) = \begin{cases} A & \text{if } n = 0, \\ k & \text{if } n > 0, \end{cases}$$

ii) if $m > 1$,

$$H^n(kQ/J^2) = \begin{cases} k & \text{if } n = 0, 2sm, 2sm+1 \text{ for any } s \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.15 *[12, page 98] If* Q *is the oriented cycle* $1 \to 2 \to \cdots \to m \to 1$, *and* $\mathrm{char} k = 2$ *then*

i) if $m = 1$, $A = k[x]/ < x^2 >$ *and* $H^n(kQ/J^2) = A$ *for any* $n \geq 0$,

ii) if $m > 1$,

$$H^n(kQ/J^2) = \begin{cases} k & \text{if } n = 0, sm, sm+1 \text{ for any } s \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

4.4 Incidence algebras

Let $(P, \leq)$ be a finite partially ordered set (poset). Without loose of generality, we may assume that $P = \{1, 2, \ldots, n\}$. Let $I(P)$ be the subalgebra of the square matrices over k, $M_n(k)$, such that

$$I(P) = \{(a_{ij}) \in M_n(k) : a_{ij} = 0 \ \text{ if } i \not\leq j\}.$$

Then $I(P)$ is the so-called *incidence algebra* associated to the poset P.
The ordinary quiver associated to an incidence algebra $I(P)$ is given as follows: the set of vertices Q_0 is P, and there is an arrow $i \to j$ in Q_1 whenever $i > j$ and there is no $s \in P$ such that $i > s > j$. We say that two paths are parallel if they have the same starting and ending points. Then $I(P) = kQ/I$, where I is the ideal generated by differences of parallel paths.

Example 4.16

1) The lower triangular square matrices algebra $T_n(k)$ *is an incidence algebra associated to the poset* $P = \{1 \leq 2 \leq \cdots \leq n\}$.

2) Let $P = \{1,2,3,4\}$ and $1 \leq 2 \leq 4$, $1 \leq 3 \leq 4$. So $I(P) = kQ/I$ with

$$\begin{tikzcd}
& 3 \arrow[dr, "\alpha_2"] & \\
4 \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & 1 \\
& 2 \arrow[ur, "\beta_2"'] &
\end{tikzcd}$$

and $I = < \alpha_2\alpha_1 - \beta_2\beta_1 >$.

Given any poset P, we may associate a simplicial complex $\Sigma_P = (C_i, d_i)$ with $C_i = \{s_0 > s_1 > \cdots > s_i : s_j \in P\}$ the set of i–simplices. Let kC_i be the k–vector space with basis the set C_i. The cohomology of Σ_P with coefficients in k is the cohomology of the following complex:

$$0 \to \mathrm{Hom}_k(kC_0, k) \xrightarrow{b_1} \mathrm{Hom}_k(kC_1, k) \xrightarrow{b_2} \mathrm{Hom}_k(kC_2, k) \to \ldots$$

with

$$b_i(f)(s_0 > \cdots > s_{i+1}) = \sum_{j=0}^{i+1} (-1)^j f(s_0 > \cdots > \hat{s_j} > \cdots > s_{i+1}).$$

Theorem 4.17 *[16, page 148], [11, page 225] $H^i(\Sigma_P, k) \simeq H^i(I(P))$.*

Proof: Denote $A = I(P)$. We apply the functor $\mathrm{Hom}_{A^e}(.,A)$ to the resolution of the radical and we use the following identification

$$\mathrm{Hom}_{A^e}(A \otimes_E (\mathrm{rad}\, A)^{\otimes_E n} \otimes_E A, A) \simeq \mathrm{Hom}_{E^e}((\mathrm{rad}\, A)^{\otimes_E n}, A) \simeq \mathrm{Hom}_k(kC_n, k).$$

