Resenhas IME-USP 2002, Vol. 5, No. 3, 171 - 230.

Hyperbolicity and Recurrence in Dynamical Systems: A Survey of Recent Results [1]

Luis Barreira

Abstract: We discuss selected topics of current research interest in the theory of dynamical systems, with emphasis on dimension theory, multifractal analysis, and quantitative recurrence. The topics include the quantitative versus the qualitative behavior of Poincaré recurrence, the product structure of invariant measures and return times, the dimension of invariant sets and invariant measures, the complexity of the level sets of local quantities from the point of view of Hausdorff dimension, and the conditional variational principles as well as their applications to problems in number theory. We present the foundations of each area, and discuss recent developments and applications. All the necessary notions from ergodic theory, hyperbolic dynamics, dimension theory, and the thermodynamic formalism are briefly recalled. We concentrate on uniformly hyperbolic dynamics, although we also refer to nonuniformly hyperbolic dynamics. Instead of always presenting the most general results, we made a selection with the purpose of illustrating the main ideas while we avoid the accessory technicalities.

Key words: dimension, hyperbolicity, recurrence.

Contents

[1] Partially supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, Lisbon, Portugal, through FCT's Funding Program

Introduction

Our main objective is to discuss selected topics of current research interest in the theory of dynamical systems, with emphasis on the study of *recurrence*, *hyperbolicity*, and *dimension*. We recall that nontrivial recurrence and hyperbolicity stand as principal mechanisms responsible for the existence of stochastic behavior. We want to proceed from nontrivial recurrence and hyperbolicity, and discuss recent results in three areas of research: *dimension theory*, *multifractal analysis*, and *quantitative recurrence*. We present a global view of the topics under discussion, although the text substantially reflects a personal taste. In view of readability, instead of always presenting the most general results, we made a selection with the purpose of illustrating the main ideas while we avoid the accessory technicalities. Furthermore, all the necessary notions from hyperbolic dynamics, ergodic theory, dimension theory, and the thermodynamic formalism are briefly recalled. We apologize if some reference was overlooked, although if this happened it was of course totally unintentional.

Recurrence

The notion of nontrivial recurrence goes back to Poincaré in his study of the three body problem. He proved in his celebrated memoir [91] of 1890 that whenever a dynamical system preserves volume almost all trajectories return arbitrarily close to their initial position and they do this an infinite number of times. This is Poincaré's recurrence theorem. The memoir is the famous one that in its first version (printed in 1889, even having circulated shortly, and of which some copies still exist today) had the error that can be seen as the main cause for the study of chaotic behavior in the theory of dynamical systems. Incidentally, Poincaré's recurrence theorem was already present in the first printed version of the memoir as then again in [91]. Already after the publication of [91], the following was observed by Poincaré about the complexity caused by the existence of homoclinic points in the restricted three body problem (as quoted for example in [18, p. 162]):

> *One is struck by the complexity of this figure that I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three-body problem and of all the problems of dynamics in general [...].*

For a detailed and compelling historical account we recommend [18].

Hyperbolicity

The study of hyperbolicity goes back to the seminal work of Hadamard [47] in 1898 concerning the geodesic flow on the unit tangent bundle of a surface with negative curvature, in particular revealing the instability of the flow with respect to the initial conditions. Hadamard observed (as quoted for example in [18, p. 209]) that:

> *[...] each stable trajectory can be transformed, by an infinitely small variation in the initial conditions, into a completely unstable trajectory extending to infinity, or, more generally, into a trajectory of any of the types given in the general discussion: for example, into a trajectory asymptotic to a closed geodesic.*

It should be noted that the geodesic flow preserves volume and as such exhibits a nontrivial recurrence that was also exploited by Hadamard. A considerable activity took place during the 1920's and 1930's in particular with the important contributions of Hedlund and Hopf who established several topological and ergodic properties of geodesic flows, also in the case of manifolds with not necessarily constant negative sectional curvature. We refer the reader to the survey [52] for details and further references. Also in [47] Hadamard laid the foundations for symbolic dynamics, subsequently developed by Morse and Hedlund and raised to a subject in its own right (see in particular their work [71] of 1938; incidentally, this is where the expression "symbolic dynamics" appeared for the first time).

Quantitative recurrence

It should be noted that while the recurrence theorem of Poincaré is a fundamental result in the theory of dynamical systems, on the other hand it only provides information of qualitative nature. In particular, it gives no information about the frequency with which each trajectory visits a given set. This drawback was surpassed by Birkhoff [21, 22] and von Neumann [125] who in 1931 established independently the first versions of the ergodic theorem. Together with its variants and generalizations, the ergodic theorem is a fundamental result in the theory of dynamical systems and in particular in ergodic theory (one of the first appearances of the expression "ergodic theory" occured in 1932 in joint work of Birkhoff and Koopman [23]). Nevertheless, the ergodic theorem considers only one aspect of the quantitative behavior of recurrence. In particular, it gives no information about the rate at which a given trajectory returns arbitrarily close to itself. There has been a growing interest in the area during the last decade, particularly with the work of Boshernitzan [26] and Ornstein and Weiss [75].

Dimension theory

In another direction, the dimension theory of dynamical systems progressively developed, during the last two decades, into an independent field. We emphasize that we are mostly concerned here with the study of dynamical systems, and in particular of their invariant sets and measures, from the point of view of dimension. The first comprehensive reference that clearly took this point of view is the book by Pesin [82]. The main objective of the dimension theory of dynamical systems is to measure the complexity from the dimensional point of view of the objects that remain invariant under the dynamical system, such as the invariant sets and measures. It turns out that the thermodynamic formalism developed by Ruelle in his seminal work [99] has a privileged relation with the dimension theory of dynamical systems.

Multifractal analysis

The multifractal analysis of dynamical systems is a subfield of the dimension theory of dynamical systems. Briefly, multifractal analysis studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. For example, we can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions, and local entropies. These functions are usually only measurable and thus their level sets are rarely manifolds. Hence, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. Multifractal analysis has also a privileged relation with the experimental study of dynamical systems. More precisely, the so-called multifractal spectra obtained from the study of the complexity of the level sets can be determined experimentally with considerable precision. As such we may expect to be able to recover some information about the dynamical system from the information contained in the multifractal spectra.

Contents of the survey

We first introduce in Section 1 the fundamental concepts of ergodic theory and hyperbolic dynamics, and in particular nontrivial recurrence and hyperbolicity. Section 2 is dedicated to the discussion of the concept of nonuniform hyperbolicity, that in particular includes the case of a dynamical system preserving a measure with all Lyapunov exponents nonzero. This theory (also called Pesin theory) is clearly recognized today as a fundamental step in the study of stochastic behavior. We consider hyperbolic measures in Section 3, i.e., measures with all Lyapunov exponents nonzero and describe their product structure that imitates the product structure observed on hyperbolic sets. We also discuss in Section 3 the relation with the dimension theory of invariant measures.

Section 4 is dedicated to the study of the dimension theory of invariant sets. This study presents complications of different nature from those in the dimension theory of invariant measures. In particular, the study of the dimension of invariant sets is often affected by number-theoretical properties. Our emphasis here is on the study of the so-called geometric constructions which can be seen as models of invariant sets of dynamical systems. There is again a privileged relation with the thermodynamic formalism and we start to describe this relation in Section 4. Section 5 is dedicated to the study of the dimension of invariant sets of hyperbolic dynamics, both invertible and noninvertible. In particular, we present the dimension formulas for repellers and hyperbolic sets in the case of conformal dynamics.

Symbolic dynamics plays a fundamental role in some studies of dimension and is also considered in Section 5. In particular, this allows us to model invariant sets by geometric constructions. Multifractal analysis is the main theme of Section 6. We describe the interplay between local and global properties in the case of hyperbolic dynamics. We also present several examples of multifractal spectra and describe their properties.

In Section 7 we discuss the properties of the set of points for which the Birkhoff averages do not converge. In view of the ergodic theorem this set has zero measure with respect to any invariant measure, and thus it is very small from the point of view of measure theory. On the other hand, it is rather large from the point of view of dimension theory and entropy theory. We also discuss how one can make rigorous a certain multifractal classification of dynamical systems. We discuss conditional variational principles in Section 8. These have a privileged relation with multifractal analysis: roughly speaking, the multifractal analysis of a given multifractal spectrum is equivalent to the existence of a corresponding conditional variational principle. We also discuss in Section 8 how these variational principles and in particular their multi-dimensional versions have applications to certain problems in number theory. In Section 9 we address the problem of quantitative recurrence allude to above. We also describe the product structure of return times thus providing an additional view of the product structure described before for invariant sets and invariant measure. We also briefly describe some applications to number theory.

Acknowledgment. I am particularly indebted to my friends and collaborators Yakov Pesin, Benoît Saussol, Jörg Schmeling, and Christian Wolf with whom I enjoyed countless mathematical as well as nonmathematical conversations.

1 Hyperbolicity and nontrivial recurrence

1.1 Dynamical systems and hyperbolicity

One of the paradigms of the theory of dynamical systems is that the local instability of trajectories influences the global behavior of the system and opens the way to the existence of stochastic behavior. Mathematically, the instability of trajectories corresponds to some degree of hyperbolicity.

Let $f\colon M \to M$ be a diffeomorphism and $\Lambda \subset M$ an f-invariant set, i.e., a set such that $f^{-1}\Lambda = \Lambda$. We say that Λ is a *hyperbolic set* for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space

$$T_xM = E^s(x) \oplus E^u(x) \tag{1}$$

varying continuously with x that satisfies

$$d_xfE^s(x) = E^s(fx) \quad \text{and} \quad d_xfE^u(x) = E^u(fx),$$

and there exist constants $\lambda \in (0,1)$ and $c > 0$ such that

$$\|d_x f^n | E^s(x)\| \le c\lambda^n \quad \text{and} \quad \|d_x f^{-n} | E^u(x)\| \le c\lambda^n$$

for each $x \in \Lambda$ and $n \in \mathbb{N}$.

Given $\varepsilon > 0$, for each $x \in M$ we consider the sets

$$V_\varepsilon^s(x) = \{y \in B(x,\varepsilon) : d(f^n y, f^n x) < \varepsilon \text{ for every } n > 0\}$$

and

$$V_\varepsilon^u(x) = \{y \in B(x,\varepsilon) : d(f^n y, f^n x) < \varepsilon \text{ for every } n < 0\},$$

where d is the distance on M and $B(x,\varepsilon) \subset M$ is the open ball centered at x of radius ε.

A hyperbolic set possesses a very rich structure.

Theorem 1 (Hadamard–Perron Theorem) *If Λ is a compact hyperbolic set for a C^1 diffeomorphism then there exists $\varepsilon > 0$ such that for each $x \in \Lambda$ the sets $V_\varepsilon^s(x)$ and $V_\varepsilon^u(x)$ are manifolds containing x and satisfying*

$$T_x V_\varepsilon^s(x) = E^s(x) \quad \text{and} \quad T_x V_\varepsilon^u(x) = E^u(x). \tag{2}$$

We refer the reader to the book by Anosov [1, §4] for references and for a detailed account of the origins of the Hadamard–Perron Theorem.

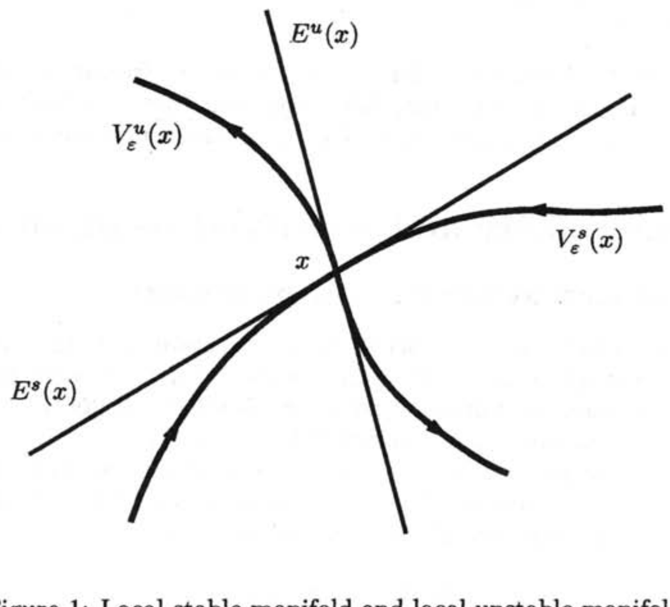

Figure 1: Local stable manifold and local unstable manifold

The manifolds $V_\varepsilon^s(x)$ and $V_\varepsilon^u(x)$ are called respectively *local stable manifold* and *local unstable manifold* at x (of size ε). It follows from (1) and (2) that these manifolds are transverse (see Figure 1). Furthermore, under the assumptions of Theorem 1 one can show that the sizes of $V^s_\varepsilon(x)$ and $V^u_\varepsilon(x)$ are uniformly bounded away from zero, i.e., there exists $\gamma = \gamma(\varepsilon) > 0$ such that

$$V^s_\varepsilon(x) \supset B^s_\gamma(x) \quad \text{and} \quad V^u_\varepsilon(x) \supset B^u_\gamma(x)$$

for every point $x \in \Lambda$, where $B^s_\gamma(x)$ and $B^u_\gamma(x)$ are the open balls centered at x of radius γ with respect to the distances induced by d respectively on $V^s_\varepsilon(x)$ and $V^u_\varepsilon(x)$. The continuous dependence of the spaces $E^s(x)$ and $E^u(x)$ in $x \in \Lambda$ guarantees that there exists $\delta = \delta(\varepsilon) > 0$ such that if $d(x,y) < \delta$ for two given points x, $y \in \Lambda$ then the intersection $V^s_\varepsilon(x) \cap V^u_\varepsilon(y)$ is composed of exactly one point. We call *product structure* to the function

$$[\cdot,\cdot]\colon \{(x,y) \in \Lambda \times \Lambda : d(x,y) < \delta\} \to M$$

defined by $[x,y] = V^s_\varepsilon(x) \cap V^u_\varepsilon(y)$ (see Figure 2).

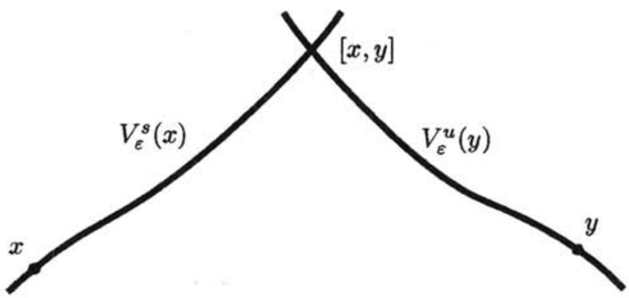

Figure 2: Product structure

When the whole manifold M is a hyperbolic set for f we say that f is an *Anosov diffeomorphism*. This class of diffeomorphisms was introduced and studied by Anosov in [1]. The notion of hyperbolic set was introduced by Smale in his seminal paper [119]. Anosov diffeomorphisms and more generally the diffeomorphisms with hyperbolic sets constitute in a certain sense the class of transformations with the "strongest possible" hyperbolicity. Moreover, hyperbolicity is one of the main mechanisms responsible for the stochastic behavior in natural phenomena, even though not always with the presence of (uniformly) hyperbolic sets as defined above. These considerations justify the search for a "weaker" concept of hyperbolicity, present in a much more general class of dynamical systems that we will call *nonuniformly hyperbolic dynamical systems* (see Section 2.1). The study of these systems is much more delicate than the study of diffeomorphisms with hyperbolic sets and namely of Anosov diffeomorphisms. However, it is still possible to establish the presence of a very rich structure and in particular the existence of families of stable and unstable manifolds (see Section 2.1).

1.2 Ergodic theory and nontrivial recurrence

We now introduce the concept of invariant measure, which constitutes another fundamental departure point for the study of stochastic behavior. Namely, the existence of a finite invariant measure causes the existence of a *nontrivial recurrence*, that is proper of stochastic behavior.

If $T\colon X \to X$ is a measurable transformation, we say that a measure μ on X is *T-invariant* if

$$\mu(T^{-1}A) = \mu(A)$$

for every measurable set $A \subset X$. The study of transformations with invariant measures is the main theme of ergodic theory.

In order to describe rigorously the concept of nontrivial recurrence, we recall one of the basic but fundamental results of ergodic theory—the Poincaré recurrence theorem. This result states that any dynamical system preserving a finite measure exhibits a nontrivial recurrence in any set A with positive measure, in the sense that the orbit of almost every point in A returns infinitely often to A.

Theorem 2 (Poincaré recurrence theorem) *Let $T\colon X \to X$ be a measurable transformation and μ a T-invariant finite measure on X. If $A \subset X$ is a measurable set with positive measure then*

$$\operatorname{card}\{n \in \mathbb{N} : T^n x \in A\} = \infty$$

for μ-almost every point $x \in A$.

A slightly modified version of Theorem 2 was first established by Poincaré in his seminal memoir on the three body problem [91] (see [18] for a detailed historical account).

The simultaneous existence of *hyperbolicity* and *nontrivial recurrence* ensures the existence of a very rich orbit structure. Roughly speaking, the nontrivial recurrence allows us to conclude that there exist orbits that return as close to themselves as desired. On the other hand, the existence of stable and unstable manifolds at these points and their transversality guarantees the existence of transverse homoclinic points, thus causing an enormous complexity through the occurrence of Smale horseshoes (see also Section 2). As such, hyperbolicity and nontrivial recurrence are two of the main mechanisms responsible for the existence of stochastic behavior in natural phenomena.

2 Nonuniform hyperbolicity

2.1 Nonuniformly hyperbolic trajectories

The concept of *nonuniform hyperbolicity* originated in the fundamental work of Pesin [79, 80, 81], in particular with the study of smooth ergodic theory that today is clearly recognized as a fundamental step in the study of stochastic behavior (see [4, 57, 68, 124] and the references therein).

