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On the Differentiable Manifold Structure of some Spaces of Maps and Applications to Variational Mechanics [1]

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Abstract: In this paper we construct a family of covariant functors from the category of finite dimensional smooth vector bundles over a fixed differentiable manifold M to the category of smooth vector bundles with differentiable structures modelled on Banach spaces. As an application, we use one of these functors to construct the differentiable manifold structures in some spaces of curves which appear naturally in the context of sub-Riemannian geometry and vakonomic mechanics.

Key words: Variational mechanics, Vakonomic mechanics, Constrained mechanical systems, Sobolev spaces of manifold valued functions, Global analysis on manifolds.

CONTENTS

§1. INTRODUCTION

Let $\overline{\mathrm{M}}$ be a finite dimensional compact differentiable manifold, possibly with boundary, and $k \in \mathbb{N}$. Let $\mathscr{F}$ and $\mathscr{G}$ be two covariant functors satisfying axioms $(\mathscr{F}1) - (\mathscr{F}4)$, $(\mathscr{G}1) - (\mathscr{G}2)$ and (μ) from section 1. The main properties stated in those axioms are:

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(a) $\mathscr{F}$ is a covariant multiplicative functor from the category of finite dimensional smooth fiber bundles over $\overline{\mathrm{M}}$ to the category of Banach manifolds, which maps the subcategory of smooth vector bundles to the subcategory of Banachable spaces. For each smooth fiber bundle $\pi_E : E \to \overline{\mathrm{M}}$, $\mathscr{F}(E)$ is a subset of the space of C^k sections of E, $\mathsf{C}^k(E)$, and the inclusion is smooth.

(b) $\mathscr{G}$ is a covariant additive functor from the category of finite dimensional C^k vector bundles over $\overline{\mathrm{M}}$ to the category of Banachable spaces, which maps each C^k vector bundle $\pi_\xi : \xi \to \overline{\mathrm{M}}$ to a linear subspace of the space $S(\xi)$ of all sections of ξ.

(c) The multiplication $\mathscr{F}(\mathbb{R}_{\overline{\mathrm{M}}}) \times \mathscr{G}(\mathbb{R}_{\overline{\mathrm{M}}}) \to \mathscr{G}(\mathbb{R}_{\overline{\mathrm{M}}})$ is well defined and bilinear continuous, where $\mathbb{R}_{\overline{\mathrm{M}}}$ is the trivial bundle $\overline{\mathrm{M}} \times \mathbb{R}$.

Then, for each smooth finite dimensional differentiable manifold M, we construct a covariant functor $\mathscr{F}\mathscr{G}$, from the category of finite dimensional smooth vector bundles over M to the category of (infinite dimensional) smooth vector bundles over $\mathscr{F}(\mathrm{M})$. For each smooth vector bundle $\pi_\xi : \xi \to \mathrm{M}$, the total space of $\mathscr{F}\mathscr{G}(\xi)$ is the set given by Definition 1, and for each VB-morphism $f : \xi \to \eta$, the morphism $\mathscr{F}\mathscr{G}(f)$ is given by $(f\circ) : z \mapsto f \circ z$. Besides, if the functor $\mathscr{G}$ satisfies the additional axiom ($\mathscr{G}3$), we can extend the functor $\mathscr{F}\mathscr{G}$ to the category of finite dimensional smooth vector bundles with smooth fiber bundle morphisms (not necessarily linear on the fibers).

The construction of these functors is based on the generalization of a technique used in [5], for the case $\overline{\mathrm{M}} = [a_0, a_1] \subset \mathbb{R}$, to define a differentiable manifold structure on the space of curves $\mathsf{H}^1\mathsf{L}^2(\mathsf{TM}) := \{(\gamma, z) : [a_0, a_1] \to \mathsf{TM} \mid \gamma \in \mathsf{H}^k(\mathrm{M}) \text{ and } z \in \mathsf{L}^2(\gamma^*\mathsf{TM})\}$. These functors appear in a somewhat natural manner in the context of sub-Riemannian geometry and constrained mechanical systems; the reason for this is the fact that, applying "$\frac{T}{dt}$" to given a curve $\gamma \in \mathsf{C}^k(\mathrm{M})$ or $\gamma \in \mathsf{H}^k(\mathrm{M})$, with $k \geqslant 1$, we obtain elements $\frac{T\gamma}{dt} \in \mathsf{C}^k\mathsf{C}^{k-1}(\mathrm{TM})$ and $\frac{T\gamma}{dt} \in \mathsf{H}^k\mathsf{H}^{k-1}(\mathrm{TM})$, respectively, on the fiber over γ.

In section section 3, we restrain ourselves to the case $\overline{\mathrm{M}} = [a_0, a_1] \subset \mathbb{R}$, $\mathscr{F} = \mathsf{H}^k$ and $\mathscr{G} = \mathsf{H}^{k-1}$, $k \geqslant 2$, applying the functor $\mathscr{F}\mathscr{G}$ to:

(1) reprove the smoothness of the Lagrangian functional $\mathcal{L} : \mathsf{H}^k(\mathrm{M}) \to \mathbb{R}$ induced by a smooth Lagrangian $\mathrm{L} : \mathsf{TM} \to \mathbb{R}$ on the tangent bundle of a smooth manifold M – see Proposition 6.

(2) show that the spaces $\mathsf{H}^k(\mathrm{M}, \mathscr{C})$, $\mathsf{H}^k(\mathrm{M}, \mathscr{C}, q)$ of horizontal curves to a constraint manifold $\mathscr{C}$ corresponding to a regular constraint $f : \mathsf{TM} \to \mathsf{S}$ admit a smooth manifold structure endowed of which they become smooth embedded submanifolds of $\mathsf{H}^k(\mathrm{M})$ – see Definition 7, Definition 8 and Theorem *A*.

We conclude the paper with a brief description of some results we have obtained in [12] using the manifold structure on the spaces of curves horizontal to the constraint manifold $\mathscr{C}$ mentioned above.

§2. A CLASS OF VECTOR BUNDLE STRUCTURES ON SPACES OF MAPS

1. Basic Notations and Definitions

Throughout this paper we will use the adjectives "smooth" or "differentiable" meaning "of class C^∞". We use the notation E_{M} to denote the trivial fiber bundle of base M and fiber E, and $\tau_{\mathrm{M}} : \mathsf{TM} \to \mathrm{M}$ to denote the tangent bundle of a smooth manifold M. Given $k \in \mathbb{N}$ and a finite dimensional compact smooth manifold $\overline{\mathrm{M}}$, possibly with boundary (which will remain fixed until the end of this section), let us consider two covariant functors $\mathscr{F}$ and $\mathscr{G}$ satisfying the following axioms:

($\mathscr{F}1$) $\mathscr{F}$ is a functor from the category of smooth finite dimensional fiber bundles over $\overline{\mathrm{M}}$ with smooth fiber bundle morphisms over $\mathrm{id}_{\overline{\mathrm{M}}}$ as the morphisms (denoted by $\mathbf{FB}(\overline{\mathrm{M}})$ henceforth) to the category of differentiable manifolds modelled on Banach spaces (denoted by **Ban Man** henceforth). We consider the category of finite dimensional smooth manifolds **Man** as a subcategory of $\mathbf{FB}(\overline{\mathrm{M}})$, identifying a manifold M with the trivial fiber bundle $\mathrm{M}_{\overline{\mathrm{M}}} := \overline{\mathrm{M}} \times \mathrm{M}$ and a smooth map $f : \mathrm{M} \to \mathrm{N}$ with the fiber bundle morphism $f \times \mathrm{id}_{\overline{\mathrm{M}}} : \mathrm{M}_{\overline{\mathrm{M}}} \to \mathrm{N}_{\overline{\mathrm{M}}}$.

($\mathscr{F}2$) $\mathscr{F}$ maps the subcategory $\mathbf{VB}(\overline{\mathrm{M}})$ of finite dimensional smooth vector bundles over $\overline{\mathrm{M}}$ to the subcategory **Ban** of Banachable spaces; we consider the category **Lin** of finite dimensional vector spaces as a subcategory of $\mathbf{VB}(\overline{\mathrm{M}})$, identifying a vector space V with the trivial vector bundle $\mathsf{V}_{\overline{\mathrm{M}}} := \overline{\mathrm{M}} \times \mathsf{V}$ and a linear map $f : \mathsf{V} \to \mathsf{W}$ with the vector bundle morphism $\mathrm{id}_{\overline{\mathrm{M}}} \times f : \mathsf{V}_{\overline{\mathrm{M}}} \to \mathsf{W}_{\overline{\mathrm{M}}}$.

($\mathscr{F}3$) For all $E \in \mathbf{FB}(\overline{\mathrm{M}})$, $\mathscr{F}(E) \subset \mathsf{C}^k(E)$, and the inclusion is smooth, where $\mathsf{C}^k(E)$ is the Banach manifold of C^k sections of the smooth fiber bundle $\pi_E : E \to \overline{\mathrm{M}}$. Moreover, given a morphism $(f : E \to F) \in \mathrm{Mor}\,\mathbf{FB}(\overline{\mathrm{M}})$, $\mathscr{F}(f)$ is given by $\mathscr{F}(f) = (f\circ) : s \mapsto f \circ s$.

($\mathscr{F}4$) $\mathscr{F}$ is multiplicative, that is, given $E_1, E_2 \in \mathbf{FB}(\overline{\mathrm{M}})$, we have $\mathscr{F}(E_1 \times_{\overline{\mathrm{M}}} E_2) \equiv \mathscr{F}(E_1) \times \mathscr{F}(E_2)$. Moreover, if $E_1 \subset E_2 \in \mathbf{FB}(\overline{\mathrm{M}})$, and the total space of E_1 is an embedded submanifold of the total space of E_2, then $\mathscr{F}(E_1)$ is an embedded submanifold of $\mathscr{F}(E_2)$ and $\mathscr{F}(E_1) = \{\gamma \in \mathscr{F}(E_2) \mid \gamma(t) \in E_1 \text{ for all } t \in \overline{\mathrm{M}}\}$.

($\mathscr{G}1$) $\mathscr{G}$ is a functor from the category $\mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$ of finite dimensional C^k vector bundles over $\overline{\mathrm{M}}$ to the category **Ban** of Banachable spaces; as in ($\mathscr{F}2$), we consider **Lin** as a subcategory of $\mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$.

($\mathscr{G}2$) For all $\xi \in \mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$, $\mathscr{G}(\xi) \subset S(\xi)$, where $S(\xi)$ is the vector space of all sections of the vector bundle ξ. Moreover, given a morphism $(f : \xi \to \eta) \in \mathrm{Mor}\,\mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$, $\mathscr{G}(f)$ is given by $\mathscr{G}(f) = (f\circ) : s \mapsto f \circ s$.

(μ) The multiplication:

$$\begin{array}{rccc} \mu: & \mathscr{F}(\mathbb{R}) \times \mathscr{G}(\mathbb{R}) & \longrightarrow & \mathscr{G}(\mathbb{R}) \\ & (f,g) & \longmapsto & f \cdot g \end{array}$$

is well defined and continuous, where $(f \cdot g)(x) := f(x)g(x)$, for all $x \in \overline{\mathrm{M}}$.

Remark 1.

(i) We are identifying sections which are equal almost everywhere on $\overline{\mathrm{M}}$.

(ii) It follows from theses axioms that the functor $\mathscr{G}$ is an additive functor and preserves exact sequences, and the same holds for the restriction $\mathscr{F} : \mathbf{VB}(\overline{\mathrm{M}}) \to \mathbf{Ban}$.

(iii) It follows from ($\mathscr{F}$3) and ($\mathscr{F}$4) that, if $E_1 \subset E_2 \in \mathbf{FB}(\overline{\mathrm{M}})$, and E_1 is an open (respectively, closed) submanifold of E_2, then $\mathscr{F}(E_1) = \{\gamma \in \mathscr{F}(E_2) \mid \gamma(t) \in E_1 \text{ for all } t \in \overline{\mathrm{M}}\}$ is an open (respectively, closed) submanifold of $\mathscr{F}(E_2)$.

Example 1. The following functors satisfy the axioms above, where $n := \dim \overline{\mathrm{M}}$:

(i) $\mathscr{F} = \mathsf{C}^s$ and $\mathscr{G} = \mathsf{C}^r$, $0 \leqslant r \leqslant k \leqslant s$.