These isomorphisms follow from the fact that the ideal I identifies parallel paths, and

$$\mathrm{rad}\, A = \oplus_{s>t} e_t A e_s \simeq \oplus_{s>t} k$$

since $\dim_k e_t A e_s = 1$. Hence

$$\begin{aligned}
(\mathrm{rad}\, A)^{\otimes_E n} &= \mathrm{rad}\, A \otimes_E \cdots \otimes_E \mathrm{rad}\, A \\
&= \oplus_{s_0 > s_1 > \cdots > s_n} e_{s_n} A e_{s_{n-1}} \otimes_k e_{s_{n-1}} A e_{s_{n-2}} \otimes_k \cdots \otimes_k e_{s_1} A e_{s_0} \\
&= \oplus_{s_0 > s_1 > \cdots > s_n} e_{s_n} A e_{s_0}
\end{aligned}$$

Moreover, these isomorphisms commute with the boundaries, hence we have a complex isomorphism. □

Remark 4.18 *The previous result says that the computation of the Hochschild cohomology is at least as complicated as the computation of the cohomology of simplicial complexes.*

Example 4.19 *Consider the incidence algebra $I(P)$ given by the quiver*

$$\begin{tikzcd}
(n,0) \arrow[d] \arrow[dr] & (n,1) \arrow[d] \arrow[dl] \\
(n-1,0) \arrow[d] \arrow[dr] & (n-1,1) \arrow[d] \arrow[dl] \\
(n-2,0) & (n-2,1) \\
\vdots \arrow[d] \arrow[dr] & \vdots \arrow[d] \arrow[dl] \\
(0,0) & (0,1)
\end{tikzcd}$$

Then the corresponding simplicial complex is $\Sigma_P \simeq S^n$ the n–sphere, and

$$H^i(I(P)) = \begin{cases} k & \text{if } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

To any poset $(P, \leq)$ we are going to associate a new poset $\overline{P}$ adding two new elements a, b such that $a > u > b$ for any $u \in P$. If $A = I(P)$ and $\overline{A} = I(\overline{P})$ are the corresponding incidence algebras, and $A = kQ/I$, then $\overline{A} = k\overline{Q}/\overline{I}$, where $\overline{Q}$ is the quiver Q with two new vertices a, b and a new arrow from a to each source vertex of Q and a new arrow from each sink vertex to b.

Theorem 4.20 *[11, page 225] $H^i(I(P)) \simeq \operatorname{Ext}^{i+2}_{\overline{A}}(S_a, S_b)$ for any $i \geq 1$.*

Proof: Since

$$\operatorname{Ext}^{i+2}_{\overline{A}}(S_a, S_b) \simeq H^{i+2}(\overline{A}, \operatorname{Hom}_k(S_a, S_b))$$

and $\operatorname{Hom}_k(S_a, S_b) \simeq e_b\overline{A}e_a$ as $\overline{A}$–bimodules, we use the resolution of the radical to compute $H^{i+2}(\overline{A}, e_b\overline{A}e_a)$. There is an isomorphism

$$\operatorname{Hom}_{\overline{E}^e}((\operatorname{rad}\overline{A})^{\otimes_{\overline{E}} i+2}, e_b\overline{A}e_a) \simeq \operatorname{Hom}_{E^e}((\operatorname{rad} A)^{\otimes_E i}, A) \simeq \operatorname{Hom}_k(kC_i, k)$$

that follows from the fact that the paths in $\overline{Q}$ from a to b, that is, the $i+2$–simplices $a > s_1 > \cdots > s_i > b$, are in correspondence with the paths in Q corresponding to the i–simplices $s_1 > \cdots > s_i$. Moreover, these isomorphisms commute with the corresponding boundaries, hence we get the desired result using Theorem 4.17. □

Corollary 4.21 *[14] If P is a poset with a unique maximal (minimal) element then $H^i(I(P)) = 0$, for all $i \geq 1$.*

Proof: Let x be the unique maximal element in P. Then

$$0 \to P_x \to P_a \to S_a \to 0$$

is a projective resolution of S_a over $\overline{A}$. So $\operatorname{Ext}^j_{\overline{A}}(S_a, .) = 0$ for any $j \geq 2$. □

J.C. Bustamante has obtained a nice generalization of the previous result, see [5].