Let $f\colon M \to M$ be a diffeomorphism. The trajectory $\{f^n x : n \in \mathbb{Z}\}$ of a point $x \in M$ is called *nonuniformly hyperbolic* if there exist decompositions

$$T_{f^n x}M = E^s_{f^n x} \oplus E^u_{f^n x}$$

for each $n \in \mathbb{Z}$, a constant $\lambda \in (0,1)$, and for each sufficiently small $\varepsilon > 0$ a positive function C_ε defined on the trajectory of x such that if $k \in \mathbb{Z}$ then:

1. $C_\varepsilon(f^k x) \le e^{\varepsilon|k|}C_\varepsilon(x)$;
2. $d_x f^k E^s_x = E^s_{f^k x}$ and $d_x f^k E^u_x = E^u_{f^k x}$;
3. if $v \in E^s_{f^k x}$ and $m > 0$ then
$$\|d_{f^k x} f^m v\| \le C_\varepsilon(f^k x)\lambda^m e^{\varepsilon m}\|v\|;$$
4. if $v \in E^u_{f^k x}$ and $m < 0$ then
$$\|d_{f^k x} f^m v\| \le C_\varepsilon(f^k x)\lambda^{|m|} e^{\varepsilon |m|}\|v\|;$$
5. $\angle(E^u_{f^k x}, E^s_{f^k x}) \ge C_\varepsilon(f^k x)^{-1}$.

The expression "nonuniform" refers to the estimates in conditions 3 and 4, that can differ from the "uniform" estimate λ^m by multiplicative terms that may grow along the orbit, although the exponential rate ε in 1 is small when compared to the constant $-\log\lambda$. It is immediate that any trajectory in a hyperbolic set is nonuniformly hyperbolic.

Among the most important properties due to nonuniform hyperbolicity is the existence of stable and unstable manifolds (with an appropriate version of Theorem 1), and their "absolute continuity" established by Pesin in [79]. The theory also describes the ergodic properties of dynamical systems with an invariant measure absolutely continuous with respect to the volume [80]. Also of importance is the Pesin entropy formula for the Kolmogorov–Sinai entropy in terms of the Lyapunov exponents [80] (see also [64]). Combining the *nonuniform hyperbolicity* with the *nontrivial recurrence* guaranteed by the existence of a finite invariant measure (see Section 1.2), the fundamental work of Katok in [55] revealed a very rich and complicated orbit structure (see also [57]).

We now state the result concerning the existence of stable and unstable manifolds, established by Pesin in [79].

Theorem 3 (Existence of invariant manifolds) *If $\{f^n x : n \in \mathbb{Z}\}$ is a nonuniformly hyperbolic trajectory of a $C^{1+\alpha}$ diffeomorphism, for some $\alpha > 0$, then for each sufficiently small $\varepsilon > 0$ there exist manifolds $V^s(x)$ and $V^u(x)$ containing x, and a function D_ε defined on the trajectory of x such that:*

1. $T_x V^s(x) = E^s_x$ *and* $T_x V^u(x) = E^u_x$*;*

2. $D_\varepsilon(f^k x) \le e^{2\varepsilon|k|} D_\varepsilon(x)$ *for each* $k \in \mathbb{Z}$*;*

3. if $y \in V^s(x)$*,* $m > 0$ *and* $k \in \mathbb{Z}$ *then*

$$d(f^{m+k}x, f^{m+k}y) \le D_\varepsilon(f^k x)\lambda^m e^{\varepsilon m} d(f^k x, f^k y); \tag{3}$$

4. if $y \in V^u(x)$*,* $m < 0$ *and* $k \in \mathbb{Z}$ *then*

$$d(f^{m+k}x, f^{m+k}y) \le D_\varepsilon(f^k x)\lambda^{|m|} e^{\varepsilon|m|} d(f^k x, f^k y). \tag{4}$$

The manifolds $V^s(x)$ and $V^u(x)$ are called respectively *local stable manifold* and *local unstable manifold* at x. Contrarily to what happens with hyperbolic sets, in the case of nonuniformly hyperbolic trajectories the "size" of these manifolds may not be bounded from below along the orbit (although they may decrease at most with an exponentially small speed when compared to the speeds in (3) and (4)). This makes their study more complicated.

In [93], Pugh constructed a C^1 diffeomorphism in a manifold of dimension 4, that is not of class $C^{1+\alpha}$ for any $\alpha > 0$ and for which there exists no manifold tangent to E^s_x such that the inequality (3) is valid in some open neighborhood of x. This example shows that the hypothesis $\alpha > 0$ is crucial in Theorem 3.

The proof of Theorem 3 in [79] is an elaboration of the classical work of Perron. This approach was extended by Katok and Strelcyn in [58] for maps with singularities. In [101], Ruelle obtained a proof of Theorem 3 based on the study of perturbations of products of matrices in the Multiplicative ergodic theorem (see Theorem 4 in Section 2.2). Another proof of Theorem 3 is due to Pugh and Shub [94] with an elaboration of the classical work of Hadamard. See [4, 45, 57] for detailed expositions.

There exist counterparts of Theorem 3 for dynamical systems in infinite dimensional spaces. Ruelle [102] established the corresponding version in Hilbert spaces and Mañé [67] considered transformations in Banach spaces under some compactness and invertibility assumptions, including the case of differentiable maps with compact derivative at each point. The results of Mañé were extended by Thieullen in [123] for a class transformations satisfying a certain asymptotic compactness. We refer the reader to the book by Hale, Magalhães and Oliva [48] for a detailed discussion of the state-of-the-art of the geometric theory of dynamical systems in infinite dimensional spaces.

2.2 Dynamical systems with nonzero Lyapunov exponents

The concept of hyperbolicity is closely related to the study of Lyapunov exponents. These numbers measure the asymptotic exponential rates of contraction and expansion in the neighborhood of each given trajectory.

Let $f\colon M \to M$ be a diffeomorphism. Given $x \in M$ and $v \in T_xM$, we define the *(forward) Lyapunov exponent* of (x, v) by

$$\chi(x,v) = \limsup_{n\to+\infty} \frac{1}{n} \log\|d_x f^n v\|,$$

with the convention that $\log 0 = -\infty$. The abstract theory of Lyapunov exponents (see [4]), guarantees that for each $x \in M$ there exist a positive integer $s(x) \le \dim M$, numbers

$$\chi_1(x) < \cdots < \chi_{s(x)}(x),$$

and linear spaces

$$\{0\} = E_0(x) \subset E_1(x) \subset \cdots \subset E_{s(x)}(x) = T_xM$$

such that if $i = 1, \ldots, s(x)$ then

$$E_i(x) = \{v \in T_xM : \chi(x,v) \le \chi_i(x)\},$$

and $\chi(x,v) = \chi_i(x)$ whenever $v \in E_i(x) \setminus E_{i-1}(x)$. Considering negative time, we can also define for each $x \in M$ and $v \in T_xM$ the *(backward) Lyapunov exponent* of (x,v) by

$$\chi^-(x,v) = \limsup_{n\to-\infty} \frac{1}{|n|} \log\|d_x f^n v\|.$$

Again, the abstract theory of Lyapunov exponents guarantees that for each $x \in M$ there exist a positive integer $s^-(x) \le \dim M$, numbers

$$\chi_1^-(x) > \cdots > \chi_{s^-(x)}^-(x),$$

and linear spaces

$$T_xM = E_1^-(x) \supset \cdots \supset E_{s^-(x)}^-(x) \supset E_{s^-(x)+1}^-(x) = \{0\}$$

such that if $i = 1, \ldots, s^-(x)$ then

$$E_i^-(x) = \{v \in T_xM : \chi^-(x,v) \le \chi_i^-(x)\},$$

and $\chi^-(x,v) = \chi_i^-(x)$ whenever $v \in E_i^-(x) \setminus E_{i+1}^-(x)$. A priori these two structures (for positive and negative time) could be totally unrelated. The following result of Oseledets [76] shows that the two structures are indeed related, in a very strong manner, in sets of full measure with respect to any finite invariant measure.

Theorem 4 (Multiplicative ergodic theorem) *Let $f\colon M \to M$ be a C^1 diffeomorphism and μ an f-invariant finite measure on M such that $\log^+\|df\|$ and $\log^+\|df^{-1}\|$ are μ-integrable. Then for μ-almost every point $x \in \Lambda$ there exist subspaces $H_j(x) \subset T_xM$ for $j = 1, \ldots, s(x)$ such that:*

1. *if* $i = 1, \ldots, s(x)$ *then* $E_i(x) = \bigoplus_{j=1}^{i} H_j(x)$ *and*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|d_x f^n v\| = \chi_i(x)$$

with uniform convergence for v *on* $\{v \in H_i(x) : \|v\| = 1\}$*;*

2. *if* $i \neq j$ *then*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\angle(H_i(f^n x), H_j(f^n x))| = 0.$$

We note that if M is a compact manifold then the functions $\log^+\|df\|$ and $\log^+\|df^{-1}\|$ are μ-integrable for any finite measure μ on M. The statement in Theorem 4 also holds in the more general case of cocyles over a measurable transformation. See [4] for a detailed exposition and for a proof of the Multiplicative ergodic theorem.

Let $f\colon M \to M$ be a C^1 diffeomorphism on a compact manifold and μ an f-invariant finite Borel measure on M. We say that f is *nonuniformly hyperbolic* with respect to μ if the set $\Lambda \subset M$ of points whose trajectories are nonuniformly hyperbolic has measure $\mu(\Lambda) > 0$. In this case the constants λ and ε in the definition of nonuniformly hyperbolic trajectory are replaced by measurable functions $\lambda(x)$ and $\varepsilon(x)$.

It follows from Theorem 4 that the following conditions are equivalent:

1. f is nonuniformly hyperbolic with respect to the measure μ;
2. $\chi(x, v) \neq 0$ for each $v \in T_x M$ and each x in a set with μ-positive measure.

Therefore, the nonuniformly hyperbolic diffeomorphisms with respect to a given measure are precisely the diffeomorphisms with all Lyapunov exponents nonzero in a set of positive measure.

One of the standing problems of the theory of nonuniformly hyperbolic dynamical systems is to understand how "common" this class is. Let M be a compact smooth Riemannian manifold. It was established by Katok in [54] when $\dim M = 2$ and by Dolgopyat and Pesin in [35] when $\dim M \geq 3$ that there exists a C^∞ diffeomorphism f such that:

1. f preserves the Riemannian volume m on M;
2. f has nonzero Lyapunov exponents at m-almost every point $x \in M$;
3. f is a Bernoulli diffeomorphism.

For any compact smooth Riemannian manifold M of dimension at least 5, Brin constructed in [33] a C^∞ Bernoulli diffeomorphism which preserves the Riemannian volume and has all but one Lyapunov exponent nonzero. On the other hand, the construction of the above diffeomorphisms is not robust. In another direction, Bochi [24] showed that on any compact surface there exists a residual set D of the C^1 area preserving diffeomorphisms such each $f \in D$ is either an Anosov diffeomorphism or has all Lyapunov exponents zero. This result was announced by Mañé but his proof was never published.

3 Hyperbolic measures and dimension theory

3.1 Product structure

Let $f: M \to M$ be a diffeomorphism. We say that an f-invariant measure μ on M is a *hyperbolic measure* (with respect to f) if all Lyapunov exponents are nonzero μ-almost everywhere, i.e., if $\chi(x, v) \neq 0$ for each $v \in T_xM$ and each x in a set with full μ-measure. One consequence of the discussion in Section 2.2 is that the existence of an invariant hyperbolic measure guarantees the presence of nonuniform hyperbolicity and thus of a considerable structure (see also Section 2.1). In this section we describe in detail the structure of the hyperbolic measures.

Let μ be an f-invariant hyperbolic measure. By Theorem 3, for μ-almost every point $x \in M$ there exist local stable and unstable manifolds $V^s(x)$ and $V^u(x)$. These manifolds somehow reproduce the product structure present in the case of diffeomorphisms with hyperbolic sets, but a priori it is unclear whether a given hyperbolic measure imitates or not the product structure. This problem became known as the Eckmann–Ruelle conjecture, claiming that locally a hyperbolic measure indeed imitates the product structure defined by stable and unstable manifolds. Even though Eckmann and Ruelle apparently never formulated the conjecture, their work [37] discusses several related problems and played a fundamental role in the development of the theory and as such the expression seems appropriate.

In order to formulate a rigorous result related to the resolution of the conjecture we need the families of conditional measures μ_x^s and μ_x^u generated by certain measurable partitions constructed by Ledrappier and Young in [65], based on former work of Ledrappier and Strelcyn in [63]. As shown by Rohklin, any measurable partition ξ of M has associated a family of conditional measures [98]: for μ-almost every point $x \in M$ there exists a probability measure μ_x defined on the element $\xi(x)$ of ξ containing x. Furthermore, the conditional measures are characterized by the following property: if $\mathcal{B}_\xi$ is a σ-subalgebra of the Borel σ-algebra generated by the unions of elements of ξ then for each Borel set $A \subset M$, the function $x \mapsto \mu_x(A \cap \xi(x))$ is $\mathcal{B}_\xi$-measurable and

$$\mu(A) = \int_A \mu_x(A \cap \xi(x))\, d\mu.$$

In [65], Ledrappier and Young obtained two measurable partitions ξ^s and ξ^u of M such that for μ-almost every point $x \in M$ we have:

1. $\xi^s(x) \subset V^s(x)$ and $\xi^u(x) \subset V^u(x)$;

2. for some $\gamma = \gamma(x) > 0$,

$$\xi^s(x) \supset V^s(x) \cap B(x,\gamma) \quad \text{and} \quad \xi^u(x) \supset V^s(x) \cap B(x,\gamma).$$

We denote by μ_x^s and μ_x^u the conditional measures associated respectively with the partitions ξ^s and ξ^u. We represent by $B^s(x,r) \subset V^s(x)$ and $B^u(x,r) \subset V^u(x)$ the open balls centered at x of radius r with respect to the distances induced respectively in $V^s(x)$ and $V^u(x)$.

The following result of Barreira, Pesin and Schmeling [7] establishes in the affirmative the Eckmann–Ruelle conjecture.

Theorem 5 (Product structure of hyperbolic measures) *Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism, for some $\alpha > 0$, and μ an f-invariant finite measure on M with compact support. If μ is hyperbolic then given $\delta > 0$ there exists a set $\Lambda \subset M$ with $\mu(\Lambda) > \mu(M) - \delta$ such that for each $x \in \Lambda$ we have*

$$r^\delta \le \frac{\mu(B(x,r))}{\mu_x^s(B^s(x,r))\mu_x^u(B^u(x,r))} \le r^{-\delta}$$

for all sufficiently small $r > 0$.

This result was previously unknown even in the case of Anosov diffeomorphisms. One of the major difficulties in the approach to the problem has to do with the regularity of the stable and unstable foliations that in general are not Lipschitz. In fact, Schmeling showed in [105] that for a generic diffeomorphism with a hyperbolic set, in some open set of diffeomorphisms, the stable and unstable foliations are only Hölder. Furthermore, the hyperbolic measure may not possess a "uniform" product structure even if the support does.

We know that the hypotheses in Theorem 5 are in a certain sense optimal: Ledrappier and Misiurewicz [62] showed that the hyperbolicity of the measure is essential, while Pesin and Weiss [85] showed that the statement in Theorem 5 cannot be extended to Hölder homeomorphisms. On the other hand one does not know what happens for C^1 diffeomorphisms that are not of class $C^{1+\alpha}$ for some $\alpha > 0$, particularly due to the nonexistence of an appropriate theory of nonuniformly hyperbolic dynamical systems of class C^1.

3.2 Dimension theory

There is a very close relation between the results described in Section 3.1 and the dimension theory of dynamical systems. In order to describe this relation we briefly introduce some basic notions of dimension theory.

Let X be a separable metric space. Given a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$ we define

$$m(Z,\alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all finite or countable covers of Z composed of open sets with diameter at most ε. The *Hausdorff dimension* of Z is defined by

$$\dim_H Z = \inf\{\alpha : m(Z,\alpha) = 0\}.$$

The *lower* and *upper box dimensions* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon},$$

where $N(Z,\varepsilon)$ denotes the number of balls of radius ε needed to cover Z. It is easy to show that

$$\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z. \tag{5}$$

In general these inequalities can be strict and the coincidence of the Hausdorff dimension and of the lower and upper box dimensions is a relatively rare phenomenon that occurs only in some "rigid" situations (see Sections 4 and 5; see [82, 2] for more details).

Let now μ be a finite measure on X. The *Hausdorff dimension* and the *lower* and *upper box dimensions* of μ are defined respectively by

$$\begin{aligned}
\dim_H \mu &= \lim_{\delta\to 0} \inf\{\dim_H Z : \mu(Z) \ge \mu(X) - \delta\},\\
\underline{\dim}_B \mu &= \lim_{\delta\to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\},\\
\overline{\dim}_B \mu &= \lim_{\delta\to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\}.
\end{aligned}$$

In general these quantities do not coincide, respectively, with the Hausdorff dimension and the lower and upper box dimensions of the support of μ, and thus contain additional information about the way in which the measure μ is distributed on its support. It follows immediately from (5) that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu. \tag{6}$$

As with the inequalities in (5), the inequalities in (6) are also strict in general. The following criterion for equality was established by Young in [131]: if μ is a finite measure on X and

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d \tag{7}$$

for μ-almost every $x \in X$ then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

The limit in (7), when it exists, is called *pointwise dimension* of μ at x.

In order to simplify the exposition we will assume that μ is an ergodic measure, i.e., that any set $A \subset M$ such that $f^{-1}A = A$ satisfies $\mu(A) = 0$ or $\mu(M \setminus A) = 0$. There is in fact no loss of generality (see for example [65, 7] for details). The following was established by Ledrappier and Young in [65].

Theorem 6 (Existence of pointwise dimensions) *Let f be a C^2 diffeomorphism and μ an ergodic f-invariant finite measure with compact support. If μ is hyperbolic then there exist constants d^s and d^u such that*

$$\lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} = d^s \quad \text{and} \quad \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r} = d^u \tag{8}$$

for μ-almost every point $x \in M$.

The limits in (8), when they exist, are called respectively *stable* and *unstable pointwise dimensions* of μ at x.

It was also established in [65] that

$$\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le d^s + d^u \tag{9}$$

for μ-almost every $x \in M$. It should be noted that Ledrappier and Young consider a more general class of measures in [65], for which some Lyapunov exponents may be zero. On the other hand, they require the diffeomorphism f to be of class C^2. The only place in [65] where f is required to be of class C^2 concerns the Lipschitz regularity of the holonomies generated by the intermediate foliations (such as any strongly stable foliation inside the stable one). In the case of hyperbolic measures a new argument was given by Barreira, Pesin and Schmeling in [7] establishing the Lipschitz regularity for $C^{1+\alpha}$ diffeomorphisms. This ensures that (8) and (9) hold almost everywhere even when f is only of class $C^{1+\alpha}$. See [7] for details.