(ii) $\mathscr{F} = \mathsf{C}^s$ and $\mathscr{G} = \mathsf{L}^p_r$, $0 \leqslant r \leqslant k \leqslant s$, $1 \leqslant p < \infty$.

(iii) $\mathscr{F} = \mathsf{L}^q_s$ and $\mathscr{G} = \mathsf{L}^p_r$, $1 \leqslant p, q < \infty$, $0 \leqslant r \leqslant k < s - \frac{n}{q}$.

As a particular case of (iii), we can take $n = 1$, $p = q = 2$ and $0 \leqslant r \leqslant k < s$; we will consider this case in the applications in the next section.

We refer the reader to [8] for details on these functors. See also [3], [10], [4] and [2]. We also refer the reader to [11], which reports a technical slip in Palais' proof of a basic lemma on functors from vector bundles over compact manifolds to Banach spaces of sections (see [8]) and proposes a slight modification in Palais' axiom (B§2) to eliminate the problem. Nevertheless, there exists another technical slip in Palais' formulation for section functors, which we are currently working out. The problem appears in Palais' construction following the proof of the "Mayer-Vietoris Theorem": given a smooth finite dimensional vector bundle ξ over a smooth compact n-dimensional manifold M, he takes charts $\varphi_i : D^n \to \mathrm{M}$, $1 \leqslant i \leqslant r$, such that $\mathrm{M} \subset \cup_{1 \leqslant i \leqslant r} \varphi_i(\mathring{D}^n)$, and local trivializations $\psi_i : \varphi_i{}^*\xi \to D^n \times \mathbb{R}^q$, where D^n is the n-disc $\{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$. Unfortunately, this cannot be done if the manifold M has boundary $\partial\mathrm{M} \neq \emptyset$. We have already devised a possible solution to this problem and we point out that this technical slip does not have any implications in the results stated here.

2. The Functor $\mathscr{F}\mathscr{G}$

Let $\mathscr{F} : \mathbf{FB}(\overline{\mathrm{M}}) \to \mathbf{Ban\,Man}$ and $\mathscr{G} : \mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}}) \to \mathbf{Ban}$ be two covariant functors satisfying the axioms of the previous subsection and let $\mathrm{M} \in \mathbf{Man}$. In this section we will construct a covariant functor $\mathscr{F}\mathscr{G} : \mathbf{VB}(\mathrm{M}) \to \mathbf{Ban\,VB}\big(\mathscr{F}(\mathrm{M})\big)$, where $\mathbf{Ban\,VB}\big(\mathscr{F}(\mathrm{M})\big)$ is the category of smooth vector bundles over $\mathscr{F}(\mathrm{M})$, modelled on Banach spaces.

DEFINITION 1. *Let* $\pi_\xi : \xi \to \mathrm{M} \in \mathbf{VB}(\mathrm{M})$. *We define:*

$$\mathscr{F}\mathscr{G}(\xi) := \{(\gamma, z) : \overline{\mathrm{M}} \to \xi \mid \gamma \in \mathscr{F}(\mathrm{M}) \textit{ and } z \in \mathscr{G}(\gamma^*\xi)\}$$

In the definition above, note that, by axiom ($\mathscr{F}$3) we have $\gamma \in \mathscr{F}(\mathrm{M}) \subset \mathsf{C}^k(\mathrm{M})$, so that $\gamma^*\xi$ is in $\mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$.

The following lemmata and definitions will be used to construct the differentiable vector bundle structure on the set $\mathscr{F}\mathscr{G}(\xi)$ defined above.

LEMMA 1. *Let* $\pi_\xi : \xi \to \mathrm{M}$ *be a finite dimensional differentiable vector bundle over* M. *Then there exists* $N \in \mathbb{N}$ *such that* ξ *is isomorphic to a smooth vector subbundle of the trivial bundle* $\mathbb{R}^N_{\mathrm{M}} = \mathrm{M} \times \mathbb{R}^N$.

A proof of this lemma can be found in [7].

Remark 2. (i) A similar result valid in infinite dimension can be found in [6].

(ii) The lemma also holds for C^k vector bundles $\pi_\xi : \xi \to \mathrm{M}$, $k \geqslant 0$, if the base M is compact (see [7]). Thus, given a smooth finite dimensional compact manifold M and a C^k vector bundle $\pi_\xi : \xi \to \mathrm{M}$, $k \geqslant 0$, there exists $N \in \mathbb{N}$ such that this vector bundle is isomorphic to a C^k vector subbundle of the trivial (smooth) bundle $\mathbb{R}^N_{\mathrm{M}}$. This allows us to apply the theory of [8], chapter 14, to the C^k vector bundle $\pi_\xi : \xi \to \mathrm{M}$.

Notation. Until the end of this subsection, let us fix a finite dimensional smooth vector bundle $\pi_\xi : \xi \to \mathrm{M}$, and let $N \in \mathbb{N}$, given by Lemma 1, such that ξ is a smooth vector subbundle of the trivial bundle $\mathbb{R}^N_{\mathrm{M}}$. Let us endow $\mathbb{R}^N_{\mathrm{M}}$ with the metric tensor induced by the canonical inner product of $\mathbb{R}^N$, and let $\pi_\zeta : \zeta \to \mathrm{M}$ be the smooth vector subbundle of $\mathbb{R}^N_{\mathrm{M}}$ such that, for each $p \in \mathrm{M}$, $\zeta_p = (\xi_p)^\perp$. Then we have $\xi \oplus_{\mathrm{M}} \zeta = \mathbb{R}^N_{\mathrm{M}}$. Let us denote by P_ξ and P_ζ (or simply P, whenever there is no confusion about which "P" we are referring to) the induced orthogonal projections. Given $p \in \mathrm{M}$, we identify the fiber $(\mathbb{R}^N)_p$ with $\mathbb{R}^N$ and we denote by $(P_\xi)_p$ the restriction $P_\xi|_{(\mathbb{R}^N)_p} : \mathbb{R}^N \to \xi_p$, and similarly for $(P_\eta)_p$.

Let us also give ourselves a metric d which defines the topology of M. Given $p \in \mathrm{M}$ and $r > 0$, we will denote by $B_r(p) \subset \mathrm{M}$ the open ball of radius r and centered on p in that metric.

DEFINITION 2. *Given* $p, q \in \mathrm{M}$, *we say that* $p \sim q$ *if* $P_p|_{\xi_q} : \xi_q \to \xi_p$ *is a linear isomorphism. Note that* ξ_q *is a linear subspace of* $(\mathbb{R}^N)_q \equiv \mathbb{R}^N$, *so that the restriction makes sense.*

LEMMA 2. *The relation* $\sim$ *on* $\mathrm{M} \times \mathrm{M}$ *is reflexive and symmetric (but it is not transitive, in general). Moreover, the set:*

$$\mathcal{W}_{P_\xi} := \{(p,q) \in \mathrm{M} \times \mathrm{M} \mid p \sim q\}$$

is open in $\mathrm{M} \times \mathrm{M}$ *(and contains the diagonal* Δ_{M}, *since* $\sim$ *is reflexive).*

Proof. (i) It is clear that $\sim$ is reflexive. To see that it is also symmetric, note that $P_p|_{\xi_q} : \xi_q \to \xi_p$ is a linear isomorphism if, and only if, $\xi_q \oplus \zeta_p = \mathbb{R}^N$. This follows from the fact that $\dim \xi_q = \dim \xi_p =: m$, $\dim \zeta_p = N - m$ and $\mathrm{Ker}\,(P_p|_{\xi_q}) = \xi_q \cap \zeta_p$.

But $\xi_q \oplus \zeta_p = \mathbb{R}^N$ if, and only if, $\xi_q^\perp \oplus \zeta_p^\perp = \mathbb{R}^N$, that is, $\zeta_q \oplus \xi_p = \mathbb{R}^N$. By the same argument, $\xi_p \oplus \zeta_q = \mathbb{R}^N$ if, and only if, $P_q|_{\xi_p} : \xi_p \to \xi_q$ is a linear isomorphism. Thus we have shown $p \sim q \Leftrightarrow q \sim p$, as asserted.

(ii) It remains to show that $\mathcal{W}_{P_\xi}$ is open in $M \times M$. We will prove that $\mathcal{W}_{P_\xi}^C = M \times M \setminus \mathcal{W}_{P_\xi}$ is closed in $M \times M$.

Indeed, let $\{(p_n, q_n)\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{W}_{P_\xi}^C$ such that $(p_n, q_n) \overset{n\to\infty}{\longrightarrow} (p,q) \in M \times M$. We have to show that $(p,q) \in \mathcal{W}_{P_\xi}^C$.

We have proven in the previous item that $(p,q) \in \mathcal{W}_{P_\xi}$ if, and only if, $\xi_q \oplus \zeta_p = \mathbb{R}^N$; since $\big(\forall (x,y) \in M \times M\big)\ \dim \xi_x + \dim \zeta_y = \dim \mathbb{R}^N$, this implies that $(p,q) \in \mathcal{W}_{P_\xi}^C$ if, and only if, $\xi_q \cap \zeta_p \neq \{\mathbb{O}\}$. Thus, we have $\big(\forall n \in \mathbb{N}\big)\ \xi_{q_n} \cap \zeta_{p_n} \neq \{\mathbb{O}\}$, and we want to show that this implies $\xi_q \cap \zeta_p \neq \{\mathbb{O}\}$. Let us endow $Gr(\mathbb{R}^N)$ with the Hausdorff metric D induced by the Euclidean distance d of $\mathbb{R}^N$ (see appendix B); since $\big(Gr(\mathbb{R}^N), D\big)$ is a compact metric space, we can assume, passing to a convergent subsequence if necessary, that $\xi_{q_n} \cap \zeta_{p_n}$ converges in D to a linear subspace $X \in Gr(\mathbb{R}^N)$. Moreover,the fact that $\big(\forall n \in \mathbb{N}\big)\ \xi_{q_n} \cap \zeta_{p_n} \neq \{\mathbb{O}\}$ and that the connected components of $Gr(\mathbb{R}^N)$ are $Gr_0(\mathbb{R}^N), \ldots, Gr_N(\mathbb{R}^N)$ implies that $X \neq \{\mathbb{O}\}$.

We assert that $X \subset \xi_q \cap \zeta_p$, so that $\xi_q \cap \zeta_p \neq \{\mathbb{O}\}$, what concludes the proof.

As a matter of fact, it is sufficient to verify that $d(\xi_{q_n} \cap \zeta_{p_n}, \xi_q \cap \zeta_p) \overset{n\to\infty}{\longrightarrow} 0$, since this implies that $d(X, \xi_q \cap \zeta_p) = \lim_{n\to\infty} d(\xi_{q_n} \cap \zeta_{p_n}, \xi_q \cap \zeta_p) = 0$, so that $X \subset \xi_q \cap \zeta_p$. But this equivalent to condition $(C1)$ (see appendix A). Given $(x_n)_{n \in \mathbb{N}}$ sequence in $\mathbb{R}^N$ such that $\big(\forall n \in \mathbb{N}\big)\ x_n \in \xi_{q_n} \cap \zeta_{p_n}$, suppose that $(x_{n_m})_m$ is a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$ with $x_{n_m} \overset{m\to\infty}{\longrightarrow} x \in \mathbb{R}^N$. Then $x \in \xi_q \cap \zeta_p$, because $\xi_{q_n} \to \xi_q$ and $\zeta_{p_n} \to \zeta_p$ in the Hausdorff metric (since ξ and ζ are smooth vector subbundles of $\mathbb{R}^N_M$), and an application of condition $(C1)$ to these two sequences gives $x \in \xi_q$ and $x \in \zeta_p$, respectively. Thus, condition $(C1)$ is verified, as asserted.

$\square$

COROLLARY 1. *Using the above notation, let* $\gamma \in \mathscr{F}(M)$. *Then there exists* $r > 0$ *such that, for all* $t \in \overline{M}$, $B_r(\gamma(t)) \times B_r(\gamma(t)) \subset \mathcal{W}_{P_\xi}$.