Example 4.22

i) $H^i(T_n(k)) = 0$ *for any* $i \geq 1$.

ii) Let $A = kQ/I$, *where* Q *is the quiver*

$$\begin{tikzcd}
& 3 \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & 4 \\
& 2 \arrow[ur] &
\end{tikzcd}$$

and I *is the parallel ideal. Then* $H^i(A) = 0$ *for any* $i \geq 1$.

5 Inductive method to compute Hochschild cohomology of triangular algebras

An algebra A is said to be triangular if the corresponding quiver has no oriented cycles. In this case, the quiver has sinks and sources, and this allows us to describe A as a one–point extension (co–extension) algebra.
Let B be a finite dimensional k–algebra, M a left B–module. The one-point extension $A = B[M]$ of B by M is by definition the finite dimensional k–algebra

$$B[M] = \begin{pmatrix} k & 0 \\ M & B \end{pmatrix}$$

with multiplication $\begin{pmatrix} a & 0 \\ m & b \end{pmatrix}\begin{pmatrix} a' & 0 \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ ma' + bm' & bb' \end{pmatrix}$ where $a, a' \in k$, $m, m' \in M$ and $b, b' \in B$.

Proposition 5.1 *Let* A *be an algebra,* Q *its ordinary quiver. The following assertions are equivalent:*

i) A *is a one–point extension algebra;*

ii) there is a simple injective A*–module* S*;*

iii) there is a vertex $i \in Q_0$ *which is a source.*

Proof: *i)* $\to$ *ii)* Assume that $A = B[M]$ is a one point extension algebra of B by M. Then $S = D(e_{11}A)$ is a simple injective A–module, where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

ii) $\rightarrow$ *iii)* Since the injective module $S = D(e_i A)$ is simple, then the corresponding vertex i is a source, that is to say, there is no arrow $\alpha \in Q_1$ ending at i.
iii) $\rightarrow$ *i)* Suppose there is a source $i \in Q_0$ and let $M = \operatorname{rad} Ae_i$ and $B = A/ < e_i >$, where $< e_i >$ denotes the two-sided ideal in A generated by the idempotent e_i. Then M is a B-module, and it is easily checked that $A \simeq B[M]$.

Example 5.2

1) Let B be the hereditary algebra with ordinary quiver

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

and let M be the B-module with representation $M(1) = 0$, $M(2) = k$, $M(3) = k$ and $M(4) = 0$. Then $A = B[M]$, the one-point extension algebra of B by M, is the algebra kQ_A/I_A with ordinary quiver Q_A

$$\begin{tikzcd} 0 \arrow[dr, "\alpha"] & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr, "\beta"] & \\ & & 4 \end{tikzcd}$$

and the ideal I_A is generated by $\beta\alpha$.

2) $T_n(k)$ the algebra of $n \times n$-upper triangular matrices over k is a one-point extension algebra of $T_{n-1}(k)$ by the $T_{n-1}(k)$-module $M = \operatorname{rad} T_n(k)e_{11}$.

Theorem 5.3 *[18, page 124] Let $A = B[M]$ be a one point extension algebra of B by M. Then there exists the following long exact sequence connecting the Hochschild cohomology of A and B*

$$0 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow \operatorname{End}_B(M)/k \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow \operatorname{Ext}^1_B(M,M) \rightarrow \dots$$

$$\dots \rightarrow \operatorname{Ext}^i_B(M,M) \rightarrow H^{i+1}(A) \rightarrow H^{i+1}(B) \rightarrow \operatorname{Ext}^{i+1}_B(M,M) \rightarrow \dots$$

Proof: Let $A = B[M]$, $M = \operatorname{rad} P_0$, $P_0 = Ae_0$, e_0 the idempotent associated to the source $0 \in Q_0$. First observe that