For an ergodic hyperbolic finite measure with compact support that is invariant under a $C^{1+\alpha}$ diffeomorphism, Theorems 5 and 6 (and the above discussion) imply that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d^s + d^u \tag{10}$$

for μ-almost every $x \in M$. Therefore, the above criterion by Young allows us to conclude that

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d^s + d^u.$$

In fact the almost everywhere existence of the limit in (10) guarantees the coincidence not only of these three dimensional characteristics but also of many other characteristics of dimensional type (see [131, 82, 7] for more details). This allows us to choose any of these dimensional characteristics according to the convenience in each application, since the common value is always $d^s + d^u$. Therefore, for hyperbolic measures Theorems 5 and 6 allow a rigorous approach to a "fractal" dimension of invariant measures, that is well adapted to applications. Furthermore, the almost everywhere existence of the limit in (10) plays the corresponding role in dimension theory to the role of the Shannon–McMillan–Breiman theorem in the entropy theory (see Section 6.2).

The μ-almost everywhere existence of the limit in (10) was established by Young [131] when M is a surface and by Ledrappier [61] when μ is an SRB-measure (after Sinai, Ruelle and Bowen; see for example [4] for the definition).

In [88], Pesin and Yue extended the approach of Ledrappier to hyperbolic measures with a "quasi-product" structure. Theorem 5 shows that any hyperbolic measure possesses a structure that is very close to the "quasi-product" structure. In [112] (see also [109]) Schmeling and Troubetzkoy obtained versions of Theorem 5 and (10) for a class of endomorphisms.

4 Dimension theory and thermodynamic formalism

4.1 Dimension theory of geometric constructions

As we mentioned in Section 3.2 there are important differences between the dimension theory of invariant sets and the dimension theory of invariant measures. In particular, while virtually all dimensional characteristics of invariant hyperbolic *measures* coincide, the study of the dimension of invariant hyperbolic *sets* revealed that the different dimensional characteristics frequently depend on other properties, and in particular on number-theoretical properties. This justifies the interest in simpler models in the context of the theory of dynamical systems. We now make a little digression into the theory of geometric constructions that precisely provides these models.

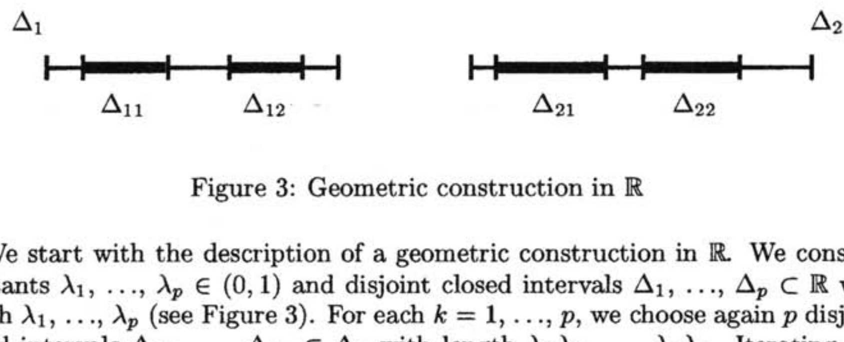

Figure 3: Geometric construction in $\mathbb{R}$

We start with the description of a geometric construction in $\mathbb{R}$. We consider constants $\lambda_1, \ldots, \lambda_p \in (0,1)$ and disjoint closed intervals $\Delta_1, \ldots, \Delta_p \subset \mathbb{R}$ with length $\lambda_1, \ldots, \lambda_p$ (see Figure 3). For each $k = 1, \ldots, p$, we choose again p disjoint closed intervals $\Delta_{k1}, \ldots, \Delta_{kp} \subset \Delta_k$ with length $\lambda_k\lambda_1, \ldots, \lambda_k\lambda_p$. Iterating this procedure, for each $n \in \mathbb{N}$ we obtain p^n disjoint closed intervals $\Delta_{i_1\cdots i_n}$ with length $\prod_{k=1}^{n} \lambda_{i_k}$. We define the set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_1\cdots i_n} \Delta_{i_1\cdots i_n}. \tag{11}$$

In [70], Moran showed that $\dim_H F = s$ where s is the unique real number satisfying the identity

$$\sum_{k=1}^{p} {\lambda_k}^s = 1. \tag{12}$$

It is remarkable that the Hausdorff dimension of F does not depend on the location of the intervals $\Delta_{i_1\cdots i_n}$ but only on their length. Pesin and Weiss [85] extended the result of Moran to arbitrary symbolic dynamics in $\mathbb{R}^m$, using the thermodynamic formalism (see Section 4.2).

To model hyperbolic invariant sets, we need to consider geometric constructions described in terms of symbolic dynamics. Given an integer $p > 0$, we consider the family of sequences $X_p = \{1, \ldots, p\}^{\mathbb{N}}$ and equip this space with the distance

$$d(\omega, \omega') = \sum_{k=1}^{\infty} e^{-k} |\omega_k - \omega'_k|. \tag{13}$$

We consider the shift map $\sigma\colon X_p \to X_p$ such that $(\sigma\omega)_n = \omega_{n+1}$ for each $n \in \mathbb{N}$. A *geometric construction* in $\mathbb{R}^m$ is defined by:

1. a compact set $Q \subset X_p$ such that $\sigma^{-1}Q \supset Q$ for some $p \in \mathbb{N}$;

2. a decreasing sequence of compact sets $\Delta_{\omega_1\cdots\omega_n} \subset \mathbb{R}^m$ for each $\omega \in Q$ with diameter $\operatorname{diam} \Delta_{\omega_1\cdots\omega_n} \to 0$ as $n \to \infty$.

We also assume that

$$\operatorname{int} \Delta_{i_1\cdots i_n} \cap \operatorname{int} \Delta_{j_1\cdots j_n} \neq \varnothing$$

whenever $(i_1 \cdots i_n) \neq (j_1 \cdots j_n)$. We define the *limit set* F of the geometric construction by (11) with the union taken over all vectors $(i_1, \ldots, i_n)$ such that $i_k = \omega_k$ for each $k = 1, \ldots, n$ and some $\omega \in Q$.

The geometric constructions include as a particular case the iterated functions systems, that have been one of the main objects of study of dimension theory, unfortunately sometimes with emphasis on the form and not on the content. The situation considered here has in mind applications to the dimension theory and the multifractal analysis of dynamical systems (see the following sections for a detailed description).

We now consider the case in which all the sets $\Delta_{i_1\cdots i_n}$ are balls (see Figure 4). Write $r_{i_1\cdots i_n} = \operatorname{diam} \Delta_{i_1\cdots i_n}$. The following result was established by Barreira in [2].

Theorem 7 (Dimension of the limit set) *For a geometric construction modelled by $Q \subset X_p$ for which the sets $\Delta_{i_1\cdots i_n}$ are balls, if there exists a constant $\delta > 0$ such that*

$$r_{i_1\cdots i_{n+1}} \geq \delta r_{i_1\cdots i_n} \quad \textit{and} \quad r_{i_1\cdots i_{n+m}} \leq r_{i_1\cdots i_n} r_{i_{n+1}\cdots i_m}$$

for each $(i_1 i_2 \cdots) \in Q$ and each n, $m \in \mathbb{N}$ then

$$\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s,$$

where s is the unique real number satisfying the identity

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} {r_{i_1\cdots i_n}}^s = 0. \tag{14}$$

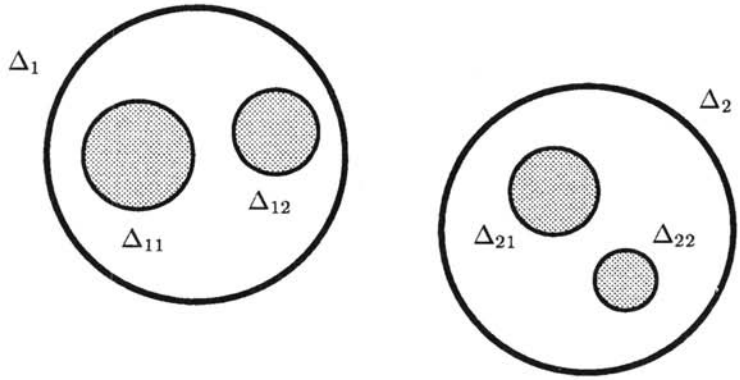

Figure 4: Geometric construction with balls

We observe that this result contains as particular cases the results of Moran and of Pesin and Weiss mentioned above (see also Section 4.2), for which

$$r_{i_1 \cdots i_n} = \prod_{k=1}^{n} \lambda_{i_k}.$$

The value of the dimension is also independent of the location of the sets $\Delta_{i_1 \cdots i_n}$. We note that the hypotheses in Theorem 7 naturally occur in a class of invariant sets of uniformly hyperbolic dynamics (see Section 5.1).

We now illustrate with an example how certain number-theoretical properties can be relevant in dimension theory. We consider a geometric construction in $\mathbb{R}^2$ for which the sets

$$\Delta_{i_1 \cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})([0,1] \times [0,1])$$

are rectangles of sides a^n and b^n, obtained through the composition of the functions

$$f_1(x,y) = (ax, by) \quad \text{and} \quad f_2(x,y) = (ax - a + 1, by - b + 1),$$

where a, $b \in (0,1)$ with $b < 1/2$ (see Figure 5). In particular, the projection of $\Delta_{i_1 \cdots i_n}$ on the horizontal axis is an interval with right endpoint given by

$$a^n + \sum_{k=0}^{n-1} j_k a^k, \tag{15}$$

where $j_k = 0$ if $i_k = 1$ and $j_k = 1 - a$ if $i_k = 2$. We assume now that $a = (\sqrt{5} - 1)/2$. In this case we have $a^2 + a = 1$ and thus for each $n > 2$ there exist several combinations $(i_1 \cdots i_n)$ with the same value in (15). This duplication causes a larger concentration of the sets $\Delta_{i_1 \cdots i_n}$ in certain regions of the limit set F. Thus, in view of computing the Hausdorff dimension of F, when we take an open cover (see Section 3.2) it may be possible to replace, in the regions of larger concentration of the sets $\Delta_{i_1 \cdots i_n}$, several elements of the cover by a unique element. This procedure can cause F to have a smaller Hausdorff dimension than expected (with respect to the generic value obtained by Falconer in [40]). This was established by Neunhäuserer in [72]. See also [92, 89] for former related results. Additional complications can occur when f_1 and f_2 are replaced by functions that are not affine.

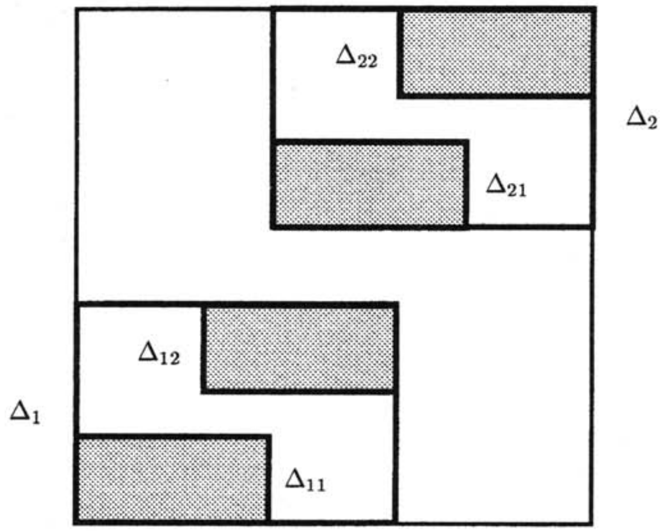

Figure 5: Number-theoretical properties and dimension theory

4.2 Thermodynamic formalism

The proof of Theorem 7 is based on a "nonadditive" version of the topological pressure. We first briefly introduce the classical concept of topological pressure.

Given a compact set $Q \subset X_p$ such that $\sigma^{-1}Q \supset Q$ and a continuous function $\varphi\colon Q \to \mathbb{R}$ we define the *topological pressure* of φ (with respect to σ) by

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \cdots i_n} \exp \sup \left(\sum_{k=0}^{n-1} \varphi \circ \sigma^k \right), \tag{16}$$

where the supremum is taken over all sequences $(j_1 j_2 \cdots) \in Q$ such that $(j_1 \cdots j_n) = (i_1 \cdots i_n)$. We define the *topological entropy* of $\sigma|Q$ by

$$h(\sigma|Q) = P(0).$$

One can easily verify that

$$h(\sigma|Q) = \lim_{n\to\infty} \frac{1}{n} \log N(Q,n),$$

where $N(Q,n)$ is the number of vectors $(i_1,\ldots,i_n)$ whose components constitute the first n entries of some element of Q.

The topological pressure is one of the fundamental notions of the thermodynamic formalism developed by Ruelle. The topological pressure was introduced by Ruelle in [99] for expansive transformations and by Walters in [126] in the general case. For more details and references see [30, 56, 100, 128].

We now present an equivalent description of the topological pressure. Let μ be a σ-invariant measure on Q and ξ a countable partition of Q into measurable sets. We write

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C)\log\mu(C),$$

with the convention that $0\log 0 = 0$. We define the *Kolmogorov–Sinai entropy* of $\sigma|Q$ with respect to μ by

$$h_\mu(\sigma|Q) = \sup_{\xi} \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} \sigma^{-k}\xi\right), \tag{17}$$

where $\bigvee_{k=0}^{n-1} \sigma^{-k}\xi$ is the partition of Q into sets of the form

$$C_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} \sigma^{-k} C_{i_{k+1}} \tag{18}$$

with $C_{i_1}, \ldots, C_{i_n} \in \xi$ (it can be shown that there exists the limit when $n \to \infty$ in (17)). The topological pressure satisfies the *variational principle* (see [128, 56] for details and references)

$$P(\varphi) = \sup_{\mu} \left\{ h_\mu(\sigma|Q) + \int_Q \varphi\, d\mu \right\}, \tag{19}$$

where the supremum is taken over all σ-invariant probability measures on Q. A σ-invariant probability measure on Q is called an *equilibrium measure* for φ (with respect to $\sigma|Q$) if the supremum in (19) is attained by this measure, i.e., if

$$P(\varphi) = h_\mu(\sigma|Q) + \int_Q \varphi\, d\mu.$$

There exists a very close relation between dimension theory and the thermodynamic formalism. To illustrate this relation we consider numbers $\lambda_1, \ldots, \lambda_p$ and define the function $\varphi\colon Q \to \mathbb{R}$ by

$$\varphi(i_1 i_2 \cdots) = \log \lambda_{i_1}. \tag{20}$$

We have

$$\begin{aligned} P(s\varphi) &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n}\exp\left(s\sum_{k=1}^{n}\log\lambda_{i_k}\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n}\prod_{k=1}^{n}{\lambda_{i_k}}^{s} \\ &= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i=1}^{p}{\lambda_i}^{s}\right)^{n} \\ &= \log\sum_{i=1}^{p}{\lambda_i}^{s}. \end{aligned}$$

Therefore, the equation in (12) is equivalent to the following equation involving the topological pressure:

$$P(s\varphi) = 0. \tag{21}$$

This equation was introduced by Bowen in [32] and is called *Bowen equation* (although it would be appropriate to call it instead *Bowen–Ruelle equation*). It has a rather universal character: virtually all known equations to compute or estimate the dimension of invariant sets of dynamical systems are particular cases of this equation or of appropriate generalizations. For example, the result of Pesin and Weiss in [85] mentioned in Section 4.1 can be formulated as follows.

Theorem 8 (Dimension of the limit set) *For a geometric construction modelled by $Q \subset X_p$ in which the sets $\Delta_{i_1\cdots i_n}$ are balls of diameter $\prod_{k=1}^{n}\lambda_{i_k}$, we have*

$$\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s,$$

where s is the unique real number satisfying $P(s\varphi) = 0$ with φ as in (20).

However, the classical topological pressure is not adapted to all geometric constructions. Namely, comparing the equations in (14) and (16) it appears that it would be appropriate to replace the sequence of functions

$$\varphi_n = \sum_{k=0}^{n-1}\varphi\circ\sigma^k \tag{22}$$

in (16) by the new sequence

$$\psi_n = s\log\operatorname{diam}\Delta_{i_1\cdots i_n}.$$

We note that while the sequence φ_n satisfies the identity

$$\varphi_{n+m} = \varphi_n + \varphi_m\circ\sigma^n,$$

the new sequence ψ_n may not satisfy any additivity between its terms. Due to technical problems related with the existence of the limit in (16) for sets that are not necessarily compact, a different approach was used by Barreira in [2] to introduce the nonadditive topological pressure. It is similar to the introduction of the Hausdorff dimension, and uses the theory of Carathéodory characteristics developed by Pesin (see [82] for references and full details).

Let (X,ρ) be a compact metric space and $f\colon X \to X$ a continuous transformation. Given an open cover $\mathcal{U}$ of X we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of vectors $\mathbf{U} = (U_0,\dots,U_n)$ of sets U_0, …, $U_n \in \mathcal{U}$ and write $m(\mathbf{U}) = n$. For each $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ we define the open set

$$X(\mathbf{U}) = \bigcap_{k=0}^{n} f^{-k}U_k.$$

We consider a sequence of functions $\Phi = \{\varphi_n\colon X \to \mathbb{R}\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we define

$$\gamma_n(\Phi,\mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x,\, y \in X(\mathbf{U}) \text{ for some } \mathbf{U} \in \mathcal{W}_n(\mathcal{U})\}$$

and assume that

$$\limsup_{\operatorname{diam}\mathcal{U}\to 0}\ \limsup_{n\to\infty} \frac{\gamma_n(\Phi,\mathcal{U})}{n} = 0. \tag{23}$$

In the additive case (i.e., when Φ is composed by continuous functions obtained as in (22)) the condition (23) is always satisfied.