Proof. Let:

$$\begin{array}{rccc} i_{\Delta_M}: & M & \longrightarrow & M \times M \\ & x & \longmapsto & (x,x) \end{array}$$

and $\widetilde{\gamma} := i_{\Delta_M} \circ \gamma : \overline{M} \to M \times M$. Then $\widetilde{\gamma}$ is continuous, since $\gamma \in \mathscr{F}(M) \subset C^0(M)$, so that $\widetilde{\gamma}(\overline{M}) = \{\big(\gamma(t), \gamma(t)\big) \mid t \in \overline{M}\}$ is compact and contained in the open set $\mathcal{W}_{P_\xi} \subset M \times M$, and the assertion follows immediately by a compactness argument. $\square$

DEFINITION 3. *Using the above notation, let* $\gamma \in \mathscr{F}(M)$, *and let* $r > 0$ *given by the previous corollary. The metric* d *of* M *induces an admissible metric* D *for the topology of* $C^0(M)$, *given by:*

$$D(\phi, \psi) := \sup_{t \in \overline{M}} d\big(\phi(t), \psi(t)\big)$$

for all $\phi, \psi \in C^0(M)$.

Since $\mathscr{F}(\mathrm{M}) \subset \mathsf{C}^0(\mathrm{M})$, by axiom ($\mathscr{F}3$), we have $\gamma \in \mathsf{C}^0(\mathrm{M})$. Let $\mathcal{U}_\gamma := B_r(\gamma)$ be the open ball of radius r centered at γ in $\big(\mathsf{C}^0(\mathrm{M}), D\big)$. Then $\mathcal{U}_\gamma$ is also an open neighborhood of γ in $\mathscr{F}(M)$, since, also by axiom ($\mathscr{F}3$), the inclusion of $\mathscr{F}(\mathrm{M})$ in $\mathsf{C}^k(\mathrm{M}) \subset \mathsf{C}^0(\mathrm{M})$ is continuous. Moreover, by the choice of r, it follows from the previous corollary that, for all $q_1, q_2 \in \mathcal{U}_\gamma$ and all $t \in \overline{\mathrm{M}}$, we have $\big(q_1(t), q_2(t)\big) \in \mathcal{W}_{P_\xi}$. Let us choose, for each $\gamma \in \mathscr{F}(\mathrm{M})$, such an $r > 0$ and such an open neighborhood $\mathcal{U}_\gamma$, so that we have an open covering $\mathcal{A} = \{\mathcal{U}_\gamma \mid \gamma \in \mathscr{F}(\mathrm{M})\}$ of $\mathscr{F}(\mathrm{M})$.

Let $\mathscr{F}\mathscr{G}(\xi)$ be the set given by Definition 1, and let:

$$\begin{array}{rccc} \pi: & \mathscr{F}\mathscr{G}(\xi) & \longrightarrow & \mathscr{F}(\mathrm{M}) \\ & (\gamma, z) & \longmapsto & \gamma \end{array}$$

Finally, for each $\mathcal{U}_\gamma \in \mathcal{A}$, let us define:

$$\begin{array}{rccc} \Phi_{\mathcal{U}_\gamma}: & \pi^{-1}(\mathcal{U}_\gamma) & \longrightarrow & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\xi) \\ & (q, z) & \longmapsto & (q, P_\gamma \cdot z) \end{array} \tag{1}$$

where:

$$\begin{array}{rccc} P_\gamma \cdot z: & \overline{\mathrm{M}} & \longrightarrow & \gamma^*\xi \\ & t & \longmapsto & P_{\gamma(t)} \cdot z(t) \in \xi_{\gamma(t)} \end{array}$$

PROPOSITION 1. *The maps $\Phi_{\mathcal{U}_\gamma}$ are well defined and $\{\Phi_{\mathcal{U}_\gamma} \mid \mathcal{U}_\gamma \in \mathcal{A}\}$ is a differentiable VB-atlas in $\mathscr{F}\mathscr{G}(\xi)$, so that $\pi : \mathscr{F}\mathscr{G}(\xi) \to \mathscr{F}(\mathrm{M})$ is a smooth vector bundle, that is, it is an object of the category* **Ban VB**$\big(\mathscr{F}(\mathrm{M})\big)$.

Proof. (i) Each $\Phi_{\mathcal{U}_\gamma}$ is well defined, that is, for all $(q, z) \in \pi^{-1}(\mathcal{U}_\gamma)$, $P_\gamma \cdot z \in \mathscr{G}(\gamma^*\xi)$. Indeed:

(1) Let:

$$\begin{array}{rccc} \widetilde{P}: & \mathrm{M} & \longrightarrow & \mathfrak{gl}(\mathbb{R}^N) \\ & x & \longmapsto & P_x \in \mathsf{L}(\mathbb{R}^N, \xi_x) \end{array}$$

Since:

$$\begin{array}{rccc} \xi: & \mathrm{M} & \longrightarrow & Gr_m(\mathbb{R}^N_{\mathrm{M}}) \\ & x & \longmapsto & \xi_x \end{array}$$

is a smooth section (or, equivalently, ξ is a smooth vector subbundle of rank m of $\mathbb{R}^N_{\mathrm{M}}$), it follows that the map $\widetilde{P}$ is smooth. Therefore, we can apply the functor $\mathscr{F}$ to this map, yielding the smooth map:

$$\mathscr{F}(\widetilde{P}) = (\widetilde{P}\circ) : \mathscr{F}(\mathrm{M}) \to \mathscr{F}\big(\mathfrak{gl}(\mathbb{R}^N)\big)$$

(2) Let us consider:

$$\begin{array}{rccc} \delta: & \mathscr{F}\big(\mathfrak{gl}(\mathbb{R}^N)\big) \times \mathscr{G}(\mathbb{R}^N) & \longrightarrow & \mathscr{G}(\mathbb{R}^N) \\ & (A, x) & \longmapsto & A \cdot x \end{array} \tag{2}$$

By axiom (μ), this map is well defined and bilinear continuous.

(3) It follows from the two previous items that, given $q \in \mathcal{U}_\gamma$ and $z \in \mathscr{G}(q^*\xi) \subset \mathscr{G}(q^*\mathbb{R}^N_{\mathrm{M}}) \equiv \mathscr{G}(\mathbb{R}^N)$, we have:

$$P_\gamma \cdot z = \delta\big((\widetilde{P}\circ)\cdot\gamma, z\big) \in \mathscr{G}(\mathbb{R}^N)$$

But, for almost all $t \in \overline{\mathrm{M}}$, $P_{\gamma(t)} \cdot z(t) \in \xi_{\gamma(t)}$, and $\mathscr{G}(\mathbb{R}^N) = \mathscr{G}(\gamma^*\xi) \oplus \mathscr{G}(\gamma^*\zeta)$, since $\mathscr{G}$ is an additive functor; thus, taking the projection on the first factor induced by this direct sum, which is given by $(\gamma^* P_\xi)\circ$, we conclude that, in fact, $P_\gamma \cdot z \in \mathscr{G}(\gamma^*\xi)$, as asserted.

(ii) Each $\Phi_{\mathcal{U}_\gamma}$ is bijective.

As a matter of fact, let:

$$\begin{array}{rccc} \Psi_{\mathcal{U}_\gamma}: & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\xi) & \longrightarrow & \pi^{-1}(\mathcal{U}_\gamma) \\ & (q,z) & \longmapsto & \big(q, (P_\gamma|_{\xi_q})^{-1}\cdot z\big) \end{array}$$

where, for all $t \in \overline{\mathrm{M}}$, $\big(q,(P_\gamma|_{\xi_q})^{-1}\cdot z\big)(t) = \big(q(t), (P_{\gamma(t)}|_{\xi_{q(t)}})^{-1}\cdot z(t)\big)$.

Once we have proven that $\Psi_{\mathcal{U}_\gamma}$ is a well defined map (i.e., that its image lies, in fact, in $\pi^{-1}(\mathcal{U}_\gamma)$), it is clear that this map is the inverse of $\Phi_{\mathcal{U}_\gamma}$. Therefore, we just have to verify that it is, in fact, well defined:

(1) Firstly, note that, by our choice of the sets $\mathcal{U}_\gamma$ and by Corollary 1, for all $q \in \mathcal{U}_\gamma$ and for all $t \in \overline{\mathrm{M}}$, the map $P_{\gamma(t)}|_{\xi_{q(t)}} : \xi_{q(t)} \to \xi_{\gamma(t)}$ is a linear isomorphism, so that, given $z \in \mathscr{G}(\gamma^*\xi)$, the following map is well defined:

$$\begin{array}{rccc} (P_\gamma|_{\xi_q})^{-1}\cdot z: & \overline{\mathrm{M}} & \longrightarrow & \mathbb{R}^N \\ & t & \longmapsto & (P_{\gamma(t)}|_{\xi_{q(t)}})^{-1}\cdot z(t) \end{array}$$

(2) Let:

$$\begin{array}{rccc} \widetilde{P}: & \mathcal{W}_{P_\xi} \subset \mathrm{M}\times\mathrm{M} & \longrightarrow & \mathfrak{gl}(\mathbb{R}^N) \\ & (x,y) & \longmapsto & (P_x|_{\xi_y})^{-1}\circ P_x \in \mathsf{L}(\mathbb{R}^N, \xi_y) \end{array}$$

It follows from the fact that ξ_x varies smoothly with x (or, in other words, that $\xi : \mathrm{M} \to Gr_m(\mathbb{R}^N)$ is smooth) that $\widetilde{P}$ is a smooth map. Therefore, we can apply to this map the functor $\mathscr{F}$, yielding the smooth map:

$$(\widetilde{P}\circ) : \mathscr{F}(\mathcal{W}_{P_\xi}) \to \mathscr{F}\big(\mathfrak{gl}(\mathbb{R}^N)\big)$$

(3) Given $q \in \mathcal{U}_\gamma$, we have $(\gamma, q) \in \mathscr{F}(\mathrm{M}) \times \mathscr{F}(\mathrm{M})$ and, for all $t \in \overline{\mathrm{M}}$, $\big(\gamma(t), q(t)\big) \in \mathcal{W}_{P_\xi}$. Thus, by axiom ($\mathscr{F}$4), we have $(\gamma, q) \in \mathscr{F}(\mathcal{W}_{P_\xi}) \subset \mathscr{F}(\mathrm{M}\times\mathrm{M}) \equiv \mathscr{F}(\mathrm{M}) \times \mathscr{F}(\mathrm{M})$. Besides, a direct computation shows that, for all $q \in \mathcal{U}_\gamma$ and for all $z \in \mathscr{G}(\gamma^*\xi) \subset \mathscr{G}(\mathbb{R}^N)$:

$$(P_\gamma|_{\xi_q})^{-1}\cdot z = \delta\big((\widetilde{P}\circ)(\gamma,q), z\big) \in \mathscr{G}(\mathbb{R}^N)$$

where δ is given by equation (2). Since, for almost all $t \in \overline{\mathrm{M}}$, $(P_{\gamma(t)}|_{\xi_{q(t)}})^{-1} \cdot z(t) \in \xi_{q(t)}$, we have shown that $(P_\gamma|_{\xi_q})^{-1} \cdot z$ actually belongs to $\mathscr{G}(q^*\xi)$, so that $\Psi_{\mathcal{U}_\gamma}$ is well defined, as asserted.

(iii) Given $\gamma_0, \gamma_1 \in \mathscr{F}(\mathrm{M})$ such that $\mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \neq \emptyset$, we assert that the map:

$$\begin{array}{rccc} \Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}}: & \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \times \mathscr{G}(\gamma_0{}^*\xi) & \longrightarrow & \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \times \mathscr{G}(\gamma_1{}^*\xi) \\ & (q,z) & \longmapsto & \left(q, P_{\gamma_1} \cdot (P_{\gamma_0}|_{\xi_q})^{-1} \cdot z\right) \end{array}$$

is a smooth vector bundle morphism, what concludes the proof.