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0 \tag{3}$$

is a short exact sequence of A^e–modules, where $I = A^e(e_0 \otimes e_0)$, and

$$0 \to M \to P_0 \to S_0 \to 0 \tag{4}$$

is a short exact sequence of A–modules.
The proof will be done in several steps:

i) apply the functor $\mathrm{Hom}_{A^e}(A, .)$ to (3);

ii) apply the functor $\mathrm{Hom}_{A^e}(., B)$ to (3);

iii)a) apply the functor $\mathrm{Hom}_A(., P_0)$ to (4);

iii)b) apply the functor $\mathrm{Hom}_A(M, .)$ to (4).

i) Since $H^i(A) = \mathrm{Ext}^i_{A^e}(A, A)$ we get the long exact sequence

$$0 \to \mathrm{Hom}_{A^e}(A, I) \to H^0(A) \to \mathrm{Hom}_{A^e}(A, B) \to \mathrm{Ext}^1_{A^e}(A, I) \to H^1(A)$$
$$\to \mathrm{Ext}^1_{A^e}(A, B) \to \mathrm{Ext}^2_{A^e}(A, I) \to \dots$$

ii) The A^e–module I is projective, so $\mathrm{Ext}^i_{A^e}(I, .) = 0$ for all $i > 0$. Moreover, $\mathrm{Hom}_{A^e}(I, B) = 0$. So, applying $\mathrm{Hom}_{A^e}(., B)$ to (3) we get that $\mathrm{Ext}^i_{A^e}(A, B) = \mathrm{Ext}^i_{A^e}(B, B)$. But B^e is a convex subcategory of A^e, so

$$\mathrm{Ext}^i_{A^e}(B, B) = \mathrm{Ext}^i_{B^e}(B, B) = H^i(B).$$

iii) Observe that

$$\mathrm{Ext}^i_{A^e}(A, I) = H^i(A, I) = H^i(A, \mathrm{Hom}_k(S_0, P_0)) = \mathrm{Ext}^i_A(S_0, P_0)$$

since $I \simeq \mathrm{Hom}_k(S_0, P_0))$ as A–bimodules, and the last equality follows from [7, Corollary 4.4, page 170]. So,

a) applying the functor $\mathrm{Hom}_A(., P_0)$ to (4) we get that $\mathrm{Ext}^{i+1}_A(S_0, P_0) = \mathrm{Ext}^i_A(M, P_0)$ for all $i > 0$ and
$\mathrm{Ext}^1_A(S_0, P_0) = \mathrm{Hom}_k(M, P_0)/\mathrm{Hom}_k(P_0, P_0)$.

b) since S_0 is A–injective and $\mathrm{Hom}_A(M, S_0) = 0$, applying the functor $\mathrm{Hom}_A(M, .)$ to (4) we get that $\mathrm{Ext}^i_A(M, P_0) = \mathrm{Ext}^i_A(M, M)$ for all $i \geq 0$. □

Example 5.4

i) Let $A = kQ/I$ be the algebra with $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha, \beta\}$, where $\alpha : 1 \to 2$, $\beta : 2 \to 2$. Let $I = <\beta^2>$. Then $A = B[M]$, where $B = k[x]/<x^2>$ and $M = \mathrm{rad}\, P_1$. Now, M is B-projective, $\mathrm{Hom}_B(M, M) = k^2$, $H^0(B) = k^2$ and $H^0(A) = k$. So $H^i(A) = H^i(B)$ for all $i > 0$, and we have already computed $H^i(B)$ in Theorem 4.14.

ii) Let $A = kQ/I$ where Q is the quiver

$$\begin{tikzcd}
 & & 3 \arrow[dr, "\gamma"] & \\
1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\delta"'] & & 5 \\
 & & 4 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

and I is the ideal generated by $\beta\alpha$. Then $A = B[M]$, $B = A/ < e_1 >$ an hereditary algebra, M the B–module with representation $M(2) = k$, $M(3) = 0$, $M(4) = k$, $M(5) = k$, $M(\delta) = id_k$, $M(\epsilon) = id_k$. Now,

$$0 \to P_3 \to P_2 \to M \to 0$$

is the projective resolution of M over B. Applying $\mathrm{Hom}_B(.,M)$ to this short exact sequence, we get the long exact sequence

$$0 \to \mathrm{Hom}_B(M,M) \to \mathrm{Hom}_B(P_2,M) \to \mathrm{Hom}_B(P_3,M)$$
$$\to \mathrm{Ext}^1_B(M,M) \to 0.$$