Given $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we write

$$\varphi(\mathbf{U}) = \begin{cases} \sup_{X(\mathbf{U})}\varphi_n & \text{if } X(\mathbf{U}) \neq \varnothing \\ -\infty & \text{otherwise} \end{cases}.$$

Given $Z \subset X$ and $\alpha \in \mathbb{R}$ we define

$$M(Z,\alpha,\Phi,\mathcal{U}) = \lim_{n\to\infty}\inf_{\Gamma}\sum_{\mathbf{U}\in\Gamma}\exp\left(-\alpha m(\mathbf{U}) + \varphi(\mathbf{U})\right),$$

where the infimum is taken over all finite and infinite countable families $\Gamma \subset \bigcup_{k\geq n}\mathcal{W}_k(\mathcal{U})$ satisfying $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$. We define

$$P_Z(\Phi,\mathcal{U}) = \inf\{\alpha : M(Z,\alpha,\Phi,\mathcal{U}) = 0\}.$$

The following properties were established in [2]:

1. there exists the limit

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi,\mathcal{U});$$

2. if there are constants $c_1, c_2 > 0$ such that $c_1 n \le \varphi_n \le c_2 n$ for each $n \in \mathbb{N}$ and $h(f) < \infty$, then there exists a unique number $s \in \mathbb{R}$ such that $P_Z(s\Phi) = 0$;
3. if there exists a continuous function $\psi\colon X \to \mathbb{R}$ such that
$$\varphi_{n+1} - \varphi_n \circ f \to \psi \text{ uniformly on } X$$
then
$$P_X(\Phi) = \sup_\mu \left\{ h_\mu(f) + \int_X \psi \, d\mu \right\},$$
where the supremum is taken over all f-invariant probability measures on X.

We call $P_Z(\Phi)$ the *nonadditive topological pressure* of Φ on the set Z (with respect to f). The nonadditive topological pressure is a generalization of the classical topological pressure and contains as a particular case the subadditive version introduced by Falconer in [41] under more restrictive assumptions. In the additive case we recover the notion of topological pressure introduced by Pesin and Pitskel' in [83]. The quantity $P_Z(0)$ coincides with the notion of topological entropy for noncompact sets introduced in [83], and is equivalent to the notion introduced by Bowen in [29] (see [82]).

The equation $P_Z(s\Phi) = 0$ is a nonadditive version of Bowen's equation in (21). In particular, one can show that the equation (14) is equivalent to $P_Q(s\Phi) = 0$, where Φ is the sequence of functions $\varphi_n\colon Q \to \mathbb{R}$ defined by
$$\varphi_n(i_1 i_2 \cdots) = \log \operatorname{diam} \Delta_{i_1 \cdots i_n}. \tag{24}$$

5 Hyperbolic sets and dimension theory

5.1 Repellers and symbolic dynamics

As we observed in Section 4.1, one of the main motivations for the study of geometric constructions is the study of the dimension theory of invariant sets of dynamical systems. This approach can be effected with the use of Markov partitions.

We first consider the case of expanding maps. These constitute a noninvertible version of the diffeomorphisms with hyperbolic sets. Let $g\colon M \to M$ be a differentiable map of a compact manifold. We consider a g-invariant set $J \subset M$, i.e., a set such that $g^{-1}J = J$. We say that J is a *repeller* of g and that g is an *expanding map* on J if there exist constants $c > 0$ and $\beta > 1$ such that
$$\|d_x g^n v\| \ge c\beta^n \|v\|$$
for each $n \in \mathbb{N}$, $x \in J$ and $v \in T_x M$.

Let now J be a repeller of the differentiable map $g\colon M \to M$. A finite cover of J by nonempty closed sets $R_1, \ldots, R_p$ is called a *Markov partition* of J if:

1. $\overline{\operatorname{int} R_i} = R_i$ for each i;

2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \neq j$;

3. $gR_i \supset R_j$ whenever $g(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing$.

The interior of each set R_i is computed with respect to the topology induced on J. Any repeller has Markov partitions with arbitrarily small diameter (see [103]).

We can now use Markov partitions to model repellers by geometric constructions. Let J be a repeller of $g\colon M \to M$ and let $R_1, \ldots, R_p$ be the elements of a Markov partition of J. We define a $p \times p$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } g(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing \\ 0 & \text{if } g(\operatorname{int} R_i) \cap \operatorname{int} R_j = \varnothing \end{cases}.$$

Consider the space of sequences $X_p = \{1, \ldots, p\}^{\mathbb{N}}$ and the shift map $\sigma\colon X_p \to X_p$ (see Section 4.1). We call *topological Markov chain* with *transition matrix* A to the restriction of σ to the set

$$X_A = \{(i_1 i_2 \cdots) \in X_p : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{N}\}.$$

We recall that a transformation g is *topologically mixing on* J if given open sets U and V with nonempty intersection with J there exists $n \in \mathbb{N}$ such that $g^m U \cap V \cap J \neq \varnothing$ for every $m > n$. If g is topologically mixing on J then there exists $k \in \mathbb{N}$ such that A^k has only positive entries.

It is easy to show that one can define a coding map $\chi\colon X_A \to J$ by

$$\chi(i_1 i_2 \cdots) = \bigcap_{k=0}^{\infty} g^{-k} R_{i_{k+1}}. \tag{25}$$

Furthermore, χ is surjective, satisfies

$$\chi \circ \sigma = g \circ \chi \tag{26}$$

(i.e., the diagram in Figure 6 is commutative), and is Hölder continuous (with respect to the distance in X_p introduced in (13)).

$$\begin{array}{ccc} X_A & \xrightarrow{\sigma} & X_A \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi} \\ J & \xrightarrow{g} & J \end{array}$$

Figure 6: Symbolic coding of a repeller

Even though in general χ is not invertible (although $\operatorname{card} \chi^{-1} x \leq p^2$ for every x), the identity in (26) allows us to see χ as a dictionary transferring the symbolic dynamics $\sigma|X_A$ (and often the results on the symbolic dynamics) to the dynamics of g on J. In particular, the function χ allows us to see each repeller as a geometric construction (see Section 4.1) defined by the sets

$$\Delta_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} g^{-k} R_{i_{k+1}}.$$

We say that g is *conformal* on J if $d_x g$ is a multiple of an isometry for every $x \in J$. When J is a repeller of a conformal transformation of class $C^{1+\alpha}$ one can show that there is a constant $C > 0$ such that

$$C^{-1} \prod_{k=0}^{n-1} \exp \varphi(g^k x) \le \operatorname{diam} \Delta_{i_1\cdots i_n} \le C \prod_{k=0}^{n-1} \exp \varphi(g^k x) \tag{27}$$

for every $x \in \Delta_{i_1\cdots i_n}$, where the function $\varphi\colon J \to \mathbb{R}$ is defined by

$$\varphi(x) = -\log\|d_x g\|. \tag{28}$$

We now use the topological pressure defined by (16) with $Q = X_A$.

Theorem 9 (Dimension of conformal repellers) *If J is a repeller of a $C^{1+\alpha}$ transformation f, for some $\alpha > 0$, such that f is conformal on J then*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s is the unique real number such that $P(s\varphi) = 0$.

Ruelle established in [103] that $\dim_H J = s$ (under the additional assumption that g is topologically mixing on J). The coincidence between the Hausdorff dimension and the box dimensions is due to Falconer [42]. The result in Theorem 9 was independently extended to expanding transformations of class C^1 by Gatzouras and Peres in [46] and by Barreira in [2] using different approaches. Under the additional assumption that g is of class $C^{1+\alpha}$ and topologically mixing on J, it was also shown by Ruelle in [103] that if μ is the unique equilibrium measure of $-s\varphi$ then

$$\dim_H J = \dim_H \mu. \tag{29}$$

His proof consists in showing that μ is equivalent to the s-dimensional Hausdorff measure on J (in fact with Radon–Nikodym derivative bounded and bounded away from zero).

Using (27) we find that

$$P(s\varphi) = P_{X_A}(s\Phi)$$

with φ as in (28) and being Φ the sequence of functions defined by (24). Furthermore, the conformality of g on J allows us to show that even though the sets $\Delta_{i_1\cdots i_n}$ may not be balls they essentially behave as if they were. In fact one would be able to reproduce with little changes the proof of Theorem 7 (and also the proof of Theorem 8) to establish Theorem 9. Nevertheless, there is a technical difficulty related to the possible noninvertibility of the coding map χ (see (25)). More generally, under the assumptions in Theorem 9 it was shown by Schmeling in [108] that

$$\dim_H(\chi^{-1}B) = \dim_H B$$

for *any* subset $B \subset J$, provided that X_A is given the distance induced by the distance on J, so that

$$\operatorname{diam} C_{i_1\cdots i_n} = \prod_{k=0}^{n-1} \exp(\varphi \circ \chi)(\sigma^k\omega) = \|d_{\sigma\omega}g^n\|^{-1}$$

for each $\omega = (i_1 i_2 \cdots) \in X_A$ (see (18)).

5.2 Dimension theory in hyperbolic dynamics

We now move to the study of the dimension of hyperbolic sets.

Let Λ be a hyperbolic set for a diffeomorphism $f\colon M \to M$. We consider the functions $\varphi_s\colon \Lambda \to \mathbb{R}$ and $\varphi_u\colon \Lambda \to \mathbb{R}$ defined by

$$\varphi_s(x) = \log\|d_x f|E^s(x)\| \quad \text{and} \quad \varphi_u(x) = -\log\|d_x f|E^u(x)\|.$$

Recall that Λ is said to be *locally maximal* if there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U.$$

The following result is a version of Theorem 9 in the case of hyperbolic sets.

Theorem 10 (Dimension of hyperbolic sets on surfaces) *If Λ is a locally maximal compact hyperbolic set of a C^1 surface diffeomorphism, and* $\dim E^s(x) = \dim E^u(x) = 1$ *for every $x \in \Lambda$, then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t_s + t_u,$$

where t_s and t_u are the unique real numbers such that

$$P(t_s\varphi_s) = P(t_u\varphi_u) = 0.$$

It follows from work of McCluskey and Manning [69] that $\dim_H \Lambda = t_s + t_u$. The coincidence between the Hausdorff dimension and the lower and upper box dimensions is due to Takens [120] for C^2 diffeomorphisms (see also [78]) and to Palis and Viana [77] in the general case. Barreira [2] and Pesin [82] presented new proofs of Theorem 10 entirely based on the thermodynamic formalism.

One can also ask whether there is an appropriate generalization of property (29) in the present context, that is, whether there exists an invariant measure μ

supported on Λ that satisfies $\dim_H \Lambda = \dim_H \mu$. However, the answer is "almost always" negative. More precisely, McCluskey and Manning [69] showed that such a measure exists if and only if there exists a continuous function $\psi \colon \Lambda \to \mathbb{R}$ such that

$$t_s\varphi_s - t_u\varphi = \psi \circ f - f$$

on Λ. By Livschitz theorem (see for example [56]), this happens if and only if

$$\|d_x f|E^s(x)\|^{t_s}\|d_x f|E^u(x)\|^{t_u} = 1$$

for every $x \in \Lambda$ and every $n \in \mathbb{N}$ such that $f^n x = x$. One can instead ask whether the supremum

$$\delta(f) = \sup\{\dim_H \nu : \nu \text{ is an } f\text{-invariant measure on } \Lambda\}$$

is attained. Any invariant measure attaining this supremum would attain the maximal complexity from the point of view of dimension theory. The main difficulty of this problem is that the map $\nu \mapsto \dim_H \nu$ is not upper semi-continuous: simply consider the sequence $(\nu + (n-1)\delta)/n$ where $\dim_H \nu > 0$ and δ is an atomic measure. It was shown by Barreira and Wolf in [16] (also using results in [17]) that the supremum is indeed attained and by an ergodic measure, that is,

$$\delta(f) = \max\{\dim_H \nu : \nu \text{ is an ergodic } f\text{-invariant measure on } \Lambda\}.$$

See [130] for a precursor result in the special case of polynomial automorphisms of $\mathbb{C}^2$.

We now sketch the proof of Theorem 10. McCluskey and Manning showed in [69] that

$$\dim_H(V_\varepsilon^s(x) \cap \Lambda) = t_s \quad \text{and} \quad \dim_H(V_\varepsilon^u(x) \cap \Lambda) = t_u \tag{30}$$

for each $x \in \Lambda$. Furthermore, Palis and Viana showed in [77] that

$$\dim_H(V_\varepsilon^s(x) \cap \Lambda) = \underline{\dim}_B(V_\varepsilon^s(x) \cap \Lambda) = \overline{\dim}_B(V_\varepsilon^s(x) \cap \Lambda) \tag{31}$$

and

$$\dim_H(V_\varepsilon^u(x) \cap \Lambda) = \underline{\dim}_B(V_\varepsilon^u(x) \cap \Lambda) = \overline{\dim}_B(V_\varepsilon^u(x) \cap \Lambda) \tag{32}$$

for each $x \in \Lambda$. Using these results, the completion of the proof of Theorem 10 depends in a crucial way on the fact that the stable and unstable manifolds have dimension 1. In fact, the product structure $[\cdot,\cdot]$ restricted to $(V_\varepsilon^s(x) \cap \Lambda) \times (V_\varepsilon^u(x) \cap \Lambda)$ is locally a Hölder homeomorphism with Hölder inverse (and in general is not more than Hölder, for generic diffeomorphisms in a given open set, in view of work of Schmeling [105]; see also [110]). However, when the stable and unstable manifolds have dimension 1, the product structure $[\cdot,\cdot]$ is locally a Lipschitz homeomorphism with Lipschitz inverse (see for example [56]). This allows us to conclude that

$$\dim_H[V_\varepsilon^s(x) \cap \Lambda, V_\varepsilon^u(x) \cap \Lambda] = \dim_H((V_\varepsilon^s(x) \cap \Lambda) \times (V_\varepsilon^u(x) \cap \Lambda)),$$

with corresponding identities between the lower and upper box dimensions. Since the inequalities

$$\dim_H A + \dim_H B \le \dim_H(A \times B) \quad \text{and} \quad \overline{\dim}_B(A \times B) \le \overline{\dim}_B A + \overline{\dim}_B B$$

are valid for any subsets A and B of $\mathbb{R}^m$, it follows from the identities in (30), (31) and (32) that

$$\begin{aligned}\dim_H[V^s_\varepsilon(x) \cap \Lambda, V^u_\varepsilon(x) \cap \Lambda] &= \underline{\dim}_B[V^s_\varepsilon(x) \cap \Lambda, V^u_\varepsilon(x) \cap \Lambda] \\ &= \overline{\dim}_B[V^s_\varepsilon(x) \cap \Lambda, V^u_\varepsilon(x) \cap \Lambda] \\ &= t_s + t_u.\end{aligned} \tag{33}$$

On the other hand, since Λ is locally maximal one can easily show that $[x, y] \in \Lambda$ for all sufficiently close x, $y \in \Lambda$, or simply that

$$[V^s_\varepsilon(x) \cap \Lambda, V^u_\varepsilon(x) \cap \Lambda] \subset \Lambda$$

for all $x \in \Lambda$ and all sufficiently small ε. Choosing points x_1, x_2, ... in Λ such that

$$\Lambda = \bigcup_{n \in \mathbb{N}} [V^s_\varepsilon(x_n) \cap \Lambda, V^u_\varepsilon(x_n) \cap \Lambda],$$

Theorem 10 follows now immediately from (33).

We say that $f\colon M \to M$ is *conformal* on Λ when $d_x f|E^s(x)$ and $d_x f|E^u(x)$ are multiples of isometries for every point $x \in \Lambda$ (for example, if M is a surface and $\dim E^s(x) = \dim E^u(x) = 1$ for every $x \in \Lambda$ then f is conformal on Λ). The proof of Theorem 10 given by Pesin in [82] includes the case of conformal diffeomorphisms on manifolds of arbitrary dimension (the statement can also be obtained from results in [2]). In this situation the product structure is still locally a Lipschitz homeomorphism with Lipschitz inverse (see [51, 82] for details) and thus we can use the same approach as above.

The study of the dimension of repellers and hyperbolic sets of nonconformal transformations is not yet as developed. The main difficulty has to do with the possibility of existence of distinct Lyapunov exponents associated to directions that may change from point to point. There exist however some partial results, for certain classes of repellers and hyperbolic sets, starting essentially with the seminal work by Douady and Oesterlé in [36]. Namely, Falconer [43] computed the Hausdorff dimension of a class of nonconformal repellers (see also [40]), while Hu [53] computed the box dimension of a class of nonconformal repellers that leave invariant a strong unstable foliation. Related ideas were applied by Simon and Solomyak in [117] to compute the Hausdorff dimension of a class of hyperbolic sets in $\mathbb{R}^3$. Falconer also studied a class of limit sets of geometric constructions obtained from a composition of affine transformations that are not necessarily conformal [40]. In another direction, Bothe [27] and Simon [116] (also using his methods in [115] for noninvertible transformations) studied the dimension of solenoids (see [82, 113] for details). A solenoid is a hyperbolic set of the form $\Lambda = \bigcap_{n=1}^{\infty} f^n T$, where $T \subset \mathbb{R}^3$ is diffeomorphic to a "solid torus" $S^1 \times D$ for some closed disk $D \subset \mathbb{R}^2$ and $f\colon T \to T$ is a diffeomorphism such that for each $x \in S^1$ the intersection $f(T) \cap (\{x\} \times D)$ is a disjoint union of p sets homeomorphic to a closed disk.

In a similar way that in Section 5.1 the proof of the identities in (30) can be obtained with the use of Markov partitions. We briefly recall the notion of Markov partition for a hyperbolic set. A nonempty closed set $R \subset \Lambda$ is called a *rectangle* if $\operatorname{diam} R < \delta$ (where δ is given by the product structure; see Section 1.1), $\overline{\operatorname{int} R} = R$, and $[x, y] \in R$ whenever $x, y \in R$. A finite cover of Λ by rectangles $R_1, \ldots, R_p$ is called a *Markov partition* of Λ if:

1. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \neq j$;
2. if $x \in f(\operatorname{int} R_i) \cap \operatorname{int} R_j$ then

$$f^{-1}(V_\varepsilon^u(fx) \cap R_j) \subset V_\varepsilon^u(x) \cap R_i \quad \text{and} \quad f(V_\varepsilon^s(x) \cap R_i) \subset V_\varepsilon^s(fx) \cap R_j.$$

The interior of each set R_i is computed with respect to the topology induced on Λ. Any hyperbolic set possesses Markov partitions with arbitrarily small diameter (see [30] for references and full details).