Indeed:

(1) Let:

$$\begin{array}{rccc} \widetilde{P}: & M \times \mathcal{W}_{P_\xi} \subset \mathrm{M} \times \mathrm{M} \times \mathrm{M} & \longrightarrow & \mathfrak{gl}(\mathbb{R}^N) \\ & (x,y,z) & \longmapsto & P_x \circ (P_y|_{\xi_z})^{-1} \circ P_y \in \mathsf{L}(\mathbb{R}^N, \xi_x) \end{array}$$

Again, the fact that ξ_x varies smoothly with x implies that $\widetilde{P}$ is a smooth map. Hence, we can apply to this map the functor $\mathscr{F}$ to obtain the smooth map:

$$(\widetilde{P}\circ): \mathscr{F}(\mathrm{M}) \times \mathscr{F}(\mathcal{W}_{P_\xi}) \to \mathscr{F}\left(\mathfrak{gl}(\mathbb{R}^N)\right)$$

Therefore, the following composition is also smooth (using axiom ($\mathscr{F}4$) again to ensure $(\gamma_0, q) \in \mathscr{F}(\mathcal{W}_{P_\xi})$ for $q \in \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1}$):

$$\begin{array}{c} \widehat{P}: \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \longrightarrow \mathscr{F}(\mathrm{M}) \times \mathscr{F}(\mathcal{W}_{P_\xi}) \xrightarrow{(\widetilde{P}\circ)} \mathscr{F}\left(\mathfrak{gl}(\mathbb{R}^N)\right) \\ q \longmapsto (\gamma_1, \gamma_0, q) \longmapsto P_{\gamma_1} \circ (P_{\gamma_0}|_{\xi_q})^{-1} \circ P_{\gamma_0} \end{array} \tag{3}$$

what, in turn, implies that the following composition is also smooth:

$$\begin{array}{c} \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \times \mathscr{G}(\gamma_0{}^*\xi) \xrightarrow{\widehat{P}\times i} \mathscr{F}\left(\mathfrak{gl}(\mathbb{R}^N)\right) \times \mathscr{G}(\mathbb{R}^N) \xrightarrow{\quad\delta\quad} \mathscr{G}(\mathbb{R}^N) \\ (q,z) \longmapsto P_{\gamma_1} \circ (P_{\gamma_0}|_{\xi_q})^{-1} \cdot z \end{array}$$

where i is the inclusion and δ is given by equation (2). Note that, since $z \in \mathscr{G}(\gamma_0{}^*\xi)$, we have $P_{\gamma_0} \cdot z = z$. Thus, since the image of $\Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}}$ is actually contained in $\mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1} \times \mathscr{G}(\gamma_1{}^*\xi)$, and since $\mathscr{G}(\gamma_1{}^*\xi)$ is a Banachable subspace (hence, an embedded submanifold) of $\mathscr{G}(\mathbb{R}^N)$, we have shown that $\Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}}$ is a smooth map.

(2) It remains to show that $\Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}}$ is linear continuous on the fibers.
In fact, given $q \in \mathcal{U}_{\gamma_0} \cap \mathcal{U}_{\gamma_1}$, we have:

$$\begin{array}{rccc} (\Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}})_q: & \mathscr{G}(\gamma_0{}^*\xi) & \longrightarrow & \mathscr{G}(\gamma_1{}^*\xi) \\ & z & \longmapsto & P_{\gamma_1} \circ (P_{\gamma_0}|_{\xi_q})^{-1} \cdot z \end{array}$$

It follows from what we have seen in the previous item that the map:

$$\begin{array}{rccc} \delta(\widehat{P}(q), \cdot): & \mathscr{G}(\mathbb{R}^N) & \longrightarrow & \mathscr{G}(\mathbb{R}^N) \\ & z & \longmapsto & P_{\gamma_1} \circ (P_{\gamma_0}|_{\xi_q})^{-1} \circ P_{\gamma_0} \cdot z \end{array}$$

is linear continuous, where $\widehat{P}$ is given by equation (3). Since the restriction of this map to the Banachable subspace $\mathscr{G}(\gamma_0{}^*\xi)$ of $\mathscr{G}(\mathbb{R}^N)$ is just $(\Phi_{\mathcal{U}_{\gamma_1}} \circ \Phi^{-1}_{\mathcal{U}_{\gamma_0}})_q$, it follows that this map is linear continuous, and the assertion follows.

□

The next step is to show that the differentiable vector bundle structure of $\mathscr{F}\mathscr{G}(\xi)$ is intrinsic, in the sense that it does not depend on the embedding $: \xi \to \mathbb{R}^N_{\mathrm{M}}$. In the sequel, to complete the construction of the functor $\mathscr{F}\mathscr{G}$, we will map each differentiable vector bundle morphism $f : \xi \to \eta$ to a differentiable vector bundle morphism $(f\circ) : \mathscr{F}\mathscr{G}(\xi) \to \mathscr{F}\mathscr{G}(\eta)$, to be defined in the next proposition.

DEFINITION 4. *Let ξ and η be finite dimensional differentiable vector bundles over* M, *of ranks m and n, respectively. Let $\varphi : \xi \to \mathbb{R}^{N_1}_{\mathrm{M}}$ and $\psi : \eta \to \mathbb{R}^{N_2}_{\mathrm{M}}$ be differentiable vector bundle monomorphisms, and let us endow $\mathbb{R}^{N_1}_{\mathrm{M}}$ and $\mathbb{R}^{N_2}_{\mathrm{M}}$ with metric tensors induced by the canonical inner products of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively. Denote by $P_\xi : \mathbb{R}^{N_1}_{\mathrm{M}} \to \xi$ and $P_\eta : \mathbb{R}^{N_2}_{\mathrm{M}} \to \eta$ the respective orthogonal projections (and again we will write just P if it is clear to which P we are referring to) and by $\mathcal{W}_{P_\xi}$ and $\mathcal{W}_{P_\eta}$ the corresponding open sets of* $\mathrm{M} \times \mathrm{M}$ *defined like in Lemma 2. Let $(\mathscr{F}\mathscr{G}(\xi), \varphi)$ and $(\mathscr{F}\mathscr{G}(\eta), \psi)$ denote the differentiable vector bundle structures induced, respectively, in $\mathscr{F}\mathscr{G}(\xi)$ and $\mathscr{F}\mathscr{G}(\eta)$ by the vector bundle atlases $\{\Phi_{\mathcal{U}_\gamma} \mid \mathcal{U}_\gamma \in \mathcal{A}_\varphi\}$ and $\{\Psi_{\mathcal{U}_\gamma} \mid \mathcal{U}_\gamma \in \mathcal{A}_\psi\}$ defined like in Definition 3, using the embeddings φ and ψ, respectively.*

This notation is temporary; we will drop the "φ" of $(\mathscr{F}\mathscr{G}(\xi), \varphi)$ after we prove that the vector bundle structure actually does not depend on the embedding φ.

PROPOSITION 2. *Using the above notation, let $f : \xi \to \eta$ be a smooth vector bundle morphism. Let us define the map:*

$$\begin{array}{rccc} \mathscr{F}\mathscr{G}(f) = (f\circ): & (\mathscr{F}\mathscr{G}(\xi), \varphi) & \longrightarrow & (\mathscr{F}\mathscr{G}(\eta), \psi) \\ & (q, z) & \longmapsto & f \circ (q, z) = (q, f_q \cdot z) \end{array}$$

where $(\forall x \in \mathrm{M})\ f_x := f|_{\xi_x} : \xi_x \to \eta_x$ and $(\forall t \in \overline{\mathrm{M}})\ (f_q \cdot z)(t) := f_{q(t)} \cdot z(t)$.
Then $(f\circ)$ is well defined and it is a smooth vector bundle morphism.

Proof. (i) $(f\circ)$ is well defined (that is, for all $q \in \mathscr{F}(\mathrm{M})$ and for all $z \in \mathscr{G}(q^*\xi)$, in fact it is true that $f_q \cdot z \in \mathscr{G}(q^*\eta)$) and linear continuous on the fibers.

Indeed, given $q \in \mathscr{F}(M)$, we have $q \in \mathsf{C}^k(M)$ by axiom ($\mathscr{F}3$), hence $q^*f : q^*\xi \to q^*\eta$ is a C^k VB-morphism. Applying to this morphism the functor $\mathscr{G}$, we obtain the linear continuous map:

$$\begin{array}{rccc} \mathscr{G}(q^*f): & \mathscr{G}(q^*\xi) & \longrightarrow & \mathscr{G}(q^*\eta) \\ & z & \longmapsto & f\circ z = f_q \cdot z \end{array}$$

and the assertion follows immediately.

(ii) $(f\circ)$ is smooth.

Indeed, given $\gamma \in \mathscr{F}(\mathrm{M})$, let $(\mathcal{U}_\gamma, \Phi)$ and $(\mathcal{U}_\gamma, \Psi)$ be VB-charts of $(\mathscr{F}\mathscr{G}(\xi), \varphi)$ and $(\mathscr{F}\mathscr{G}(F), \psi)$, respectively, where $\mathcal{U}_\gamma$ is a neighborhood of γ in $\mathscr{F}(\mathrm{M})$. We have:

$$\begin{array}{rccc} \Phi: & \pi^{-1}_{\mathscr{F}\mathscr{G}(\xi)}(\mathcal{U}_\gamma) & \longrightarrow & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\xi) \\ & (q,z) & \longmapsto & (q, (P_\xi)_\gamma \cdot z) \end{array}$$

and:

$$\begin{array}{rccc} \Psi: & \pi^{-1}_{\mathscr{F}\mathscr{G}(\eta)}(\mathcal{U}_\gamma) & \longrightarrow & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\eta) \\ & (q,z) & \longmapsto & (q, (P_\eta)_\gamma \cdot z) \end{array}$$

so that:

$$\begin{array}{rccc} \Psi\circ(f\circ)\circ\Phi^{-1}: & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\xi) & \longrightarrow & \mathcal{U}_\gamma \times \mathscr{G}(\gamma^*\eta) \\ & (q,z) & \longmapsto & \left(q, (P_\eta)_\gamma \circ f_q \circ \left((P_\xi)_\gamma|_{\xi_q}\right)^{-1} \cdot z\right) \end{array}$$

where, for all $t \in \overline{\mathrm{M}}$:

$$\left((P_\eta)_\gamma \circ f_q \circ \left((P_\xi)_\gamma|_{\xi_q}\right)^{-1} \cdot z\right)(t) = (P_\eta)_{\gamma(t)} \circ f_{q(t)} \circ \left((P_\xi)_{\gamma(t)}|_{\xi_{q(t)}}\right)^{-1} \cdot z(t)$$

We will show that the second component of this map is smooth. Since $\gamma \in \mathscr{F}(\mathrm{M})$ was arbitrarily taken, this is sufficient to prove that $(f\circ)$ is smooth.

Consider the following maps:

(1)

$$\begin{array}{rccc} \widetilde{f}: & \mathcal{W}_{P_\xi} \subset \mathrm{M}\times\mathrm{M} & \longrightarrow & \mathsf{L}(\mathbb{R}^{N_1}, \mathbb{R}^{N_2}) \\ & (x,y) & \longmapsto & (P_\eta)_x \circ f_y \circ \left((P_\xi)_x|_{\xi_y}\right)^{-1} \circ (P_\xi)_x \in \mathsf{L}(\mathbb{R}^{N_1}, \eta_x) \end{array}$$

The fact that ξ_x and η_x vary smoothly with x (that is, that $\xi : \mathrm{M} \to Gr_m(\mathbb{R}^{N_1})$ and $\eta : \mathrm{M} \to Gr_n(\mathbb{R}^{N_2})$ are smooth) implies that $\widetilde{f}$ is smooth. Note that $\widetilde{f}$ is well defined, since, by the definition of $\mathcal{W}_{P_\xi}$, for all $(x,y) \in \mathcal{W}_{P_\xi}$ the map $(P_\xi)_x|_{\xi_y} : \xi_y \to \xi_x$ is a linear isomorphism.

Since $\widetilde{f}$ is smooth, we can apply to it the functor $\mathscr{F}$ to obtain the smooth map:

$$(\widetilde{f}\circ) : \mathscr{F}(\mathcal{W}_{P_\xi}) \subset \mathscr{F}(\mathrm{M}) \times \mathscr{F}(\mathrm{M}) \to \mathscr{F}\left(\mathsf{L}(\mathbb{R}^{N_1}, \mathbb{R}^{N_2})\right)$$

(2)

$$\begin{array}{rcll} \delta: & \mathscr{F}\big(\mathsf{L}(\mathbb{R}^{N_1},\mathbb{R}^{N_2})\big)\times\mathscr{G}(\mathbb{R}^{N_1}) & \longrightarrow & \mathscr{G}(\mathbb{R}^{N_2}) \\ & (A,x) & \longmapsto & A\cdot x \end{array} \tag{4}$$

Applying axiom (μ), we conclude that δ is well defined and bilinear continuous, hence smooth.