But $\mathrm{Hom}_B(M,M) = k$, $\mathrm{Hom}_B(P_2,M) = M(2) = k$ and $\mathrm{Hom}_B(P_3,M) = M(3) = 0$. So $\mathrm{Ext}^i_B(M,M) = 0$ for all $i > 0$. By the previous theorem, we have that $H^i(A) = H^i(B)$ for all $i \geq 0$. Since B is hereditary, we know that $H^0(B) = H^1(B) = k$ and $H^i(B) = 0$ for all $i > 1$, see Proposition 4.4.

iii) Let $A = kQ/I$ where Q is the quiver

$$\begin{tikzcd}
 & & 3 \arrow[dr, "\gamma"] & \\
1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\delta"'] & & 5 \\
 & & 4 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

and I is the ideal generated by $\gamma\beta\alpha - \epsilon\beta\alpha$. Then $A = B[M]$, $B = A/ < e_1 >$ an hereditary algebra, M the B–module with representation $M(2) = k$, $M(3) = k$, $M(4) = k$, $M(5) = k$, $M(\beta) = id_k$, $M(\gamma) = id_k$, $M(\epsilon) = id_k$, $M(\delta) = id_k$. Now,

$$0 \to S_5 \to P_2 \to M \to 0$$

is the projective resolution of M over B. Applying $\mathrm{Hom}_B(.,M)$ to this short exact sequence, we get the long exact sequence

$$0 \to \mathrm{Hom}_B(M,M) \to \mathrm{Hom}_B(P_2,M) \to \mathrm{Hom}_B(S_5,M) \to \mathrm{Ext}^1_B(M,M) \to 0.$$

But $\mathrm{Hom}_B(M,M) = k$, $\mathrm{Hom}_B(P_2,M) = M(2) = k$ and $\mathrm{Hom}_B(S_5,M) = k$. So $\mathrm{Ext}^1_B(M,M) = k$ and $\mathrm{Ext}^i_B(M,M) = 0$ for all $i > 1$. Since B is *hereditary, we know, from Proposition 4.4, that* $H^0(B) = H^1(B) = k$ *and* $H^i(B) = 0$ *for all* $i > 1$. *From Theorem 5.3 we have that* $H^i(A) = H^i(B)$ *for* $i = 0$ *and* $i > 1$, *and we also have the exact sequence*

$$0 \to H^1(A) \to H^1(B) \to \mathrm{Ext}^1_B(M, M) \to H^2(A) \to 0.$$

So $H^1(A) = H^2(A)$ *and* $\dim_k H^1(A) \leq \dim_k H^1(B) = 1$. *Hence* $H^1(A) = H^2(A) = 0$ *or* k.
In fact, A is a tilted algebra, that is, $A \simeq \mathrm{End}_{kQ}(T)$, *with Q the quiver*

$$\begin{tikzcd}
3 \arrow[dr] & & 4 \arrow[dl] \\
& 1 & \\
2 \arrow[ur] & & 5 \arrow[ul]
\end{tikzcd}$$

that is a tree. So $H^1(A) = H^1(kQ) = 0$, *see Theorem 4.8 and Corollary 4.5. This says that the non-inner derivations in B can not be extended to A.*

3) Let $A = I(P)$ *be the incidence algebra associated to the poset P. Let* $\overline{P} = P \cup \{a\}$ *be the poset such that* $a > u$ *for all* $u \in P$. *Let* $\overline{A} = I(\overline{P}) = A[M]$, *with* $M = \mathrm{rad}\, P_a$. *Since* $H^i(\overline{A}) = 0$ *for all* $i > 0$ *(see Corollary 4.21) and* $\mathrm{End}_k(M) = k$, *then* $H^i(A) = \mathrm{Ext}^i_A(M, M)$.

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