Let $\mathcal{R} = \{R_1, \ldots, R_p\}$ be a Markov partition of a hyperbolic set. It is well known that $\partial\mathcal{R} = \bigcup_{i=1}^{p} \partial R_i$ has zero measure with respect to any equilibrium measure. This is a simple consequence of the fact that $\partial\mathcal{R}$ is a closed set with dense complement. On the other hand, it is also interesting to estimate the measure of neighborhoods of $\partial\mathcal{R}$. This can be simpler when each element of $\mathcal{R}$ has a piecewise regular boundary (as in the case of hyperbolic automorphisms of $\mathbb{T}^2$), but it is well known that Markov partitions may have a very complicated boundary. In particular, it was discovered by Bowen [31] that $\partial\mathcal{R}$ is not piecewise regular in the case of hyperbolic automorphisms of $\mathbb{T}^3$. It was shown by Barreira and Saussol in [11] that if μ is an equilibrium measure of a Hölder continuous function then there exist constants $c > 0$ and $\nu > 0$ such that

$$\mu(\{x \in \Lambda\colon d(x, \partial\mathcal{R}) < \varepsilon\}) \leq c\varepsilon^\nu$$

for every $\varepsilon > 0$. This provides a control of the measure near $\partial\mathcal{R}$. Furthermore, it is possible to consider any $\nu > 0$ such that

$$\nu < \frac{P_\Lambda(\varphi) - P_I(\varphi)}{\log \max\{\|d_x f\| : x \in \Lambda\}},$$

where φ in chosen in such a way that μ is an equilibrium measure of φ, and

$$I = \bigcup_{n \in \mathbb{Z}} f^n(\partial\mathcal{R})$$

is the invariant hull of $\partial\mathcal{R}$.

6 Multifractal analysis

6.1 Hyperbolic dynamics and Birkhoff averages

As we described in Section 3.2, if $f: M \to M$ is a $C^{1+\alpha}$ diffeomorphism and μ is an ergodic f-invariant hyperbolic finite measure, then

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d^s + d^u \tag{34}$$

for μ-almost every point $x \in M$, where the numbers d^s and d^u are as in (8). Of course that this does not mean that *all* points necessarily satisfy (34). Multifractal analysis precisely studies the properties of the level sets

$$\left\{x \in M : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha\right\} \tag{35}$$

for each $\alpha \in \mathbb{R}$. In this section we present the main components of multifractal analysis and describe its relation with the theory of dynamical systems.

Birkhoff's ergodic theorem—one of the basic but fundamental results of ergodic theory—states that if $S: X \to X$ is a measurable transformation preserving a finite measure μ on X, then for each function $\varphi \in L^1(X,\mu)$ the limit

$$\varphi_S(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(S^k x)$$

exists for μ-almost every point $x \in X$. Furthermore, if μ is ergodic (see Section 3.2) then

$$\varphi_S(x) = \frac{1}{\mu(X)} \int_X \varphi \, d\mu \tag{36}$$

for μ-almost every $x \in X$. Again this does not mean that the identity in (36) is valid for *every* point $x \in X$ for which $\varphi_S(x)$ is well-defined. For each $\alpha \in \mathbb{R}$ we define the level set

$$K_\alpha(\varphi) = \left\{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(S^k x) = \alpha\right\},$$

i.e., the set of points $x \in X$ such that $\varphi_S(x)$ is well-defined and equal to α. We also consider the set

$$K(\varphi) = \left\{x \in X : \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(S^k x) < \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(S^k x)\right\}. \tag{37}$$

It is clear that

$$X = K(\varphi) \cup \bigcup_{\alpha\in\mathbb{R}} K_\alpha(\varphi). \tag{38}$$

Furthermore, the sets in this union (possibly uncountable) are pairwise disjoint. We call the decomposition of X in (38) a *multifractal decomposition.*

One way to measure the complexity of the sets $K_\alpha(\varphi)$ is to compute their Hausdorff dimension. We define a function

$$\mathcal{D}\colon \{\alpha \in \mathbb{R} : K_\alpha(\varphi) \neq \varnothing\} \to \mathbb{R}$$

by

$$\mathcal{D}(\alpha) = \dim_H K_\alpha(\varphi).$$

We also define the numbers

$$\underline{\alpha} = \inf\left\{\int_X \varphi\, d\mu : \mu \in \mathcal{M}\right\} \quad \text{and} \quad \overline{\alpha} = \sup\left\{\int_X \varphi\, d\mu : \mu \in \mathcal{M}\right\},$$

where $\mathcal{M}$ represents the family of S-invariant probability measures on X. It is easy to verify that $K_\alpha(\varphi) = \varnothing$ whenever $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$. We also define the function $T\colon \mathbb{R} \to \mathbb{R}$ by

$$T(q) = P(q\varphi) - qP(\varphi)$$

(where P denotes the topological pressure). For topological Markov chains (see Section 5.1) the function T is analytic (see the book by Ruelle [100]). Under the assumptions in Theorem 11 below there exists a unique equilibrium measure ν_q of $q\varphi$ (see Section 4.2).

The following result shows that in the case of topological Markov chains the set $K_\alpha(\varphi)$ is nonempty for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ and that the function $\mathcal{D}$ is analytic and strictly convex.

Theorem 11 (Multifractal analysis of Birkhoff averages) *If $\sigma|X$ is a topologically mixing topological Markov chain and $\varphi\colon X \to \mathbb{R}$ is a Hölder continuous function then:*

1. *$K_\alpha(\varphi)$ is dense in X for each $\alpha \in (\underline{\alpha}, \overline{\alpha})$;*
2. *the function $\mathcal{D}\colon (\underline{\alpha}, \overline{\alpha}) \to \mathbb{R}$ is analytic and strictly convex;*
3. *the function $\mathcal{D}$ is the Legendre transform of T, i.e.,*

$$\mathcal{D}(-T'(q)) = T(q) - qT'(q)$$

for each $q \in \mathbb{R}$;

4. *if $q \in \mathbb{R}$ then $\nu_q(K_{-T'(q)}(\varphi)) = 1$ and*

$$\lim_{r \to 0} \frac{\log \nu_q(B(x,r))}{\log r} = T(q) - qT'(q)$$

for ν_q-almost every point $x \in K_{-T'(q)}(\varphi)$.

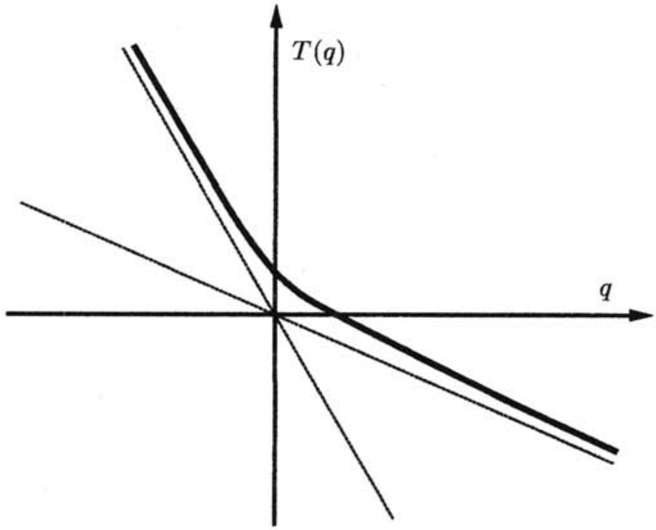

Figure 7: Graph of the function T

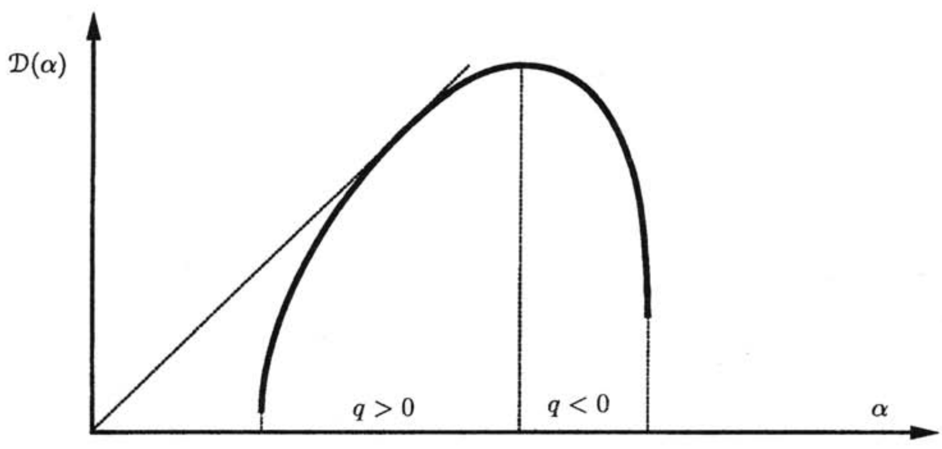

Figure 8: Graph of the function $\mathcal{D}$

See Figures 7 and 8 for typical graphs of the functions T and $\mathcal{D}$.

Statement 1 in Theorem 11 is an exercise (note that we are considering one-sided topological Markov chains, although all the results readily extend to the case of two-sided topological Markov chains). The remaining statements in Theorem 11 are a particular case of results formulated by Barreira and Schmeling in [15]. These were obtained as a consequence of results of Pesin and Weiss in [86], where they effect a multifractal analysis for conformal repellers (see Section 5.1). In [107], Schmeling showed that the domain of $\mathcal{D}$ coincides with $[\underline{\alpha}, \overline{\alpha}]$, i.e., that $K_\alpha(\varphi) \neq \varnothing$ if and only if $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [49]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [34] for a class of measures invariant under one-dimensional Markov maps. In [66], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [96], Rand studied Gibbs measures for a class of repellers. We refer the reader to the book by Pesin [82] for a more detailed discussion and further references.

Theorem 11 reveals an enormous complexity of multifractal decompositions that is not foreseen by Birkhoff's ergodic theorem. In particular it shows that the multifractal decomposition in (38) is composed by an *uncountable* number of (pairwise disjoint) dense invariant sets, each of them having positive Hausdorff dimension. We will see in Section 7.1 that the set $K(\varphi)$ in (37) is also very complex (even though it has zero measure with respect to *any* finite invariant measure, as a simple consequence of Birkhoff's ergodic theorem).

We now come back to the study of the level sets in (35). Let M be a surface and $\Lambda \subset M$ a locally maximal compact hyperbolic set for a $C^{1+\alpha}$ diffeomorphism $f\colon M \to M$. We assume that f is topologically mixing on Λ (see Section 5.1). Consider an equilibrium measure μ of a Hölder continuous function $\varphi\colon \Lambda \to \mathbb{R}$. Under these assumptions μ is unique and thus it is ergodic (see for example [56]).

We define functions $T_s\colon \Lambda \to \mathbb{R}$ and $T_u\colon \Lambda \to \mathbb{R}$ by

$$T_s(q) = P(-q \log\|df|E^s\| + q\varphi) - qP(\varphi)$$

and

$$T_u(q) = P(q \log\|df|E^u\| + q\varphi) - qP(\varphi).$$

In [118], Simpelaere showed that

$$\dim_H \left\{ x \in M : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} = T_s(q) - qT_s'(q) + T_u(q) - qT_u'(q),$$

where $q \in \mathbb{R}$ is the unique real number such that $\alpha = -T_s'(q) - T_u'(q)$. Another proof of this statement was given by Pesin and Weiss in [87] (see also [82]). Again we observe an enormous complexity that is not precluded by the μ-almost everywhere existence of the pointwise dimension.

In the case of hyperbolic flows versions of these results were obtained by Barreira and Saussol [8] (in the case of entropy spectra; see Section 6.2) and by

Pesin and Sadovskaya [84], using in particular the symbolic dynamics developed by Bowen [28] and Ratner [97].

6.2 General concept of multifractal analysis

In fact the approach of multifractal analysis extends to many other classes of dynamical systems and to other local quantities. With the purpose of unifying the theory, Barreira, Pesin and Schmeling [5] proposed a general concept of multifractal analysis that we now describe.

We consider a function $g\colon Y \to [-\infty, +\infty]$ defined on a subset Y of X. The level sets

$$K_\alpha^g = \{x \in X : g(x) = \alpha\}$$

are pairwise disjoint and we obtain a *multifractal decomposition* of X given by

$$X = (X \setminus Y) \cup \bigcup_{\alpha \in [-\infty, +\infty]} K_\alpha^g. \tag{39}$$

Let now G be a function defined on the subsets of X. We define the *multifractal spectrum* $\mathcal{F}\colon [-\infty, +\infty] \to \mathbb{R}$ of the pair (g, G) by

$$\mathcal{F}(\alpha) = G(K_\alpha^g).$$

When X is a compact manifold and g is differentiable, each level set K_α^g is a hyper-surface for all values of α that are not critical values of g. In multifractal analysis we are mostly interested in the study of level sets of functions that are not differentiable (and typically are only measurable), that naturally appear in the theory of dynamical systems.

Furthermore, the multifractal spectra encode precious information about these functions and ultimately about the dynamical system that originated them. In applications we have frequently no information about the "microscopic" nature of the dynamical system but only information about "macroscopic" quantities such as for example about multifractal spectra. It is therefore important to try to recover information about the dynamical system through the information given by these "macroscopic" quantities and in particular by the multifractal spectra (see also the discussion in Section 7.2).

We now describe some of the functions g and G that naturally occur in dynamical systems. Let X be a separable metric space and $f\colon X \to X$ a continuous function. We define functions G_D and G_E by

$$G_D(Z) = \dim_H Z \quad \text{and} \quad G_E(Z) = h(f|Z).$$

We call *dimension spectra* and *entropy spectra* respectively to the spectra generated by G_D and G_E.

Let μ be a finite Borel measure on X and $Y \subset X$ the set of points $x \in X$ for which the pointwise dimension

$$g_D(x) = g_D^{(\mu)}(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

is well-defined. We obtain two multifractal spectra

$$\mathcal{D}_D = \mathcal{D}_D^{(\mu)} \quad \text{and} \quad \mathcal{D}_E = \mathcal{D}_E^{(\mu)}$$

specified respectively by the pairs (g_D, G_D) and (g_D, G_E). For $C^{1+\alpha}$ diffeomorphisms and hyperbolic invariant measures, Theorem 5 ensures that $\mu(X \setminus Y) = 0$.

Let now X be a separable metric space and $f\colon X \to X$ a continuous transformation preserving a probability measure μ on X. Given a partition ξ of X, for each $n \in \mathbb{N}$ we define a new partition of X by $\xi_n = \bigvee_{k=0}^{n-1} f^{-k}\xi$. We consider the set Y formed by the points $x \in X$ for which the *local entropy*

$$g_E(x) = g_E^{(\mu)}(x) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

is well-defined, where $\xi_n(x)$ denotes the element of ξ_n containing x. By the Shannon–McMillan–Breiman theorem in the entropy theory (see for example [56]) we have $\mu(X \setminus Y) = 0$. Furthermore, if ξ is a generating partition and μ is ergodic then $g_E(x) = h_\mu(f)$ for μ-almost every point $x \in X$. We obtain two multifractal spectra

$$\mathcal{E}_D = \mathcal{E}_D^{(\mu)} \quad \text{and} \quad \mathcal{E}_E = \mathcal{E}_E^{(\mu)}$$

specified respectively by the pairs (g_E, G_D) and (g_E, G_E).

We can also consider functions defined by the Lyapunov exponents. In this case Theorem 4 guarantees that the involved limits exist almost everywhere. We only consider a particular case here. Let X be a differentiable manifold and $f\colon X \to X$ a C^1 map. Consider the set $Y \subset X$ of points $x \in X$ for which the limit

$$\lambda(x) = \lim_{n\to+\infty} \frac{1}{n} \log\|d_x f^n\|$$

exists. By Theorem 4 (or by Kingman's subadditive ergodic theorem), if μ is an f-invariant probability Borel measure, then $\mu(X \setminus Y) = 0$. We define the function g_L on Y by

$$g_L(x) = \lambda(x).$$

We obtain two multifractal spectra $\mathcal{L}_D$ and $\mathcal{L}_E$ specified respectively by the pairs (g_L, G_D) and (g_L, G_E).

The spectrum $\mathcal{D}_D = \mathcal{D}$ was already considered in Section 6.1. We now describe the spectrum $\mathcal{E}_E$. Given a compact hyperbolic set Λ and a continuous function $\varphi\colon \Lambda \to \mathbb{R}$ we define the function $T_E\colon \mathbb{R} \to \mathbb{R}$ by

$$T_E(q) = P(q\varphi) - qP(\varphi).$$

Under the assumptions in Theorem 12 below there exists a unique equilibrium measure ν_q of $q\varphi$. We also define the numbers

$$\underline{\beta} = \inf\left\{-\int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}\right\} \quad \text{and} \quad \overline{\beta} = \sup\left\{-\int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}\right\},$$

where $\mathcal{M}$ denotes the family of f-invariant probability measures on Λ.

Theorem 12 (Multifractal analysis of the spectrum $\mathcal{E}_E$) *Let Λ be a compact hyperbolic set of a C^1 diffeomorphism $f\colon M \to M$, such that f is topologically mixing on Λ. If μ is the equilibrium measure of a Hölder continuous function $\varphi\colon \Lambda \to \mathbb{R}$ then:*

1. *$K_\alpha^{g_E}$ is dense in Λ for each $\alpha \in (\underline{\beta}, \overline{\beta})$;*
2. *the function $\mathcal{E}_E$ is analytic and strictly convex on $(\underline{\beta}, \overline{\beta})$;*
3. *the function $\mathcal{E}_E$ is the Legendre transform of T_E;*
4. *for each $q \in \mathbb{R}$ we have*

$$g_E^{(\nu_q)}(x) = T_E(q) - qT_E'(q)$$

for ν_q-almost every point $x \in \Lambda$.

Theorem 12 is an immediate consequence of results of Barreira, Pesin and Schmeling in [6] (see also [15]).

We note that in Theorem 12 the manifold is not necessarily two-dimensional, contrarily to what happens in Section 6.1 when we described the spectrum $\mathcal{D}_D$. In the case of conformal repellers, Pesin and Weiss [86] obtained a multifractal analysis of $\mathcal{D}_D$ and Barreira, Pesin and Schmeling [5] obtained a multifractal analysis of $\mathcal{E}_E$. In [121], Takens and Verbitski obtained a multifractal analysis of the spectrum $\mathcal{E}_E$ for expansive homeomorphisms with specification and a certain class of continuous functions (note that these systems need not have Markov partitions).