Then it follows that the second component of $\Psi\circ(f\circ)\circ\Phi^{-1}$ is given by the following composition of smooth maps:

$$\begin{array}{l} \mathcal{U}_\gamma\times\mathscr{G}(\gamma^*\xi)\xrightarrow{(\tilde{f}\circ)(\gamma,\cdot)\times i}\mathscr{F}\big(\mathsf{L}(\mathbb{R}^{N_1},\mathbb{R}^{N_1})\big)\times\mathscr{G}(\mathbb{R}^{N_1})\xrightarrow{\quad\delta\quad}\mathscr{G}(\mathbb{R}^{N_2}) \\ (q,z)\longmapsto (P_\eta)_\gamma\circ f_q\circ\big((P_\xi)_\gamma|_{\xi_q}\big)^{-1}\cdot z \end{array}$$

where i is the inclusion. We have used the fact that, for all $z\in\mathscr{G}(\gamma^*\xi)$, $(P_\xi)_\gamma\cdot z=z$. Since, for all $q\in\mathcal{U}_\gamma$ and for all $z\in\mathscr{G}(q^*\xi)$, $(P_\eta)_\gamma\circ f_q\circ\big((P_\xi)_\gamma|_{\xi_q}\big)^{-1}\cdot z\in\mathscr{G}(\gamma^*\eta)$, and since $\mathscr{G}(\gamma^*\eta)$ is a Banachable subspace (hence an embedded submanifold) of $\mathscr{G}(\mathbb{R}^{N_2})$, we have shown that the second component of $\Psi\circ(f\circ)\circ\Phi^{-1}$ is smooth. Thus, $\Psi\circ(f\circ)\circ\Phi^{-1}$ is smooth, as asserted.

□

COROLLARY 2. *Using the same notation, the differentiable vector structure of $\mathscr{F}\mathscr{G}(\xi)$ does not depend on the embedding* : $\xi\to\mathbb{R}^N_{\mathrm{M}}$.

Proof. Let $\varphi_1:\xi\to\mathbb{R}^{N_1}_{\mathrm{M}}$ and $\varphi_2:\xi\to\mathbb{R}^{N_2}_{\mathrm{M}}$ be two differentiable vector bundle monomorphisms, and let $\big(\mathscr{F}\mathscr{G}(\xi),\varphi_1\big)$ and $\big(\mathscr{F}\mathscr{G}(\xi),\varphi_2\big)$ be the respective induced differentiable vector bundle structures. Then, by the previous proposition, the identity $\mathrm{id}_\xi:\xi\to\xi$ induces a differentiable vector bundle isomorphism:

$$(\mathrm{id}_\xi\circ):\big(\mathscr{F}\mathscr{G}(\xi),\varphi_1\big)\to\big(\mathscr{F}\mathscr{G}(\xi),\varphi_2\big)$$

what concludes the proof. □

COROLLARY 3. *The functor $\mathscr{F}\mathscr{G}:\mathbf{VB}(\mathrm{M})\to\mathbf{Ban\,VB}\big(\mathscr{F}(\mathrm{M})\big)$ is additive and preserves exact sequences. Moreover, given $\eta\in\mathbf{VB}(\mathrm{M})$, $(\gamma,z)\in\mathscr{F}\mathscr{G}(\eta)$, and a smooth vector subbundle ξ of η, then $(\gamma,z)\in\mathscr{F}\mathscr{G}(\xi)$ if, and only if, $\big(\gamma(t),z(t)\big)\in\xi$ for almost all $t\in\overline{\mathrm{M}}$.*

Proof. The first assertion is clear; to prove the second, let $\zeta\in\mathbf{VB}(\mathrm{M})$ such that $\xi\oplus_{\mathrm{M}}\zeta=\eta$, and let P_ξ be the projection on the first factor induced by this Whitney sum. Then we have $\mathscr{F}\mathscr{G}(\eta)=\mathscr{F}\mathscr{G}(\xi)\oplus_{\mathscr{F}(\mathrm{M})}\mathscr{F}\mathscr{G}(\zeta)$, and the projection on the first factor induced by this Whitney sum is given by $(P_\xi\circ)$. Hence, if $(\gamma,z)\in\mathscr{F}\mathscr{G}(\eta)$ and $\big(\gamma(t),z(t)\big)\in\xi$ for almost all $t\in\overline{\mathrm{M}}$, we have $(\gamma,z)=(P_\xi\circ)\cdot(\gamma,z)\in\mathscr{F}\mathscr{G}(\xi)$. □

3. An Extension of the Functor $\mathscr{F}\mathscr{G}$ to the Category $\mathbf{FVB}(\mathrm{M})$

In this subsection we will show that, if the functor $\mathscr{G}$ satisfies the axiom $(\mathscr{G}3)$, below, then, given $\mathrm{M} \in \mathbf{Man}$, the functor $\mathscr{F}\mathscr{G}$ can be extended to a functor from the category $\mathbf{FVB}(\mathrm{M})$, of finite dimensional smooth vector bundles over M with smooth fiber bundle morphisms, to the category $\mathbf{Ban\,FVB}\big(\mathscr{F}(\mathrm{M})\big)$ of smooth vector bundles over $\mathscr{F}(\mathrm{M})$ modelled on Banach spaces, with smooth fiber bundle morphisms.

$(\mathscr{G}3)$ The restriction of the functor $\mathscr{G}$ to the subcategory $\mathbf{VB}(\overline{\mathrm{M}})$ of $\mathsf{C}^k\mathbf{VB}(\overline{\mathrm{M}})$ can be extended to a functor $\mathscr{G} : \mathbf{FB}(\overline{\mathrm{M}}) \to \mathbf{Ban\,Man}$ satisfying axioms $(\mathscr{F}3)$ with $k = 0$ and $(\mathscr{F}4)$. Besides, for all $E \in \mathbf{FB}(\overline{\mathrm{M}})$, we have a smooth inclusion $\mathscr{F}(E) \subset \mathscr{G}(E)$.

Example 2. Axioms $(\mathscr{F}1) - (\mathscr{F}4)$, $(\mathscr{G}1) - (\mathscr{G}3)$, (μ) are satisfied in the following cases:

(i) $\mathscr{F} = \mathsf{C}^s$ and $\mathscr{G} = \mathsf{C}^r$, $0 \leqslant r \leqslant k \leqslant s$.

(ii) $\mathscr{F} = \mathsf{C}^s$ and $\mathscr{G} = \mathsf{L}^p_r$, $\frac{n}{p} < r \leqslant k \leqslant s$, $1 \leqslant p < \infty$.

(iii) $\mathscr{F} = \mathsf{L}^q_s$ and $\mathscr{G} = \mathsf{L}^p_r$, $1 \leqslant p,q < \infty$, $\frac{n}{p} < r \leqslant k < s - \frac{n}{q}$.

We refer the reader to [8], [3], [10] and [2] for the verification of axiom $(\mathscr{G}3)$ in these examples.

As a particular case of (iii), we can take $n = 1$, $p = q = 2$ and $1 \leqslant r \leqslant k < s$; we will consider this case in the applications in the next section.

PROPOSITION 3. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ satisfy axioms $(\mathscr{F}1) - (\mathscr{F}4)$, $(\mathscr{G}1) - (\mathscr{G}3)$ and (μ). Let ξ, η be finite dimensional smooth vector bundles over a smooth manifold* M*, $k \in \mathbb{N}$, and $f : \xi \to \eta$ a smooth fiber bundle morphism. Then the map:*

$$\begin{array}{rccc} \mathscr{F}\mathscr{G}(f) = (f\circ): & \mathscr{F}\mathscr{G}(\xi) & \longrightarrow & \mathscr{F}\mathscr{G}(\eta) \\ & (q,z) & \longmapsto & f\circ(q,z) \end{array}$$

is a smooth fiber bundle morphism.

Proof. Using the notation of Definition 4, let:

$$\begin{array}{rccc} \widetilde{f}: & \mathrm{M}\times\mathbb{R}^{N_1} & \longrightarrow & \mathbb{R}^{N_2} \\ & (x,z) & \longmapsto & f\circ(P_\xi)_x\cdot z \in \eta_x \subset \mathbb{R}^{N_2} \end{array}$$

The fact that ξ_x and η_x vary smoothly with x (that is, that $\xi : \mathrm{M} \to Gr_m(\mathbb{R}^{N_1})$ and $\eta : \mathrm{M} \to Gr_n(\mathbb{R}^{N_2})$ are smooth) implies that $\widetilde{f}$ is smooth.

Since $\widetilde{f}$ is smooth, we can apply to it the $\mathscr{G}$ functor to obtain the smooth map:

$$(\widetilde{f}\circ) : \mathscr{G}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1}) \to \mathscr{G}(\mathbb{R}^{N_2})$$

By axiom $(\mathscr{G}3)$, the inclusion : $\mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1}) \to \mathscr{G}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1})$ is smooth; therefore, the following composition of maps is smooth:

$$\begin{array}{ccccc} \mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1}) & \longrightarrow & \mathscr{G}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1}) & \xrightarrow{(\tilde{f}\circ)} & \mathscr{G}(\mathbb{R}^{N_2}) \\ (q,z) & \longmapsto & (q,z) & \longmapsto & f \circ (P_\xi)_q \cdot z \end{array}$$

what implies that:

$$\begin{array}{rccc} : & \mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_1}) & \longrightarrow & \mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^{N_2}) \\ & (q,z) & \longmapsto & (q, f \circ (P_\xi)_q \cdot z) \end{array}$$

preserves fibers and is smooth. But the image of this map lies in the split vector subbundle $\mathscr{F}\mathscr{G}(\eta)$ of $\mathscr{F}\mathscr{G}(\mathbb{R}^{N_2}_{\mathrm{M}})$, since $f(\xi) \subset \eta$, and its restriction to the vector subbundle $\mathscr{F}\mathscr{G}(\xi)$ of $\mathscr{F}\mathscr{G}(\mathbb{R}^{N_1}_{\mathrm{M}})$ coincides with $(f\circ)$. Thus, we have shown that $(f\circ) : \mathscr{F}\mathscr{G}(\xi) \to \mathscr{F}\mathscr{G}(\eta)$ is well defined, smooth and preserves fibers.

□

4. Tangent Spaces and Tangent Maps

Let $\mathscr{F}$ and $\mathscr{G}$ be two covariant functors satisfying axioms $(\mathscr{F}1)-(\mathscr{F}4)$, $(\mathscr{G}1)-(\mathscr{G}2)$ and (μ). In this subsection, given $\xi \in \mathbf{VB}(\mathrm{M})$, we give a description of the tangent spaces $\mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\xi)$, for $(\gamma,z) \in \mathscr{F}\mathscr{G}(\xi)$. This description will be useful in the applications in the next section. We also compute the tangent map $\mathsf{T}(f\circ)$ of $(f\circ)$, for a morphism $f : \xi \to \eta$ belonging to Mor $\mathbf{VB}(\mathrm{M})$ or to Mor $\mathbf{FVB}(\mathrm{M})$ if $\mathscr{G}$ also satisfies axiom $(\mathscr{G}3)$.