We note that the spectra $\mathcal{D}_D$ and $\mathcal{E}_E$ are of different nature from that of the spectra $\mathcal{D}_E$ and $\mathcal{E}_D$. Namely, the first two relate pointwise quantities—the pointwise dimension and the local entropy—with global quantities that are naturally associated to them—respectively the Hausdorff dimension and the Kolmogorov–Sinai entropy. On the other hand, the spectra $\mathcal{D}_E$ and $\mathcal{E}_D$ mix local and global quantities of distinct nature. We refer to them as *mixed spectra*. It is reasonable to expect that the mixed spectra contain additional information about the dynamical system. It is also possible to describe the multifractal properties of these spectra although this requires a different approach (see Section 8.1 for details).

The spectrum $\mathcal{L}_D$ was studied in [6, 129]. The spectrum $\mathcal{L}_E$ was studied in [5, 6] (it was introduced in [38]). See also [90] for the study of transformations of the interval with an infinite number of branches.

7 Irregular sets and multifractal rigidity

7.1 Multifractal analysis and irregular sets

In the last section we described the main components of multifractal analysis for several multifractal spectra. These spectra are obtained from decompositions as that in (38) and more generally as that in (39). In particular we possess a very detailed information from the ergodic, topological, and dimensional point of view about the level sets in each multifractal decomposition. However, we gave no information about the "irregular" set in these decompositions, i.e., the set $K(\varphi)$ in (38) and the set $X \setminus Y$ in (39).

For example, when $\varphi\colon X \to \mathbb{R}$ is a continuous function, which is thus in $L^1(X,\mu)$ for any finite (invariant) measure μ on X, it follows from Birkhoff's ergodic theorem that the set $K(\varphi)$ in (38) has zero measure with respect to *any* S-invariant finite measure on X. Therefore, at least from the point of view of measure theory, the set $K(\varphi)$ is very small. However, we will see that, remarkably, from the point of view of dimension theory, this set is as large as the whole space, revealing once more a considerable complexity (now for the "irregular" part of the multifractal decomposition).

We first make a little digression about the concept of cohomology in dynamical systems. Let $S\colon X \to X$ be a continuous transformation of the topological space X. For each function $\varphi\colon X \to \mathbb{R}$ we consider the *irregular set* $K(\varphi)$ in (38). Two continuous functions $\varphi_1\colon X \to \mathbb{R}$ and $\varphi_2\colon X \to \mathbb{R}$ are said to be *cohomologous* if there exists a continuous function $\psi\colon X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 = \psi - \psi \circ S + c$$

on X. It is easy to verify that if φ_1 and φ_2 are cohomologous then $K(\varphi_1) = K(\varphi_2)$ and $c = P(\varphi_1) - P(\varphi_2)$. In particular, if the function φ is cohomologous to a constant then $K(\varphi) = \varnothing$. The following result of Barreira and Schmeling in [15] shows that if φ is not cohomologous to a constant then $K(\varphi)$ possesses a considerable complexity from the point of view of entropy and Hausdorff dimension. Recall that $h(f|X)$ denotes the topological entropy of $f|X$ (see Section 4.2).

Theorem 13 (Irregular sets) *If X is a repeller of a $C^{1+\alpha}$ transformation, for some $\alpha > 0$, such that f is conformal and topologically mixing on X, and $\varphi\colon X \to \mathbb{R}$ is a Hölder continuous function, then the following properties are equivalent:*

1. *φ is not cohomologous to a constant;*
2. *$K(\varphi)$ is a nonempty dense set in X with*

$$h(f|K(\varphi)) = h(\sigma|X) \quad \textit{and} \quad \dim_H K(\varphi) = \dim_H X. \tag{40}$$

For topological Markov chains, the first identity in (40) was extended by Fan, Feng and Wu [44] to arbitrary continuous functions. We note that in this case the first and second identities in (40) are equivalent. See also [15] for an appropriate version of Theorem 13 in the case of hyperbolic sets.

We recall that under the hypotheses in Theorem 13 the set $K(\varphi)$ has zero measure with respect to *any* invariant measure (in particular $K(\varphi) \neq X$). Theorem 13 provides a necessary and sufficient condition for the set $K(\varphi)$ to be as large as the whole space from the point of view of entropy and Hausdorff dimension. Of course that a priori property 1 in Theorem 13 could be rare. However, precisely the opposite happens. Let $C^\theta(X)$ be the space of Hölder continuous functions on X with Hölder exponent $\theta \in (0,1]$ equipped with the norm

$$\begin{aligned}\|\varphi\|_\theta &= \sup\{|\varphi(x)| : x \in X\} \\ &\quad + \sup\left\{C > 0 : \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\theta} \le C \text{ for each } x, y \in X\right\}.\end{aligned}$$

It is shown in [15] that for each $\theta \in (0,1]$ the family of functions in $C^\theta(X)$ that are not cohomologous to a constant forms an open dense set. Therefore, given $\theta \in (0,1]$ and a generic function φ in $C^\theta(X)$ the set $K(\varphi)$ is nonempty, dense in X, and satisfies the identities in (40).

Let now $K = \bigcup_\varphi K(\varphi)$ where the union is taken over all Hölder continuous functions $\varphi\colon X \to \mathbb{R}$. Under the hypotheses of Theorem 13 we immediately conclude that

$$h(\sigma|K) = h(\sigma|X) \quad \text{and} \quad \dim_H K = \dim_H X.$$

These identities were established by Pesin and Pitskel in [83] when σ is a Bernoulli shift with two symbols, i.e., when $A = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ is the transition matrix. Their methods are different from those in [15]. Until now it was impossible to extend the approach in [83] even to the Bernoulli shift with three symbols.

A related result of Shereshevsky in [114] shows that for a generic C^2 surface diffeomorphism with a locally maximal compact hyperbolic set Λ, and an equilibrium measure μ of a Hölder continuous generic function in the C^0 topology, the set

$$I = \left\{x \in \Lambda : \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}\right\}$$

has positive Hausdorff dimension. This result is a particular case of results in [15] showing that in fact $\dim_H I = \dim_H \Lambda$ (under those generic assumptions).

7.2 Multifractal classification of dynamical systems

The former sections illustrate an enormous complexity that occurs in a natural way in the study of the multifractal properties of dynamical systems. On the other hand, in the "experimental" study of dynamical systems it is common to have only partial information. The multifractal spectra present themselves as "observable" quantities and can be determined within fairly arbitrary precision at the expense of the macroscopic observation of the phase space. It is thus of interest to investigate how to recover partially or even fully the information about of a given dynamical system, using the information contained in the multifractal spectra. This problem belongs to the theory of *multifractal rigidity.* In this section we want to illustrate with a simple example how it is possible to make this approach rigorous.

Let g and h be piecewise linear transformations of the interval $[0,1]$ with repellers given by

$$J_g = \bigcap_{n=0}^{\infty} g^{-n}(A_g \cup B_g) \quad \text{and} \quad J_h = \bigcap_{n=0}^{\infty} h^{-n}(A_h \cup B_h),$$

where A_g, B_g, A_h and B_h are closed intervals in $[0,1]$ such that

$$g(A_g) = g(B_g) = h(A_h) = h(B_h) = [0,1] \quad \text{and} \quad A_g \cap B_g = A_h \cap B_h = \varnothing.$$

See Figure 9. Both repellers can be coded by a Bernoulli shift with two symbols. We consider two Bernoulli measures μ_g and μ_h (each with two symbols) invariant respectively under g and h.

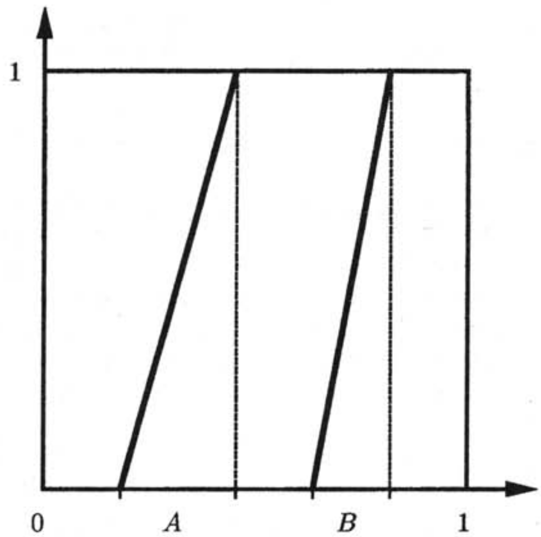

Figure 9: Piecewise linear expanding transformation

Theorem 14 (Multifractal rigidity) *If $\mathcal{D}_D^{(\mu_g)} = \mathcal{D}_D^{(\mu_h)}$ then there exists a homeomorphism $\chi\colon J_g \to J_h$ such that $dg = dh \circ \chi$ and $\mu_g = \mu_h \circ \chi$.*

Theorem 14 is due to Barreira, Pesin and Schmeling [5] and provides a multifractal classification based on the spectrum $\mathcal{D}_D$. The coincidence of the spectra of g and h guarantees in particular that the derivatives of g and h are equal at corresponding points of the repellers and namely at the periodic points. There is also a version of this result for hyperbolic sets [6]. A local version of Theorem **14** was obtained in [3] for a more general class of dynamical systems and arbitrary equilibrium measures.

For more complex dynamical systems it may be necessary to use more than one multifractal spectrum in order to obtain a multifractal classification analogous to that given by Theorem 14. This is one of the motivations for the study of other multifractal spectra and namely of the mixed spectra (see Sections 6.2 and 8.1).

8 Variational principles and number theory

8.1 Variational principles and dimension theory

As we mentioned in Section 6.2, one can consider several other multifractal spectra and in particular the mixed spectra $\mathcal{D}_E$ and $\mathcal{E}_D$. These two spectra combine local and global characteristics of distinct nature, which depend not only on the dynamics but also on the local structure provided by a given invariant measure. As we described above (see Section 6), the spectra $\mathcal{D}_D$ and $\mathcal{E}_E$ are analytic in several situations. Furthermore, they coincide with the Legendre transform of certain functions, defined in terms of the topological pressure. In particular, this allows one to show that they are always convex.

In order to explain why the study of the mixed multifractal spectra is different from the study of the nonmixed spectra, we recall the level sets

$$K_\alpha^{g_D} = \{x \in X : g_D(x) = \alpha\} \quad \text{and} \quad K_\alpha^{g_E} = \{x \in X : g_E(x) = \alpha\}$$

(see Section 6.2). The main difficulty when we study the mixed spectra is that these two families of sets need not satisfy any Fubini type decomposition.

In the case of conformal repellers, Barreira and Saussol [10] obtained the following characterization of the mixed spectra $\mathcal{D}_E$ and $\mathcal{E}_D$.

Theorem 15 (Characterization of the mixed spectra) *For a repeller X of a $C^{1+\varepsilon}$ transformation f, for some $\varepsilon > 0$, such that f is conformal and topologically mixing on X, if $\varphi\colon X \to \mathbb{R}$ is a Hölder continuous function with $P(\varphi) = 0$ and μ is an equilibrium measure of φ then*

$$\mathcal{D}_E^{(\mu)}(\alpha) = \max\left\{h_\nu(f) : \nu \text{ is ergodic and } -\frac{\int_X \varphi\, d\nu}{\int_X \log\|df\|\, d\nu} = \alpha\right\} \tag{41}$$

and

$$\mathcal{E}_D^{(\mu)}(\alpha) = \max\left\{\dim_H \nu : \nu \text{ is ergodic and } -\int_X \varphi\, d\nu = \alpha\right\}. \tag{42}$$

We call *conditional variational principle* to each of the identities in (41) and (42). We can also obtain conditional variational principles for the spectra $\mathcal{D}_D$ and $\mathcal{E}_E$ (although in this case the results are essentially equivalent to the corresponding multifractal analysis described in the former sections):

$$\mathcal{D}_D^{(\mu)}(\alpha) = \max\left\{\dim_H \nu : \nu \text{ is ergodic and } -\frac{\int_X \varphi\, d\nu}{\int_X \log\|df\|\, d\nu} = \alpha\right\} \tag{43}$$

and

$$\mathcal{E}_E^{(\mu)}(\alpha) = \max\left\{h_\nu(f) : \nu \text{ is ergodic and } -\int_X \varphi\, d\nu = \alpha\right\}. \tag{44}$$

Some of the consequences that can be obtained from Theorem 15 (see [10] for details) are the following:

1. the functions $\mathcal{D}_E$ and $\mathcal{E}_D$ are analytic;
2. the functions $\mathcal{D}_E$ and $\mathcal{E}_D$ are in general not convex and thus cannot be expressed as Legendre transforms.

The last statement strongly contrasts with what happens with the nonmixed spectra, which are always convex.

In the case of the full shift, the identity in (44) was first established by Olivier [73, 74], for the more general class of the so-called g-measures. This class, introduced by Keane in [59], is composed of equilibrium measures of a class of continuous functions that need not be Hölder continuous. It is known that any Gibbs measure is a g-measure (see [74] for details). We note that in the case of the shift map the identity (44) is equivalent to any of the identities (41), (42), and (43): simply model the full shift with m symbols by the piecewise expanding map of the interval $x \mapsto mx$ (mod 1) and observe that

$$\dim_H A = \frac{h(\sigma|A)}{\log m} \quad \text{and} \quad \dim_H \nu = \frac{h_\nu(\sigma)}{\log m}$$

for any ergodic σ-invariant measure ν.

In [10] the authors obtained conditional variational principles in the more general case of transformations with upper semi-continuous entropy (i.e., transformations f for which $\nu \mapsto h_\nu(f)$ is upper semi-continuous), for functions with a unique equilibrium measure (and thus for a dense family of functions; see the book by Ruelle [100]). In [122], Takens and Verbitski established a conditional variational principle for the spectrum $\mathcal{E}_E$ for equilibrium measures that are not necessarily unique.

For example, when $f : X \to X$ is a one-sided or two-sided topological Markov that is topologically mixing, or is an expansive homeomorphism that satisfies specification, the entropy is upper semi-continuous. If, in addition, φ is a continuous function with a certain "bounded variation" then it has a unique equilibrium measure; see [56, 60] for details. On the other hand, one can exhibit plenty transformations that do not satisfy specification but for which the entropy is still upper semi-continuous. For example, all β-shifts are expansive and thus the metric entropy is upper semi-continuous (see [60] for details), but in [106] Schmeling showed that for β in a residual set with full Lebesgue measure (although the complement has full Hausdorff dimension) the corresponding β-shift does not satisfy specification. It follows from work of Walters [127] that for every β-shift each Lipschitz function has a unique equilibrium measure.

In [12], Barreira and Saussol obtained conditional variational principles for hyperbolic flows.

8.2 Extensions and applications to number theory

In the theory of dynamical systems we are frequently interested in more that one local quantity at the same time. Examples include the Lyapunov exponents, the local entropy, and the pointwise dimension. However, the theory described above (in Sections 6 and 8.1) only allows us to consider separately each of these characteristics. This observation is a motivation to develop a multi-dimensional version of multifractal analysis. More precisely, we want to consider, for example, intersections of level sets of Birkhoff averages, such as

$$K_{\alpha,\beta} = K_\alpha(\varphi) \cap K_\beta(\psi),$$

and to describe their multifractal properties, including their "size" in terms of the topological entropy and of the Hausdorff dimension.

The corresponding multi-dimensional multifractal spectra exhibit several nontrivial phenomena that are absent in the one-dimensional case. Furthermore, the known approaches to the study of one-dimensional multifractal spectra have to be modified to treat the new situation. Nevertheless, the unifying theme continues to be the use of the thermodynamic formalism.

We now illustrate the results with a rigorous statement in the case of topological Markov chains. Let $\mathcal{M}$ be the family of σ-invariant probability measures on X and consider the set

$$\mathcal{D} = \left\{ \left(\int_X \varphi \, d\mu, \int_X \psi \, d\mu \right) \in \mathbb{R}^2 : \mu \in \mathcal{M} \right\}.$$

The following result is a conditional variational principle for the sets $K_{\alpha,\beta}$.

Theorem 16 (Conditional variational principle) *Let $\sigma|X$ be a topologically mixing topological Markov chain, and φ and ψ Hölder continuous functions on X. Then, for each $(\alpha, \beta) \in \operatorname{int} \mathcal{D}$ we have $K_{\alpha,\beta} \neq \varnothing$ and*

$$\begin{aligned} h(\sigma|K_{\alpha,\beta}) &= \sup \left\{ h_\mu(\sigma) : \left(\int_X \varphi \, d\mu, \int_X \psi \, d\mu \right) = (\alpha, \beta) \text{ with } \mu \in \mathcal{M} \right\} \\ &= \inf \left\{ P(p(\varphi - \alpha) + q(\psi - \beta)) : (p, q) \in \mathbb{R}^2 \right\}. \end{aligned} \tag{45}$$

Theorem 16 is a particular case of results of Barreira, Saussol and Schmeling [13]. Namely, they also consider the intersection of any finite number of level sets of Birkhoff averages, as well as of other local quantities such as pointwise dimensions, local entropies, and Lyapunov exponents. The first identity in (45) was obtained independently by Fan, Feng and Wu [44], also in the more general case of arbitrary continuous functions. In [122], Takens and Verbitski provided generalizations of these results. We will see below that the second identity in (45) can be applied with success to several problems in number theory.

It is also shown in [13] that if $\sigma|X$ is a topologically mixing topological Markov chain and φ and ψ are Hölder continuous functions on X, then the following properties hold:

1. if $(\alpha, \beta) \notin \overline{\mathcal{D}}$ then $K_{\alpha,\beta} = \varnothing$;
2. if for every $(p,q) \in \mathbb{R}^2$ the function $p\varphi + q\psi$ is not cohomologous to a constant then $\mathcal{D} = \overline{\operatorname{int} \mathcal{D}}$;
3. the function $(\alpha, \beta) \mapsto h(\sigma|K_{\alpha,\beta})$ is analytic on $\operatorname{int} \mathcal{D}$;
4. there is an ergodic equilibrium measure $\mu_{\alpha,\beta} \in \mathcal{M}$ with $\int_X \varphi \, d\mu = \alpha$ and $\int_X \psi \, d\mu = \beta$, such that
$$\mu_{\alpha,\beta}(K_{\alpha,\beta}) = 1 \quad \text{and} \quad h_{\mu_{\alpha,\beta}}(\sigma) = h(\sigma|K_{\alpha,\beta}).$$

In particular, the second property provides a condition which guarantees that the identities in Theorem 16 are valid for an open and dense set of pairs $(\alpha, \beta) \in \mathcal{D}$.