Let $N \in \mathbb{N}$ such that ξ is a differentiable vector subbundle of $\mathbb{R}^N_{\mathrm{M}}$, so that, for each $v_p \in \xi$, $\mathsf{T}_{v_p}\xi$ is a linear subspace of $\mathsf{T}_p\mathrm{M} \times \mathbb{R}^N$. We have already seen that $\mathscr{F}\mathscr{G}(\xi)$ is a closed differentiable vector subbundle of $\mathscr{F}\mathscr{G}(\mathbb{R}^N_{\mathrm{M}}) \equiv \mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^N)$ and it splits; therefore, given $(\gamma,z) \in \mathscr{F}\mathscr{G}(\xi)$, $\mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\xi)$ is a closed Banachable subspace of $\mathsf{T}_\gamma\mathscr{F}(\mathrm{M}) \times \mathsf{T}_z\mathscr{G}(\mathbb{R}^N) \equiv \mathsf{T}_\gamma\mathscr{F}(\mathrm{M}) \times \mathscr{G}(\mathbb{R}^N)$ and it splits. More precisely, we have:

PROPOSITION 4. *With the notation above, we have:*

$$\mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\xi) = \{X \in \mathsf{T}_\gamma\mathscr{F}(\mathrm{M}) \times \mathsf{T}_z\mathscr{G}(\mathbb{R}^N) \mid X(t) \in \mathsf{T}_{z(t)}\xi \text{ a.e. on } \overline{\mathrm{M}}\}$$

This means that we can interpret the elements of $\mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\xi)$ as maps $X : \overline{\mathrm{M}} \to \mathsf{T}\xi$ with $X(t) \in \mathsf{T}_{z(t)}\xi$ for almost all $t \in \overline{\mathrm{M}}$. We use this characterization of the tangent spaces to compute the tangent maps $\mathsf{T}(f\circ)$ for morphisms $f : \xi \to \eta$:

PROPOSITION 5. *Let $f : \xi \to \eta$ be a morphism belonging to* Mor $\mathbf{VB}(\mathrm{M})$ *and $(\gamma,z) \in \mathscr{F}\mathscr{G}(\xi)$. Then, using the notation above, the tangent map $\mathsf{T}_{(\gamma,z)}(f\circ)$ at (γ,z) of the morphism $(f\circ) : \mathscr{F}\mathscr{G}(\xi) \to \mathscr{F}\mathscr{G}(\eta)$ coincides with $(\mathsf{T}f\circ)$, that is:*

$$\begin{array}{rccc} \mathsf{T}_{(\gamma,z)}(f\circ) : & \mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\xi) & \longrightarrow & \mathsf{T}_{(\gamma,z)}\mathscr{F}\mathscr{G}(\eta) \\ & X & \longmapsto & \mathsf{T}f \circ X \end{array}$$

The same holds for a morphism $f \in \mathbf{FVB}(\mathrm{M})$ if $\mathscr{G}$ satisfies axiom $(\mathscr{G}3)$.

Remark 3. [The tangent spaces at the null section of $\mathscr{F}\mathscr{G}(\xi)$]

Using the notation above, let us denote by $\mathbb{O}_{\mathscr{F}\mathscr{G}(\xi)}$ the null section of $\mathscr{F}\mathscr{G}(\xi)$ and by $\mathbb{O}_\gamma$ the null vector of the fiber of $\mathscr{F}\mathscr{G}(\xi)$ over $\gamma \in \mathscr{F}(\mathrm{M})$. Then, identifying $\mathsf{T}_{\mathbb{O}_\gamma}\mathbb{O}_{\mathscr{F}\mathscr{G}(\xi)}$ with $\mathsf{T}_\gamma\mathscr{F}(\mathrm{M})$ and identifying the vertical subspace $\mathrm{Ver}_{\mathbb{O}_\gamma}\mathscr{F}\mathscr{G}(\xi)$ of $\mathsf{T}_\gamma\mathscr{F}\mathscr{G}(\xi)$ with $\mathscr{G}(\gamma^*\xi)$ (that is, with the fiber of $\mathscr{F}\mathscr{G}(\xi)$ over γ), we have:

$$\mathsf{T}_{\mathbb{O}_\gamma}\mathscr{F}\mathscr{G}(\xi) \equiv \mathsf{T}_\gamma\mathscr{F}(\mathrm{M}) \times \mathscr{G}(\gamma^*\xi)$$

Moreover, denoting by $P_{\mathbb{O}_\gamma} : \mathsf{T}_{\mathbb{O}_\gamma}\mathscr{F}\mathscr{G}(\xi) \to \mathscr{G}(\gamma^*\xi)$ the projection induced by this decomposition, it is clear that, given $X \in \mathsf{T}_{\mathbb{O}_\gamma}\mathscr{F}\mathscr{G}(\xi)$, for almost all $t \in \overline{\mathrm{M}}$:

$$\left(P_{\mathbb{O}_\gamma} \cdot X\right)(t) = P_{\mathbb{O}_{\gamma(t)}} \cdot X(t)$$

where $P_{\mathbb{O}_{\gamma(t)}} : \mathsf{T}_{\mathbb{O}_{\gamma(t)}}\xi \to \xi_{\gamma(t)}$ is the projection induced by the decomposition:

$$\mathsf{T}_{\mathbb{O}_{\gamma(t)}}\xi \equiv \mathsf{T}_{\gamma(t)}\mathrm{M} \times \xi_{\gamma(t)}$$

obtained by identifying $\mathsf{T}_{\mathbb{O}_{\gamma(t)}}\mathbb{O}_\xi$ with $\mathsf{T}_{\gamma(t)}\mathrm{M}$ and $\mathrm{Ver}_{\mathbb{O}_{\gamma(t)}}\xi \subset \mathsf{T}_{\gamma(t)}\xi$ with $\xi_{\gamma(t)}$.

§3. APPLICATIONS

In this section, we particularize the theory of the previous section to the case $\overline{\mathrm{M}} = [a_0, a_1] \subset \mathbb{R}$, $\mathscr{F} = \mathsf{H}^k$ and $\mathscr{G} = \mathsf{H}^r$, with $k, r \in \mathbb{N}$, $0 \leqslant r < k$. Then $\mathscr{F}$ and $\mathscr{G}$ satisfy axioms $(\mathscr{F}1) - (\mathscr{F}4)$, $(\mathscr{G}1) - (\mathscr{G}2)$ and (μ); if $r \geqslant 1$, they also satisfy $(\mathscr{G}3)$.

1. The Lagrangian Functional as a Smooth Map

As a first application, we reprove that the Lagrangian functional associated to a given smooth Lagrangian $\mathrm{L} : \mathsf{TM} \to \mathbb{R}$ defined on the tangent bundle of a finite dimensional differentiable manifold M is a smooth map if defined on some convenient spaces of maps. These are well-known results (see, for instance, [5] and [9]), but we will include them here to illustrate our theory.

DEFINITION 5. *Let* $\mathrm{L} : \mathsf{TM} \to \mathbb{R}$ *be a smooth Lagrangian defined on the tangent bundle of a finite dimensional differentiable manifold* M. *Let* $k \in \mathbb{N}$, $k \geqslant 2$, *and let us define:*

$$\begin{array}{cccc} \mathcal{L}: & \mathsf{H}^k(\mathrm{M}) & \longrightarrow & \mathbb{R} \\ & \gamma & \longmapsto & \int_0^1 \mathrm{L}\left(\frac{T}{dt}\gamma\right) \end{array}$$

We call $\mathcal{L}$ *the* Lagrangian functional *associated to* L.

PROPOSITION 6. *The Lagrangian functional* $\mathcal{L} : \mathsf{H}^k(\mathrm{M}) \to \mathbb{R}$ *is a smooth map.*

Proof. To show that $\mathcal{L}$ is smooth, let us identify $\mathsf{C}^\infty(\mathsf{TM}, \mathbb{R})$ with the set of differentiable fiber bundle morphisms $\mathrm{Hom}(\mathsf{TM}, \mathbb{R}_\mathrm{M})$ in the obvious way, so that we can look at L as a smooth fiber bundle morphism $\mathrm{L} : \mathsf{TM} \to \mathbb{R}_\mathrm{M}$. Thus, it makes sense to apply the functor $\mathsf{H}^k\mathsf{H}^{k-1}$ to L, yielding the smooth map:

$$(\mathrm{L}\circ) : \mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{TM}) \to \mathsf{H}^k(\mathrm{M}) \times \mathsf{H}^{k-1}(\mathbb{R})$$

Using this smooth map (L∘), we can write $\mathcal{L}$ as a composition of smooth maps:

$$\mathcal{L} = \left(\int_{a_0}^{a_1}\right) \circ \pi_2 \circ (\mathrm{L}\circ) \circ \left(\frac{T}{dt}\right)$$

where:

$$\begin{array}{rccc} \int_{a_0}^{a_1}: & \mathrm{H}^{k-1}(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \gamma & \longmapsto & \int_{a_0}^{a_1}\gamma \end{array}$$

which is linear continuous (hence smooth),

$$\pi_2 : \mathrm{H}^k(\mathrm{M}) \times \mathrm{H}^{k-1}(\mathbb{R}) \to \mathrm{H}^{k-1}(\mathbb{R})$$

is the projection on the second factor, which is also linear continuous, and:

$$\begin{array}{rccc} \left(\frac{T}{dt}\right): & \mathrm{H}^k(\mathrm{M}) & \longrightarrow & \mathrm{H}^k\mathrm{H}^{k-1}(\mathrm{TM}) \\ & \gamma & \longmapsto & \frac{T\gamma}{dt} \end{array}$$

which is smooth (this is obvious for $\mathrm{M} = \mathbb{R}^n$; to check the general case, embed M in $\mathbb{R}^n$ by Whitney's theorem). □

Remark 4. (i) The same result also holds for time-dependent Lagrangians, and also for the Lagrangian functional defined on $\mathrm{C}^k(\mathrm{M})$, $k \geqslant 1$, and the proof is similar.

(ii) If (M, g) is a Riemannian manifold and the Lagrangian is given by $\mathrm{L}(v_q) := \frac{1}{2}\langle v_q, v_q\rangle + \mathrm{V}(q)$, where $\mathrm{V} \in \mathrm{C}^\infty(\mathrm{M})$, the same result also holds for $\mathcal{L} : \mathrm{H}^1(\mathrm{M}) \to \mathbb{R}$.

2. The Setting for Vakonomic Mechanics

In this subsection we will show that, given a finite dimensional differentiable manifold M and a smooth submanifold $\mathscr{C}$ of the tangent bundle TM of M satisfying certain conditions (a regular constraint, in the sense of Definition 7), for $k \geqslant 2$ the set $\mathrm{H}^k(\mathrm{M}, \mathscr{C})$ formed by the H^k curves $\gamma : [a_0, a_1] \to \mathrm{M}$ which are horizontal (see Definition 8), admits a differentiable manifold structure, endowed with which it becomes a closed differentiable embedded submanifold of $\mathrm{H}^k(\mathrm{M})$. This is also true for $k = 1$ if $\mathscr{C}$ is a smooth vector subbundle of TM (in this case, this manifold structure in the spaces of horizontal curves is well known - see [5]). The same holds for $\mathrm{H}^k(\mathrm{M}, \mathscr{C}, p) \subset \mathrm{H}^k(\mathrm{M}, p)$, where $\mathrm{H}^k(\mathrm{M}, p)$ is the submanifold of $\mathrm{H}^k(\mathrm{M})$ defined in the following definition:

DEFINITION 6. *Let* M *be a finite dimensional differentiable manifold and* $p, q \in \mathrm{M}$. *For* $k \in \mathbb{N}, k \geqslant 1$, *let us define the sets:*

$$\mathrm{H}^k(\mathrm{M}, p) := \{\gamma \in \mathrm{H}^k(\mathrm{M}) \mid \gamma(a_0) = p\} \subset \mathrm{H}^k(\mathrm{M})$$
$$\mathrm{H}^k(\mathrm{M}, p, q) := \{\gamma \in \mathrm{H}^k(\mathrm{M}) \mid \gamma(a_0) = p, \gamma(a_1) = q\} \subset \mathrm{H}^k(\mathrm{M}, p)$$

It is well known that $\mathrm{H}^k(\mathrm{M}, p)$ and $\mathrm{H}^k(\mathrm{M}, p, q)$ are closed smooth embedded submanifolds of $\mathrm{H}^k(\mathrm{M})$. Moreover, given $\gamma \in \mathrm{H}^k(\mathrm{M}, p)$, we have:

$$\mathrm{T}_\gamma \mathrm{H}^k(\mathrm{M}, p) = \{X \in \mathrm{T}_\gamma \mathrm{H}^k(\mathrm{M}) \mid X(a_0) = 0\}$$

and, for $\gamma \in \mathsf{H}^k(\mathrm{M},p,q)$, we have:

$$\mathsf{T}_\gamma \mathsf{H}^k(\mathrm{M},p,q) = \{X \in \mathsf{T}_\gamma \mathsf{H}^k(\mathrm{M}) \mid X(a_0) = 0, X(a_1) = 0\}$$

This follows at once from the fact that the maps $\delta_{a_0} : \mathsf{H}^k(\mathrm{M}) \to \mathrm{M}$ and $\delta_{a_1} : \mathsf{H}^k(\mathrm{M}) \to \mathrm{M}$ defined by $\gamma \mapsto \gamma(a_0)$ and $\gamma \mapsto \gamma(a_1)$, respectively, are smooth submersions onto M.