Besides their own interest, these results have several applications and namely applications to number theory. Instead of formulating general statements here we will describe explicit examples that illustrate well the nature of the results obtained by Barreira, Saussol and Schmeling in [14]. Given an integer $m > 1$, for each number $x \in [0,1]$ we denote by $x = 0.x_1x_2\cdots$ the base-m representation of x. It is immediate that this representation is unique except for a countable set of points. Since countable sets have zero Hausdorff dimension, the nonuniqueness of the representation does not affect the study of the dimensional properties.

For each $k \in \{0, \ldots, m-1\}$, $x \in [0,1]$ and $n \in \mathbb{N}$ we define

$$\tau_k(x,n) = \operatorname{card}\{i \in \{1, \ldots, n\} : x_i = k\}.$$

Whenever there exists the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n}$$

it is called the *frequency* of the number k in the base-m representation of x. A classical result of Borel [25] says that for Lebesgue-almost every $x \in [0,1]$ we have $\tau_k(x) = 1/m$ for every k. Furthermore, for $m = 2$, Hardy and Littlewood [50] showed that for Lebesgue-almost every $x \in [0,1]$, $k = 0, 1$, and all sufficiently large n,

$$\left|\frac{\tau_k(x,n)}{n} - \frac{1}{2}\right| < \sqrt{\frac{\log n}{n}}.$$

In particular, Lebesgue-almost all numbers are normal in every integer base. This remarkable result (that today is an immediate consequence of Birkhoff's ergodic theorem) does not imply that the set of numbers for which this does not happen is empty.

Consider now the set

$$F_m(\alpha_0, \ldots, \alpha_{m-1}) = \{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for } k = 0, \ldots, m-1\},$$

whenever $\alpha_0 + \cdots + \alpha_{m-1} = 1$ with $\alpha_i \in [0,1]$ for each i. It is composed of the numbers in $[0,1]$ having a ratio α_k of digits equal to k in its base-m representation for each k. A precursor result concerning the size of these sets from the point of view of dimension theory is due to Besicovitch [19]. For $m = 2$, he showed that if $\alpha \in (0, \frac{1}{2})$ then

$$\dim_H \left\{x \in [0,1] : \limsup_{n\to\infty} \frac{\tau_1(x,n)}{n} \le \alpha\right\} = -\frac{\alpha\log\alpha + (1-\alpha)\log(1-\alpha)}{\log 2}.$$

More detailed information was later obtained by Eggleston [39], who showed that

$$\dim_H F_m(\alpha_0, \ldots, \alpha_{m-1}) = -\sum_{k=0}^{m-1} \alpha_k \log_m \alpha_k. \tag{46}$$

We note that it is easy to show—and this does not require the above result—that each set $F_m(\alpha_0, \ldots, \alpha_{m-1})$ is dense in $[0,1]$. The identity in (46) can be established by applying Theorem 11 when $m = 2$, Theorem 16 when $m = 3$, and an appropriate generalization of Theorem 16 when $m \ge 4$, thus providing a new proof of Eggleston's result (see [13] for details).

We now consider sets of more complicated nature. Let $m = 3$ and consider the set

$$F = \{x \in [0,1] : \tau_1(x) = 5\tau_0(x)\}.$$

This is the set of numbers in $[0,1]$ for which the base-3 representation has a ratio of ones that is five times the ratio of zeros. The ratio of the digit two is arbitrary. It follows from work in [14] that

$$\dim_H F = \frac{\log(1 + 6/5^{5/6})}{\log 3} \approx 0.85889\cdots. \tag{47}$$

In order to explain how this result is obtained, we first observe that

$$F = \bigcup_{\alpha\in[0,1/6]} F_3(\alpha, 5\alpha, 1-6\alpha). \tag{48}$$

It is easy to show that the constant in (47) is a lower estimate for $\dim_H F$. Namely, it follows from (46) and (48) that, since $F \supset F_3(\alpha, 5\alpha, 1-6\alpha)$ for each α,

$$\dim_H F \geq \max_{\alpha\in[0,1/6]} -\frac{\alpha\log\alpha + 5\alpha\log(5\alpha) + (1-6\alpha)\log(1-6\alpha)}{\log 3}. \tag{49}$$

The maximum in (49) is attained at $\alpha = 1/(5^{5/6}+6)$ and it is easy to verify that it is equal to the constant in (47). This establishes a lower estimate for the Hausdorff dimension.

The corresponding upper estimate is more delicate, namely because the union in (48) is composed of an *uncountable* number of pairwise disjoint sets. The approach in [14] uses a generalization of the conditional variational principle in Theorem 16, now for quotients of Birkhoff averages (see [14] for details). In the particular case considered here this variational principle states that for each $k \neq \ell$ and $\beta \geq 0$ we have

$$\dim_H \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\} = \max \left\{ -\sum_{j=0}^{m-1} \alpha_j \log_m \alpha_j : \frac{\alpha_k}{\alpha_\ell} = \beta \right\}.$$

We conclude that for each $k \neq \ell$ and $\beta \geq 0$ there exists a set

$$F_m(\alpha_0, \ldots, \alpha_{m-1}) \subset \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\}$$

with

$$\dim_H F_m(\alpha_0, \ldots, \alpha_{m-1}) = \dim_H \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\}.$$

In particular, letting $m = 3$, $k = 1$, $\ell = 0$, and $\beta = 5$ we conclude from (48) that the inequality in (49) is in fact an identity, and we establish (47).

These applications to number theory are special cases of results established in [14]: these include the study of sets defined in terms of relations between the numbers $\tau_0(x), \ldots, \tau_{m-1}(x)$, and the study of sets defined in terms of frequencies of blocks of digits, or even for which some blocks are forbidden (thus generalizing work of Billingsley [20]).

9 Quantitative recurrence and dimension theory

9.1 Quantitative recurrence

The Poincaré recurrence theorem (Theorem 2), as described in Section 1.2, is one of the basic but fundamental results of the theory of dynamical systems. Unfortunately it only provides information of qualitative nature. In particular it does not consider, for example, any of the following natural problems:

1. with which frequency the orbit of a point visits a given set of positive measure;

2. with which rate the orbit of a point returns to an arbitrarily small neighborhood of the initial point.

Birkhoff's ergodic theorem gives a complete answer to the first problem. The second problem experienced a growing interest during the last decade, also in connection with other fields, including compression algorithms, and numerical study of dynamical systems.

We consider a transformation $f\colon M \to M$. The *return time* of a point $x \in M$ to the ball $B(x,r)$ (with respect to f) is given by

$$\tau_r(x) = \inf\{n \in \mathbb{N} : d(f^n x, x) < r\}.$$

The *lower* and *upper recurrence rates* of x (with respect to f) are defined by

$$\underline{R}(x) = \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}. \tag{50}$$

Whenever $\underline{R}(x) = \overline{R}(x)$ we denote the common value by $R(x)$ and call it the *recurrence rate* of x (with respect to f).

In the present context, the study of the quantitative behavior of recurrence started with the work of Ornstein and Weiss [75], closely followed by the work of Boshernitzan [26]. In [75] the authors considered the case of symbolic dynamics (and thus the corresponding symbolic metric in (13)) and an ergodic σ-invariant measure μ, and showed that $R(x) = h_\mu(\sigma)$ for μ-almost every x (see Section 9.2 for more details). On the other hand, Boshernitzan considered an arbitrary metric space M and showed in [26] that

$$\underline{R}(x) \le \dim_H \mu \tag{51}$$

for μ-almost every $x \in M$ (although the result in [26] is formulated differently, it is shown in [9] that it can be rephrased in this manner). It is shown in [9] that the inequality (51) may be strict.

In the case of hyperbolic sets, the following result of Barreira and Saussol in [9] shows that (51) often becomes an identity.

Theorem 17 (Quantitative recurrence) *For a $C^{1+\alpha}$ diffeomorphism with a hyperbolic set Λ, for some $\alpha > 0$, if μ is an ergodic equilibrium measure of a Hölder continuous function then*

$$R(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{52}$$

for μ-almost every point $x \in \Lambda$.

Theorem 17 is a version of the result of Ornstein and Weiss in [75] in the special case of symbolic dynamics. The proof of Theorem 17 combines new ideas with the study of hyperbolic measures by Barreira, Pesin and Schmeling [7] (see Section 3.1) and results and ideas of Saussol, Troubetzkoy and Vaienti [104] and of Schmeling and Troubetzkoy [111] (see also [109]). In [11], Barreira and Saussol established a related result in the case of repellers.

We note that the identity (52) relates two quantities of very different nature. In particular, only $R(x)$ depends on the diffeomorphism and only the pointwise dimension depends on the measure.

Theorem 17 provides quantitative information about the recurrence in hyperbolic sets. Putting together (50) and (52) we obtain

$$\lim_{r\to 0}\frac{\log\inf\{n\in\mathbb{N}: d(f^n x, x) < r\}}{-\log r} = \lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \tag{53}$$

for μ-almost every point $x\in\Lambda$. Thinking as if we could erase the limits in (53), we can say that Theorem 17 shows that

$$\inf\{k\in\mathbb{N}: f^k x\in B(x,r)\} \text{ is approximately equal to } 1/\mu(B(x,r))$$

when r is sufficiently small, that is, the time that the orbit of x takes to return to the ball $B(x,r)$ is approximately equal to $1/\mu(B(x,r))$. This should be compared to Kac's lemma: since μ is ergodic it tells us that

$$\int_{B(x,r)}\tau_r(y)\,d\mu(y) = 1.$$

Hence, the average value of τ_r on $B(x,r)$ is equal to $1/\mu(B(x,r))$. Therefore, Theorem 17 can be though of as a local version of Kac's lemma.

In another direction, the results in [9, 11] motivate the introduction of a new method to compute the Hausdorff dimension of a given measure (see Section 3.2). See [9] for details.

We now consider the case of repellers and briefly present two applications to number theory. Let $x = 0.x_1x_2\cdots$ be the base-m representation of the point $x\in[0,1]$. It was shown in [11] that

$$\inf\{n\in\mathbb{N}\colon |0.x_nx_{n+1}\cdots - 0.x_1x_2\cdots| < r\} \sim \frac{1}{r} \text{ when } r\to 0$$

for Lebesgue-almost every $x\in[0,1]$, meaning that

$$\lim_{r\to 0}\frac{\log\inf\{n\in\mathbb{N}\colon |0.x_nx_{n+1}\cdots - 0.x_1x_2\cdots| < r\}}{-\log r} = 1$$

for Lebesgue-almost every $x \in [0,1]$. Another example is given by the continued fractions. Writing each number $x \in (0,1)$ as a continued fraction

$$x = [m_1, m_2, m_3, \ldots] = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}},$$

with $m_i = m_i(x) \in \mathbb{N}$ for each i (this representation is unique except for a countable subset of $(0,1)$), it is shown in [11] that

$$\inf\{n \in \mathbb{N} : |[m_n, m_{n+1}, \ldots] - [m_1, m_2, \ldots]| < r\} \sim \frac{1}{r} \text{ when } r \to 0$$

for Lebesgue-almost every $x \in (0,1)$.

9.2 Product structure and recurrence

We already described the product structure of hyperbolic sets (see Section 1.1) and the product structure of hyperbolic measures (see Section 3.1). The study of quantitative recurrence can also be used to obtain new information about the product structure.

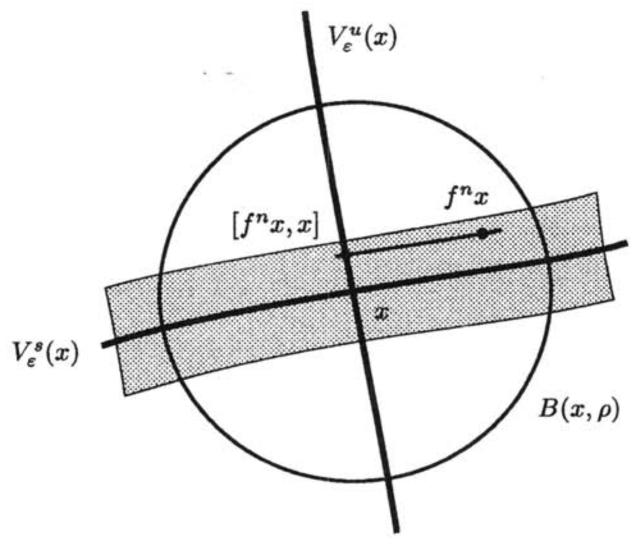

Figure 10: Definition of the unstable return time (the shaded area is the set of points at a d^u-distance of $V_\varepsilon^s(x)$ at most r)

Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism with a locally maximal compact hyperbolic set $\Lambda \subset M$, and denote by d_s and d_u the distances induced by the distance d of M respectively on each stable and unstable manifold. When $d(f^n x, x) \le \delta$, for each $\rho \le \delta$ we can define (see Figure 10)

$$\begin{aligned}\tau_r^s(x,\rho) &= \inf\{n \in \mathbb{N} : d(f^{-n}x, x) \le \rho \text{ and } d_s([x, f^{-n}x], x) < r\},\\ \tau_r^u(x,\rho) &= \inf\{n \in \mathbb{N} : d(f^{n}x, x) \le \rho \text{ and } d_u([f^{n}x, x], x) < r\}.\end{aligned}$$

We call $\tau_r^s(x,\rho)$ and $\tau_r^u(x,\rho)$ respectively *stable* and *unstable return times.* Note that the functions $\rho \mapsto \tau_r^s(x,\rho)$ and $\rho \mapsto \tau_r^u(x,\rho)$ are nondecreasing. We define the *lower* and *upper stable recurrence rates* of the point $x \in \Lambda$ (with respect to f) by

$$\underline{R}^s(x) = \lim_{\rho\to 0} \underline{R}^s(x,\rho) \quad\text{and}\quad \overline{R}^s(x) = \lim_{\rho\to 0} \overline{R}^s(x,\rho),$$

and the *lower* and *upper stable recurrence rates* of the point $x \in \Lambda$ (with respect to f) by

$$\underline{R}^u(x) = \lim_{\rho\to 0} \underline{R}^u(x,\rho) \quad\text{and}\quad \overline{R}^u(x) = \lim_{\rho\to 0} \overline{R}^u(x,\rho),$$

where

$$\begin{aligned}\underline{R}^s(x,\rho) &= \liminf_{r\to 0} \frac{\log \tau_r^s(x,\rho)}{-\log r} \quad\text{and}\quad \overline{R}^s(x,\rho) = \limsup_{r\to 0} \frac{\log \tau_r^s(x,\rho)}{-\log r},\\ \underline{R}^u(x,\rho) &= \liminf_{r\to 0} \frac{\log \tau_r^u(x,\rho)}{-\log r} \quad\text{and}\quad \overline{R}^u(x,\rho) = \limsup_{r\to 0} \frac{\log \tau_r^u(x,\rho)}{-\log r}.\end{aligned}$$

When $\underline{R}^s(x) = \overline{R}^s(x)$ we denote the common value by $R^s(x)$ and call it *stable recurrence rate* of x (with respect to f), and when $\underline{R}^u(x) = \overline{R}^u(x)$ we denote the common value by $R^u(x)$ and call it *unstable recurrence rate* of x (with respect to f).

Barreira and Saussol showed in [11] that for a $C^{1+\alpha}$ diffeomorphism that is topologically mixing on a locally maximal compact hyperbolic set Λ, and an equilibrium measure μ of a Hölder continuous function, we have

$$R^s(x) = \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} \quad\text{and}\quad R^u(x) = \lim_{r\to 0} \frac{\log \mu_x^s(B^u(x,r))}{\log r} \tag{54}$$

for μ-almost every $x \in \Lambda$, where μ_x^s and μ_x^u are the conditional measures induced by the measurable partitions ξ^s and ξ^u (see Section 3.1). Ledrappier and Young [65] showed that there exist the limits in the right-hand sides of the identities in (54) (see Section 3.2).

The following result in [11] can now be obtained using Theorems 5 and 17 and the identities in (54).

Theorem 18 (Product structure for recurrence) *Let Λ be a locally maximal compact hyperbolic set of a $C^{1+\alpha}$ diffeomorphism that is topologically mixing on Λ, for some $\alpha > 0$, and μ an equilibrium measure of a Hölder continuous function. Then, for μ-almost every point $x \in \Lambda$ the following properties hold:*

1. *the recurrence rate is equal to the sum of the stable and unstable recurrence rates, i.e.,*
$$R(x) = R^s(x) + R^u(x);$$
2. *there exists $\rho(x) > 0$ such that for each $\rho < \rho(x)$ and each $\varepsilon > 0$ there is $r(x, \rho, \varepsilon) > 0$ such that if $r < r(x, \rho, \varepsilon)$ then*
$$r^\varepsilon < \frac{\tau_r^s(x,\rho) \cdot \tau_r^u(x,\rho)}{\tau_r(x)} < r^{-\varepsilon}.$$

The second statement in Theorem 18 shows that the return time to a given set is approximately equal to the product of the return times in the stable and unstable directions, as if they were independent.

A related result was obtained by Ornstein and Weiss in the case of symbolic dynamics. Namely, they showed in [75] that if $\sigma^+ : \Sigma^+ \to \Sigma^+$ is a *one-sided* subshift and μ^+ is an ergodic σ^+-invariant probability measure on Σ^+, then
$$\lim_{k\to\infty} \frac{\log \inf\{n \in \mathbb{N} : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\}}{k} = h_{\mu^+}(\sigma) \tag{55}$$
for μ^+-almost every $(i_1 i_2 \cdots) \in \Sigma^+$. They also showed in [75] that if $\sigma : \Sigma \to \Sigma$ is a *two-sided* subshift and μ is an ergodic σ-invariant probability measure on Σ, then
$$\lim_{k\to\infty} \frac{\log \inf\{n \in \mathbb{N} : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}}{2k+1} = h_{\mu}(\sigma) \tag{56}$$
for μ-almost every $(\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$.