The differentiable manifolds $\mathsf{H}^k(\mathrm{M},\mathscr{C})$ and $\mathsf{H}^k(\mathrm{M},\mathscr{C},p)$ are the "arena" for the setting of the so called *vakonomic* or *variational mechanics* (see [12]); if $\mathscr{C}$ is a smooth vector subbundle of TM, they are also the "arena" for the setting of sub-Riemannian geometry.

Until the end of this subsection, we will use the notation and definitions from [12], which we summarize below.

Given a smooth vector bundle $\pi_\xi : \xi \to \mathrm{M}$ and a connection ∇ on ξ, with corresponding horizontal lift $H_\xi : \xi \oplus_{\mathrm{M}} \mathsf{T}\mathrm{M} \to \mathsf{T}\xi$ and horizontal subbundle $\mathrm{Hor}(\xi) \subset \mathsf{T}\xi$, we can define the connector $\kappa_\xi : \mathsf{T}\xi \to \xi$, which is a VB-epimorphism from $\tau_\xi : \mathsf{T}\xi \to \xi$ to $\pi_\xi : \xi \to M$ such that for each $z \in \mathsf{T}\xi$, $\kappa_\xi(z) \in \xi_{\pi_\xi \circ \tau_\xi(z)}$ is the unique vector, which satisfies:

$$z - H_\xi\left(\tau_\xi(z), T\pi_\xi(z)\right) = \lambda_\xi\left(\tau_\xi(z), \kappa_\xi(z)\right)$$

Note that the restriction of κ to the vertical subbundle $\mathrm{Ver}(\xi)$ is independent of the connection: actually, we don't need any connection at all to define it, that is, we can define $\kappa_\xi^V : \mathrm{Ver}(\xi) \to \xi$, which is, on each fiber $\mathrm{Ver}_{v_q}(\xi)$ of the vertical subbundle, the inverse of the vertical lift $\lambda_{v_q} : \xi_q \to \mathrm{Ver}_{v_q}(\xi)$.

Let $\pi_\xi : \xi \to \mathrm{M}$ and $\pi_\eta : \eta \to \mathrm{M}$ be vector bundles over M, and let $b : \xi \to \eta$ be a smooth fiber bundle morphism. The fiber derivative of b is the map:

$$\begin{aligned} \mathbb{F}b \quad &: \quad \xi \to \mathsf{L}(\xi,\eta) \\ &: \quad v_q \mapsto \mathbb{F}b(v_q) \end{aligned}$$

such that for all $w_q \in \xi_q$, we have:

$$\mathbb{F}b(v_q) \cdot w_q = \kappa_\eta^V \left(\frac{T}{dt}_{|t=0} b(v_q + t w_q) \right)$$

If in addition, in the vector bundles $\pi_\xi : \xi \to \mathrm{M}$ and $\pi_\eta : \eta \to \mathrm{M}$ we are given connections, with horizontal lifts H_ξ and H_η, and connectors κ_ξ and κ_η, respectively, then for any $v_q \in \xi$, we define the map $\mathbb{P}b : \xi \to \mathsf{L}(\mathsf{T}\mathrm{M},\eta)$ such that:

$$\mathbb{P}b(v_q) \cdot w_q = \kappa_\xi\left(\mathsf{T}_{v_q} \circ H_E(v_q, w_q)\right)$$

for all $w_q \in \mathsf{T}_q\mathrm{M}$. We call the map $\mathbb{P}b$ the *parallel derivative* of b.

It follows immediately from these definitions that, for each $X_{v_q} \in \mathsf{T}\xi$, we have:

$$\kappa_\eta(\mathsf{T}_{v_q} b \cdot X_{v_q}) = \mathbb{P}b(v_q)\left(\mathsf{T}\pi_\xi \cdot X_{v_q}\right) + \mathbb{F}b(v_q)\left(\kappa_\xi \cdot X_{v_q}\right)$$

DEFINITION 7. *Let* M *be a finite dimensional differentiable manifold,* S *a finite dimensional smooth vector bundle over* M *and* $f : \mathsf{TM} \to \mathsf{S}$ *a smooth fiber bundle morphism which is transversal to the null section* $\mathbb{O}_S$ *of* S, *so that* $\mathscr{C} := f^{-1}(\mathbb{O}_S)$ *is a closed differentiable embedded submanifold of* TM. *Suppose that the restriction to* $\mathscr{C}$ *of the projection of the tangent bundle* TM, τ_M, *is a submersion onto* M. *In these conditions, we call* f *a* regular constraint *and* $\mathscr{C}$ *the corresponding* constraint manifold.

The nomenclature *constraint* comes from mechanics: $\mathscr{C}$ is the set of permissible velocities of the trajectories of a constrained mechanical system.

PROPOSITION 7. *A smooth fiber bundle* $f : \mathsf{TM} \to \mathsf{S}$ *is a regular constraint if, and only if, for each* $v_p \in f^{-1}(\mathbb{O}_S)$, $\mathbb{F}f(v_p) \cdot \mathsf{T}_p\mathrm{M} = S_p$, *where* $\mathbb{F}$ *denotes the fiber derivative of* f.

We refer the reader to [12] for more details.

DEFINITION 8. *Let* M *be a finite dimensional differentiable manifold,* $p \in \mathrm{M}$ *and* $\mathscr{C} = f^{-1}(\mathbb{O}_S) \subset \mathsf{TM}$ *the constraint manifold corresponding to a regular constraint* $f : \mathsf{TM} \to \mathsf{S}$, *in the sense of the previous definition. We say that an absolutely continuous curve* $\gamma : [a_0, a_1] \to \mathrm{M}$ *is* horizontal *with respect to* $\mathscr{C}$ *if* $\dot\gamma(t) \in \mathscr{C}$ *a.e. on* $[a_0, a_1]$. *For each* $k \in \mathbb{N}$, $k \geqslant 1$, *we define the sets:*

$$\mathsf{H}^k(\mathrm{M}, \mathscr{C}) := \{\gamma \in \mathsf{H}^k(\mathrm{M}) \mid \dot\gamma(t) \in \mathscr{C} \text{ a.e. on } [a_0, a_1]\}$$
$$\mathsf{H}^k(\mathrm{M}, \mathscr{C}, p) := \{\gamma \in \mathsf{H}^k(\mathrm{M}, p) \mid \dot\gamma(t) \in \mathscr{C} \text{ a.e. on } [a_0, a_1]\}$$

THEOREM A. *With the same notation, if* $k \geqslant 2$, $\mathsf{H}^k(\mathrm{M}, \mathscr{C})$ *and* $\mathsf{H}^k(\mathrm{M}, \mathscr{C}, p)$ *are closed differentiable embedded submanifolds of* $\mathsf{H}^k(\mathrm{M})$ *and* $\mathsf{H}^k(\mathrm{M}, p)$, *respectively. Moreover, given a Riemannian metric tensor* g *on* M, *a connection* ∇^{S} *on* S *and* $\gamma \in \mathsf{H}^k(\mathrm{M}, \mathscr{C})$, *the tangent space* $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}, \mathscr{C})$ *is the subspace of* $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ *formed by the vector fields along* γ *which satisfy:*

$$\mathbb{F}f(\dot\gamma) \cdot \nabla_t X + \mathbb{P}f(\dot\gamma) \cdot X = 0 \tag{5}$$

where ∇_t *is the covariant derivative along* γ *induced by the Levi-Civita connection* ∇ *of* (M, g), $\mathbb{P}f$ *is the parallel derivative of* f *induced by* ∇ *and* ∇^{S}, *and* $\mathbb{F}f$ *is the fiber derivative of* f. *The same holds for* $\gamma \in \mathsf{H}^k(\mathrm{M}, \mathscr{C}, p)$, *that is,* $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}, \mathscr{C}, p)$ *is the subspace of* $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}, p)$ *formed by the vector fields along* γ *which satisfy* (5).

Proof. We will do the demonstration for $\mathsf{H}^k(\mathrm{M}, \mathscr{C})$. The same proof applies to $\mathsf{H}^k(\mathrm{M}, \mathscr{C}, p)$ and we will explicitly mention any important detail in the proof for this case.

Using the above notation, let us define the map:

$$\begin{array}{rccc} \mathcal{F}: & \mathsf{H}^k(\mathrm{M}) & \longrightarrow & \mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S}) \\ & \gamma & \longmapsto & (f\circ) \cdot \frac{T}{dt}\gamma \end{array}$$

that is, $\mathcal{F} = (f\circ) \circ (\frac{T}{dt})$, where $(f\circ) = \mathsf{H}^k\mathsf{H}^{k-1}(f) : \mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{TM}) \to \mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})$. Note that, as a composition of smooth maps, $\mathcal{F}$ is smooth.

We contend that $\mathcal{F}$ is transversal to the null section $\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})}$ of $\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})$. Indeed, it is sufficient to check that, given $\gamma \in \mathcal{F}^{-1}(\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})}) \subset \mathsf{H}^k(\mathrm{M})$, denoting by $\mathbb{O}_\gamma = \mathcal{F}(\gamma)$ the null vector of the fiber of $\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})$ over γ:

$$\mathsf{T}_\gamma\mathcal{F} \cdot \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) + \mathsf{T}_{\mathbb{O}_\gamma}\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})} = \mathsf{T}_{\mathbb{O}_\gamma}\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S}) \tag{6}$$

since the splitness condition is automatically fulfilled, that is, the fact that $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) \equiv \mathsf{H}^k(\gamma^*\mathsf{TM})$ is Hilbertizable implies that the closed subspace $(\mathsf{T}_\gamma\mathcal{F})^{-1}\left(\mathsf{T}_{\mathbb{O}_\gamma}\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})}\right)$ has a closed complementary subspace.

By Remark 3 and using the notation stated therein, we have $\mathsf{T}_{\mathbb{O}_\gamma}\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})} \equiv \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ and $\mathsf{T}_{\mathbb{O}_\gamma}\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S}) \equiv \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) \times \mathsf{H}^{k-1}(\gamma^*\mathsf{S})$, so that equation (6) is equivalent to:

$$P_{\mathbb{O}_\gamma} \cdot \mathsf{T}_\gamma\mathcal{F} \cdot \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) = \mathsf{H}^{k-1}(\gamma^*\mathsf{S}) \tag{7}$$

where $P_{\mathbb{O}_\gamma} : \mathsf{T}_{\mathbb{O}_\gamma}\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S}) \to \mathsf{H}^{k-1}(\gamma^*\mathsf{S})$ like in Remark 3.

Therefore, given $\eta \in \mathsf{H}^{k-1}(\gamma^*\mathsf{S})$, we have to show that there exists $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) \equiv \mathsf{H}^k(\gamma^*\mathsf{TM})$ such that:

$$P_{\mathbb{O}_\gamma} \cdot \mathsf{T}_\gamma\mathcal{F} \cdot X = \eta \tag{8}$$

or equivalently (see Remark 3), such that for each $t \in [a_0, a_1]$:

$$P_{\mathbb{O}_{\gamma(t)}} \cdot \left(\mathsf{T}_\gamma\mathcal{F} \cdot X\right)(t) = \eta(t)$$

where: $P_{\mathbb{O}_{\gamma(t)}} : \mathsf{T}_{\mathbb{O}_{\gamma(t)}}\mathsf{S} \equiv \mathsf{T}_{\gamma(t)}\mathrm{M} \times \mathsf{S}_{\gamma(t)} \to \mathsf{S}_{\gamma(t)}$ is the projection on the second factor, which is being identified with the vertical subspace $\mathrm{Ver}_{\mathbb{O}_{\gamma(t)}}\mathsf{S} \subset \mathsf{T}_{\mathbb{O}_{\gamma(t)}}\mathsf{S}$, like in Remark 3.