Given a two-sided shift $\sigma : \Sigma \to \Sigma$ it has naturally associated two one-sided shifts $\sigma^+ : \Sigma^+ \to \Sigma^+$ and $\sigma^- : \Sigma^- \to \Sigma^-$ (respectively related with the future and with the past). Furthermore, any σ-invariant measure μ on Σ induces a σ^+-invariant measure μ^+ on Σ^+ and a σ^--invariant measure μ^- on Σ^-, such that
$$h_{\mu^+}(\sigma^+) = h_{\mu^-}(\sigma^-) = h_\mu(\sigma).$$
For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$ and $k \in \mathbb{N}$ we set
$$\begin{aligned}
\tau_k^+(\omega) &= \inf\{n \in \mathbb{N} : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\},\\
\tau_k^-(\omega) &= \inf\{n \in \mathbb{N} : (i_{-n-k} \cdots i_{-n-1}) = (i_{-k} \cdots i_{-1})\},\\
\tau_k(\omega) &= \inf\{n \in \mathbb{N} : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}.
\end{aligned}$$
Let now μ be an ergodic σ-invariant measure on Σ. It follows from (55) and (56) that for μ-almost every $\omega \in \Sigma$, given $\varepsilon > 0$, if $k \in \mathbb{N}$ is sufficiently large then
$$e^{-k\varepsilon} \le \frac{\tau_k^+(\omega)\tau_k^-(\omega)}{\tau_k(\omega)} \le e^{k\varepsilon}.$$

Theorem 18 and the identities in (54) are versions of these statements in the case of dimension.

References

[1] D. Anosov, *Geodesic Flows on Closed Riemann Manifolds with Negative Curvature*, Proc. Steklov Inst. Math. 90, Amer. Math. Soc., 1969.

[2] L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), 871–927.

[3] L. Barreira, *Variational properties of multifractal spectra*, Nonlinearity **14** (2001), 259–274.

[4] L. Barreira and Ya. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series 23, Amer. Math. Soc., 2002.

[5] L. Barreira, Ya. Pesin and J. Schmeling, *On a general concept of multifractality. Multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity*, Chaos **7** (1997), 27–38.

[6] L. Barreira, Ya. Pesin and J. Schmeling, *Multifractal spectra and multifractal rigidity for horseshoes*, J. Dynam. Control Systems **3** (1997), 33–49.

[7] L. Barreira, Ya. Pesin and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), 755–783.

[8] L. Barreira and B. Saussol, *Multifractal analysis of hyperbolic flows*, Comm. Math. Phys. **214** (2000), 339–371.

[9] L. Barreira and B. Saussol, *Hausdorff dimension of measures via Poincaré recurrence*, Comm. Math. Phys. **219** (2001), 443–463.

[10] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc. **353** (2001), 3919–3944.

[11] L. Barreira and B. Saussol, *Product structure of Poincaré recurrence*, Ergodic Theory Dynam. Systems **22** (2002), 33–61.

[12] L. Barreira and B. Saussol, *Variational principles for hyperbolic flows*, in Differential Equations and Dynamical Systems, A. Galves, J. Hale and C. Rocha eds., Fields Inst. Comm. **31** (2002), 43–63.

[13] L. Barreira, B. Saussol and J. Schmeling, *Higher-dimensional multifractal analysis*, J. Math. Pures Appl. **81** (2002), 67–91.

[14] L. Barreira, B. Saussol and J. Schmeling, *Distribution of frequencies of digits via multifractal analysis*, J. Number Theory, to appear.

[15] L. Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. **116** (2000), 29–70.

[16] L. Barreira and C. Wolf, *Measures of maximal dimension for hyperbolic diffeomorphisms*, preprint.

[17] L. Barreira and C. Wolf, *Pointwise dimension and ergodic decompositions*, preprint.

[18] J. Barrow-Green, *Poincaré and the Three Body Problem*, History of Mathematics **11**, Amer. Math. Soc., 1997.

[19] A. Besicovitch, *On the sum of digits of real numbers represented in the dyadic system*, Math. Ann. **110** (1934), 321–330.

[20] P. Billingsley, *Ergodic Theory and Information*, Wiley and Sons, 1965.

[21] G. Birkhoff, *Proof of a recurrence theorem for strongly transitive systems*, Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 650–655.

[22] G. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 656–660.

[23] G. Birkhoff and B. Koopman, *Recent contributions to the ergodic theory*, Proc. Nat. Acad. Sci. U.S.A. **18** (1932), 279–282.

[24] J. Bochi, *Genericity of zero Lyapunov exponents*, preprint.

[25] E. Borel, *Sur les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo **26** (1909), 247–271.

[26] M. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.

[27] H. Bothe, *The Hausdorff dimension of certain solenoids*, Ergodic Theory Dynam. Systems **15** (1995), 449–474.

[28] R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429–460.

[29] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125–136.

[30] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics 470, Springer, 1975.

[31] R. Bowen, *Markov partitions are not smooth*, Proc. Amer. Math. Soc. **71** (1978), 130–132.

[32] R. Bowen, *Hausdorff dimension of quasi-circles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 259–273.

[33] M. Brin, *Bernoulli diffeomorphisms with nonzero exponents*, Ergodic Theory Dynam. Systems **1** (1981), 1–7.

[34] P. Collet, J. Lebowitz and A. Porzio, *The dimension spectrum of some dynamical systems*, J. Stat. Phys. **47** (1987), 609–644.

[35] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism*, Ergodic Theory Dynam. Systems **22** (2002), 409–435.

[36] A. Douady and J. Oesterlé, *Dimension de Hausdorff des attracteurs*, C. R. Acad. Sc. Paris **290** (1980), 1135–1138.

[37] J.-P. Eckmann and D. Ruelle, *Ergodic theory of chaos and strange attractors*, Rev. Modern Phys. **57** (1985), 617–656.

[38] J.-P. Eckmann and I. Procaccia, *Fluctuations of dynamical scaling indices in non-linear systems*, Phys. Rev. A (3) **34** (1986), 659–661.

[39] H. Eggleston, *The fractional dimension of a set defined by decimal properties*, Quart. J. Math. Oxford Ser. **20** (1949), 31–36.

[40] K. Falconer, *The Hausdorff dimension of self-affine fractals*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 339–350.

[41] K. Falconer, *A subadditive thermodynamic formalism for mixing repellers*, J. Phys. A: Math. Gen. **21** (1988), 1737–1742.

[42] K. Falconer, *Dimensions and measures of quasi self-similar sets*, Proc. Amer. Math. Soc. **106** (1989), 543–554.

[43] K. Falconer, *Bounded distortion and dimension for non-conformal repellers*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 315–334.

[44] A. Fan, D. Feng and J. Wu, *Recurrence, dimension and entropy*, J. London Math. Soc. (2) **64** (2001), 229–244.

[45] A. Fathi, M. Herman and J.-C. Yoccoz, *A proof of Pesin's stable manifold theorem*, in Geometric Dynamics, Rio de Janeiro, 1981, J. Palis ed., Lecture Notes in Mathematics 1007, Springer, 1983, pp. 177–215.

[46] D. Gatzouras and Y. Peres, *Invariant measures of full dimension for some expanding maps*, Ergodic Theory Dynam. Systems **17** (1997), 147–167.

[47] J. Hadamard, *Les surfaces à courbures opposées et leur lignes géodesiques*, J. Math. Pures Appl. **4** (1898), 27–73.

[48] J. Hale, L. Magalhães and W. Oliva, *Dynamics in Infinite Dimensions*, Applied Mathematical Sciences 47, Springer, 2002.

[49] T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia and B. Shraiman, *Fractal measures and their singularities: the characterization of strange sets*, Phys. Rev. A (3) **34** (1986), 1141–1151; errata in **34** (1986), 1601.

[50] G. Hardy and J. Littlewood, *Some problems on Diophantine approximations*, Acta Math. **37** (1914), 155–190.

[51] B. Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory Dynam. Systems **14** (1994), 645–666.

[52] B. Hasselblatt, *Hyperbolic dynamical systems*, in Handbook of Dynamical Systems 1A, B. Hasselblatt and A. Katok, Elsevier, 2002.

[53] H. Hu, *Box dimensions and topological pressure for some expanding maps*, Comm. Math. Phys. **191** (1998), 397–407.

[54] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. (2) **110** (1979), 529–547.

[55] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 137–173.

[56] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and Its Applications 54, Cambridge University Press, 1995.

[57] A. Katok and L. Mendoza, *Dynamical systems with nonuniformly hyperbolic behavior*, in Introduction to the Modern Theory of Dynamical Systems, A. Katok and B. Hasselblatt, Cambridge University Press, 1995.

[58] A. Katok and J.-M. Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki, *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities*, Lecture Notes in Mathematics 1222, Springer, 1986.

[59] M. Keane, *Strongly mixing g-measures*, Invent. Math. **21** (1972), 309–324.

[60] G. Keller, *Equilibrium States in Ergodic Theory*, London Mathematical Society Student Texts 42, Cambridge University Press, 1998.

[61] F. Ledrappier, *Dimension of invariant measures*, in Proceedings of the Conference on Ergodic Theory and Related Topics, II, Georgenthal, 1986, Teubner-Texte, Math. 94, 1987, pp. 116–124.

[62] F. Ledrappier and M. Misiurewicz, *Dimension of invariant measures for maps with exponent zero*, Ergodic Theory Dynam. Systems **5** (1985), 595–610.

[63] F. Ledrappier and J.-M. Strelcyn, *A proof of the estimate from below in Pesin's entropy formula*, Ergodic Theory Dynam. Systems **2** (1982), 203–219.

[64] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122** (1985), 509–539.

[65] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), 540–574.

[66] A. Lopes, *The dimension spectrum of the maximal measure*, SIAM J. Math. Anal. **20** (1989), 1243–1254.

[67] R. Mañé, *Lyapunov exponents and stable manifolds for compact transformations*, in Geometric Dynamics, Rio de Janeiro, 1981, J. Palis ed., Lecture Notes in Mathematics 1007, Springer, 1983, pp. 522–577.

[68] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3. Folge; 8, Springer, 1987.

[69] H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynam. Systems **3** (1983), 251–260.

[70] P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. **42** (1946), 15–23.

[71] M. Morse and G. Hedlund, *Symbolic dynamics*, Amer. J. Math. **60** (1938), 815–866.

[72] J. Neunhäuserer, *Number theoretical peculiarities in the dimension theory of dynamical systems*, Israel J. Math. **128** (2002), 267–283.

[73] E. Olivier, *Analyse multifractale de fonctions continues*, C. R. Acad. Sci. Paris **326** (1998), 1171–1174.

[74] E. Olivier, *Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures*, Nonlinearity **12** (1999), 1571–1585.

[75] D. Ornstein and B. Weiss, *Entropy and data compression schemes*, IEEE Trans. Inform. Theory **39** (1993), 78–83.

[76] V. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[77] J. Palis and M. Viana, *On the continuity of Hausdorff dimension and limit capacity for horseshoes*, in Dynamical Systems, Valparaiso, 1986, R. Bamón, R. Labarca and J. Palis eds., Lecture Notes in Mathematics 1331, Springer, 1988, pp. 150–160.

[78] J. Palis and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, 1993.

[79] Ya. Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **10** (1976), 1261–1305.

[80] Ya. Pesin, *Characteristic exponents and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55–114.

[81] Ya. Pesin, *Geodesic flows on closed Riemannian manifolds without focal points*, Math. USSR-Izv. **11** (1977), 1195–1228.

[82] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1997.

[83] Ya. Pesin and B. Pitskel', *Topological pressure and the variational principle for noncompact sets*, Functional Anal. Appl. **18** (1984), 307–318.

[84] Ya. Pesin and V. Sadovskaya, *Multifractal analysis of conformal axiom A flows*, Comm. Math. Phys. **216** (2001), 277–312.

[85] Ya. Pesin and H. Weiss, *On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann–Ruelle conjecture*, Comm. Math. Phys. **182** (1996), 105–153.

[86] Ya. Pesin and H. Weiss, *A multifractal analysis of Gibbs measures for conformal expanding maps and Markov Moran geometric constructions*, J. Statist. Phys. **86** (1997), 233–275.

[87] Ya. Pesin and H. Weiss, *The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples*, Chaos **7** (1997), 89–106.

[88] Ya. Pesin and C. Yue, *The Hausdorff dimension of measures with non-zero Lyapunov exponents and local product structure*, preprint.

[89] M. Pollicott and H. Weiss, *The dimensions of some self affine limit sets in the plane and hyperbolic sets*, J. Statist. Phys. **77** (1994), 841–866.

[90] M. Pollicott and H. Weiss, *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville–Pomeau transformations and applications to Diophantine approximation*, Comm. Math. Phys. **207** (1999), 145–171.

[91] H. Poincaré, *Sur le problème des trois corps et les équations de la dynamique*, Acta Math. **13** (1890), 1–270.

[92] F. Przytycki and M. Urbański, *On Hausdorff dimension of some fractal sets*, Studia Math. **93** (1989), 155–186.

[93] C. Pugh, *The $C^{1+\alpha}$ hypothesis in Pesin theory*, Inst. Hautes Études Sci. Publ. Math. **59** (1984), 143–161.

[94] C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc. **312** (1989), 1–54.

[95] M. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Israel J. Math. **32** (1979), 356–362.

[96] D. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters*, Ergodic Theory Dynam. Systems **9** (1989), 527–541.

[97] M. Ratner, *Markov partitions for Anosov flows on n-dimensional manifolds*, Israel J. Math. **15** (1973), 92–114.

[98] V. Rokhlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl. **10** (1962), 1–52.

[99] D. Ruelle, *Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. **185** (1973), 237–251.

[100] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Mathematics and Its Applications 5, Addison-Wesley, 1978.

[101] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 27–58.

[102] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. of Math. (2) **115** (1982), 243–290.

[103] D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynam. Systems **2** (1982), 99–107.

[104] B. Saussol, S. Troubetzkoy and S. Vaienti, *Recurrence, dimensions and Lyapunov exponents*, J. Statist. Phys. **106** (2002), 623–634.

[105] J. Schmeling, *Hölder continuity of the holonomy maps for hyperbolic sets. II*, Math. Nachr. **170** (1994), 211–225.

[106] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergodic Theory Dynam. Systems **17** (1997), 675–694.

[107] J. Schmeling, *On the completeness of multifractal spectra*, Ergodic Theory Dynam. Systems **19** (1999), 1595–1616.

[108] J. Schmeling, *Entropy preservation under Markov coding*, J. Stat. Phys. **104** (2001), 799–815.

[109] J. Schmeling, *Dimension theory of smooth dynamical systems*, in Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems, B. Fiedler ed., Springer, 2001.

[110] J. Schmeling and R. Siegmund-Schultze, *Hölder continuity of the holonomy maps for hyperbolic sets I*, Ergodic Theory and Related Topics III, Proc. Int. Conf. (Güstrow, Germany, October 22–27, 1990), U. Krengel, K. Richter and V. Warstat eds., Lecture Notes in Mathematics **1514**, Springer, 1992, pp. 174–191.

[111] J. Schmeling and S. Troubetzkoy, *Scaling properties of hyperbolic measures*, preprint.

[112] J. Schmeling and S. Troubetzkoy, *Pointwise dimension for regular hyperbolic measures for endomorphisms*, preprint.

[113] J. Schmeling and H. Weiss, *An overview of the dimension theory of dynamical systems*, in Smooth Ergodic Theory and Its Applications (Seattle, 1999), A. Katok, R. de la Llave, Ya. Pesin and H. Weiss eds., Proceedings of Symposia in Pure Mathematics 69, Amer. Math. Soc., 2001, pp. 429–488.

[114] M. Shereshevsky, *A complement to Young's theorem on measure dimension: the difference between lower and upper pointwise dimension*, Nonlinearity **4** (1991), 15–25.

[115] K. Simon, *Hausdorff dimension for noninvertible maps*, Ergodic Theory Dynam. Systems **13** (1993), 199–212.

[116] K. Simon, *The Hausdorff dimension of the Smale–Williams solenoid with different contraction coefficients*, Proc. Amer. Math. Soc. **125** (1997), 1221–1228.

[117] K. Simon and B. Solomyak, *Hausdorff dimension for horseshoes in $\mathbb{R}^3$*, Ergodic Theory Dynam. Systems **19** (1999), 1343–1363.

[118] D. Simpelaere, *Dimension spectrum of axiom A diffeomorphisms. II. Gibbs measures*, J. Statist. Phys. **76** (1994), 1359–1375.

[119] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[120] F. Takens, *Limit capacity and Hausdorff dimension of dynamically defined Cantor sets*, in Dynamical Systems, Valparaiso 1986, R. Bamón, R. Labarca and J. Palis eds., Lecture Notes in Mathematics 1331, Springer, 1988, pp. 196–212.

[121] F. Takens and E. Verbitski, *Multifractal analysis of local entropies for expansive homeomorphisms with specification*, Comm. Math. Phys. **203** (1999), 593–612.

[122] F. Takens and E. Verbitski, *On the variational principle for the topological entropy of certain non-compact sets*, preprint.

[123] P. Thieullen, *Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapunov. Entropie. Dimension*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **4** (1987), 49–97.

[124] M. Viana, *Stochastic Dynamics of Deterministic Systems*, Brazilian Math. Colloquium, IMPA, 1997.

[125] J. von Neumann, *Proof of the quasi-ergodic hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **18** (1932), 70–82.

[126] P. Walters, *A variational principle for the pressure of continuous transformations*, Amer. J. Math. **97** (1976), 937–971.

[127] P. Walters, *Equilibrium states for β-transformations and related transformations*, Math. Z. **159** (1978), 65–88.

[128] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer, 1981.

[129] H. Weiss, *The Lyapunov spectrum for conformal expanding maps and axiom A surface diffeomorphisms*, J. Statist. Phys. **95** (1999), 615–632.

[130] C. Wolf, *On measures of maximal and full dimension for polynomial automorphisms of* $\mathbb{C}^2$, preprint.

[131] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.

Luis Barreira
Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa
luis.barreira@math.ist.utl.pt
http://www.math.ist.utl.pt/~barreira/
Portugal