Let $\kappa : \mathsf{TTM} \to \mathsf{TM}$ be the connector induced by the Levi-Civita connection ∇ of (M, g). Since the restriction of the horizontal vector subbundle of any connection on TM (or, more generally, on any differentiable vector bundle ξ over M) to the zero section of TM (respectively, of ξ) coincides with the tangent bundle of this zero section (that is, for each $\mathbb{O}_p \in \mathsf{TM}$ we have $\mathrm{Hor}_{\mathbb{O}_p}\mathsf{TM} = \mathsf{T}_{\mathbb{O}_p}\mathbb{O}_{\mathsf{TM}}$), we have, for all $t \in [a_0, a_1]$ and for all $v \in \mathsf{T}_{\mathbb{O}_{\gamma(t)}}\mathsf{S}$:

$$P_{\mathbb{O}_{\gamma(t)}}(v) = \kappa_{\mathbb{O}_{\gamma(t)}}(v)$$

so that, by (8), the demonstration will be concluded if we prove that, given $\eta \in \mathsf{H}^{k-1}(\gamma^*\mathsf{S})$, there exists $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ such that, for all $t \in [a_0, a_1]$:

$$\kappa_{\mathbb{O}_{\gamma(t)}} \cdot \left(\mathsf{T}_\gamma\mathcal{F} \cdot X\right)(t) = \eta(t) \tag{9}$$

On the other hand, each $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ can be written as $X = \frac{Tw_s}{ds}{}_{|_{s=0}}$, where $w : (-\varepsilon, \varepsilon) \to \mathsf{H}^k(\mathrm{M})$ is a smooth curve with $w_0 = \gamma$. Thus, for each $t \in [a_0, a_1]$:

$$X(t) = \frac{Tw_s(t)}{ds}{}_{|_{s=0}}$$

and the map:

$$\begin{array}{rccc} w: & (-\varepsilon,\varepsilon) \times [a_0,a_1] & \longrightarrow & \mathrm{M} \\ & (s,t) & \longmapsto & w_s(t) \end{array}$$

is a smooth variation of $w_0 = \gamma$ by H^k curves w_s in the classical sense.

Hence, we have:

$$\begin{aligned}\left(\mathsf{T}_\gamma F \cdot X\right)(t) &= \Big(\frac{T}{ds}_{|_{s=0}} (f\circ)(\dot{w}_s)\Big)(t) = \\ &= \frac{T}{ds}_{|_{s=0}} f\big(\dot{w}_s(t)\big) = \\ &= \mathsf{T}_{\dot{\gamma}(t)} f\Big(\frac{T\dot{w}_s(t)}{ds}_{|_{s=0}}\Big)\end{aligned}$$

and, since:

$$\begin{aligned}\kappa\Big(\frac{T\dot{w}_s(t)}{ds}_{|_{s=0}}\Big) &= \nabla_t X \\ \mathsf{T}\pi_{\mathsf{TM}}\Big(\frac{T\dot{w}_s(t)}{ds}_{|_{s=0}}\Big) &= X(t)\end{aligned}$$

it follows that:

$$\kappa \cdot \big(\mathsf{T}_\gamma F \cdot X\big)(t) = \mathbb{F}f\big(\dot{\gamma}(t)\big)\cdot\nabla_t X + \mathbb{P}f\big(\dot{\gamma}(t)\big)\cdot X(t)$$

therefore, by (9), we conclude the demonstration with an application of the following lemma:

LEMMA 3. *Using the above notation, for each* $\eta \in \mathsf{H}^{k-1}(\gamma^*\mathsf{S})$, *there exists* $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) \equiv \mathsf{H}^k(\gamma^*\mathsf{TM})$, *and even* $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M},p) \subset \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ *such that, for all* $t \in [a_0,a_1]$*:*

$$\mathbb{F}f\big(\dot{\gamma}(t)\big)\cdot\nabla_t X + \mathbb{P}f\big(\dot{\gamma}(t)\big)\cdot X(t) = \eta(t) \tag{10}$$

Note that the lemma also states that there exists $X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M},p)$ satisfying equation (10), so that this demonstration also applies to the $\mathsf{H}^k(\mathrm{M},\mathscr{C},p)$ case. It is also clear that the tangent space at γ is given by:

$$\begin{aligned}\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M},\mathscr{C}) &= \mathsf{T}_\gamma F^{-1}\big(\mathsf{T}_{\mathbb{O}_\gamma}\mathbb{O}_{\mathsf{H}^k\mathsf{H}^{k-1}(\mathsf{S})}\big) \\ &= \{X \in \mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}) \mid \kappa\cdot\big(\mathsf{T}_\gamma F\cdot X\big) = 0\}\end{aligned}$$

that is, it is the subspace of $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M})$ formed by the vector fields along γ which satisfy equation (5), and the same applies to $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M},\mathscr{C},p)$. □

Proof of the lemma. Let $\operatorname{rk}\mathsf{S} = m$, $\dim\mathrm{M} = n$, and let $(e_1,\dots,e_n)$ be an H^k parallel frame field on TM along γ and $(e^1,\dots,e^n)$ be the corresponding dual coframe field along γ. Let $(s_1,\dots,s_m)$ be an H^k parallel frame field on S along γ and $(s^1,\dots,s^m)$ be the corresponding dual coframe field along γ. Such frame fields exist: we just have to choose bases of $\mathsf{T}_{\gamma(a_0)}\mathrm{M}$ and $\mathsf{S}_{\gamma(0)}$, and extend them along γ through parallel translation in the respective connections. Note that, since $\gamma \in \mathsf{H}^k(\mathrm{M})$, writing in coordinates the equation for parallel translation yields a system of linear ODE with coefficients in $\mathsf{H}^{k-1}(\mathbb{R})$, what implies that its solutions are of class H^k (see [1]).

Each $X \in \mathsf{T}_\gamma\mathsf{H}^k(M) \equiv \mathsf{H}^k(\gamma^*\mathsf{T}\mathsf{M})$ can be written uniquely as:

$$X = \sum_{i=1}^{n} X^i e_i \tag{11}$$

with $X^i \in \mathsf{H}^k(\mathbb{R})$, for $1 \leqslant i \leqslant n$.

Since $\nabla_t e_i \equiv 0$, for $1 \leqslant i \leqslant n$, we have:

$$\nabla_t X = \sum_{i=1}^{n} \dot{X}^i e_i \tag{12}$$

On the other hand, for each $t \in [a_0, a_1]$, the maps:

$$\begin{aligned} \mathbb{P}f\big(\dot\gamma(t)\big) &: \mathsf{T}_{\gamma(t)}\mathsf{M} \to \mathsf{S}_{\gamma(t)} \\ \mathbb{F}f\big(\dot\gamma(t)\big) &: \mathsf{T}_{\gamma(t)}\mathsf{M} \to \mathsf{S}_{\gamma(t)} \end{aligned}$$

are linear; taking, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$:

$$\begin{aligned} \beta_i^j(t) &:= \big\langle s^j(t), \mathbb{P}f\big(\dot\gamma(t)\big) \cdot e_i(t) \big\rangle \\ \delta_i^j(t) &:= \big\langle s^j(t), \mathbb{F}f\big(\dot\gamma(t)\big) \cdot e_i(t) \big\rangle \end{aligned} \tag{13}$$

we have $\beta_i^j, \delta_i^j \in \mathsf{H}^{k-1}(\mathbb{R})$ and, from equations (11) and (12):

$$\begin{aligned} \mathbb{P}f(\dot\gamma) \cdot X &= \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_i^j X^i s_j \\ \mathbb{F}f(\dot\gamma) \cdot \nabla_t X &= \sum_{j=1}^{m} \sum_{i=1}^{n} \delta_i^j \dot{X}^i s_j \end{aligned} \tag{14}$$

Hence, defining $\eta^i \in \mathsf{H}^{k-1}(\mathbb{R})$, for $1 \leqslant i \leqslant m$, by:

$$\eta = \sum_{i=1}^{m} \eta^i s_i \tag{15}$$

it follows from equations (14) and (15) that equation (10) is equivalent to:

$$\sum_{i=1}^{n} \delta_i^j \dot{X}^i + \sum_{s=1}^{n} \beta_s^j X^s = \eta^j \tag{16}$$

for $1 \leqslant j \leqslant m$.

Since the constraint $\mathscr{C} = f^{-1}(\mathbb{O}_\mathsf{S})$ is regular, for each $t \in [a_0, a_1]$ the map $\mathbb{F}f\big(\dot\gamma(t)\big) : \mathsf{T}_{\gamma(t)}\mathsf{M} \to \mathsf{S}_{\gamma(t)}$ is surjective. Thus, $\delta = (\delta_i^j) \in \mathsf{H}^{k-1}\big(\mathsf{L}(\mathbb{R}^n, \mathbb{R}^m)\big)$ is such that, for each $t \in [a_0, a_1]$, $\delta(t)$ has maximal rank, that is, $\operatorname{rk} \delta(t) = m$. Thus, (16) is a system of m

linear ODE in $\mathbb{R}$ with coefficients in $\mathsf{H}^{k-1}(\mathbb{R})$, and the matrix $\delta(t)$ has maximal rank for all $t \in [a_0, a_1]$, what implies that (see [1]) there exists $X \in \mathsf{H}^k(\mathbb{R}^n) \equiv \mathsf{H}^k(\gamma^*\mathsf{TM})$ with $X(0) = 0$ (that is, X belongs, in fact, to $\mathsf{T}_\gamma\mathsf{H}^k(\mathrm{M}, p)$) such that X satisfies (16). Note that we cannot guarantee that such X is unique, unless $m = n$.

□

Remark 5. The theorem also holds for $k = 1$ if $\mathscr{C}$ is a smooth vector subbundle of TM, that is, if $f : \mathsf{TM} \to \mathsf{S}$ is a smooth vector bundle epimorphism – see [5].

To close this section, we list below some of the main results we have obtained in [12] using these differentiable manifold structures on the spaces of curves horizontal to the constraint.

DEFINITION 9. *A* constrained mechanical system *is a quadruple* $(\mathrm{M}, \mathrm{K}, \mathrm{V}, f)$*, where* (M, g) *is a Riemannian manifold,*

$$\begin{array}{rccc} \mathrm{K}: & \mathsf{TM} & \longrightarrow & \mathbb{R} \\ & v_q & \longmapsto & \frac{1}{2}\mathrm{g}(v_q, v_q) \end{array}$$

is the kinetic energy*,* $f : \mathsf{TM} \to \mathsf{S}$ *is a regular constraint in the sense of Definition* 7*, and* $V : \mathrm{M} \to \mathbb{R}$ *is a smooth function called the* potential energy*. The manifold* M *is called the* configuration space*.*

DEFINITION 10. *We say that* $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$ *is an* abnormal *or* singular variational trajectory *of the constrained mechanical system* $(\mathrm{M}, \mathrm{K}, \mathrm{V}, f)$ *if it is a critical point of the endpoint mapping:*

$$\begin{array}{rccc} ev_1: & \mathsf{H}^2(\mathrm{M}, \mathscr{C}, \gamma(a_0)) & \longrightarrow & \mathrm{M} \\ & q & \longmapsto & q(a_1) \end{array}$$

We say that $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$ *is a* normal *or* regular variational trajectory *of the constrained mechanical system* $(\mathrm{M}, \mathrm{K}, \mathrm{V}, f)$ *if* $d\mathcal{L}(\gamma)$ *annihilates the closed subspace:*

$$\{X \in \mathsf{T}_\gamma\mathsf{H}^2(\mathrm{M}, \mathscr{C}, \gamma(a_0)) \mid X(a_1) = 0\}$$

of $\mathsf{T}_\gamma\mathsf{H}^2(\mathrm{M}, \mathscr{C}, \gamma(a_0))$*.*

THEOREM B. *The following conditions are equivalent, given* $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$*:*

(i) γ *is an abnormal variational trajectory.*

(ii) *There exists* $P \in \mathsf{H}^1(\gamma^*\mathsf{S})$*,* $P \neq 0$*, such that, for almost all* $t \in [a_0, a_1]$*:*

$$\nabla_t\left(\mathbb{F}^* f(\dot{\gamma}) \cdot P\right) - \mathbb{P}^* f(\dot{\gamma}) \cdot P = 0 \tag{17}$$

THEOREM C. *Let* $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$*. Then the two following conditions are equivalent:*

(i) γ *is a normal variational trajectory.*

(ii) *There exists* $P \in \mathsf{H}^1(\gamma^*\mathsf{S})$ *such that the following equation is satisfied:*

$$\nabla_t \dot{\gamma} + \operatorname{grad} \mathrm{V} \circ \gamma = -\nabla_t \left(\mathbb{F}^* f(\dot{\gamma}) \cdot P\right) + \mathbb{P}^* f(\dot{\gamma}) \cdot P \tag{18}$$

If γ is a regular point of the endpoint mapping, then $\mathsf{H}^2\big(\mathrm{M}, \mathscr{C}, \gamma(a_0), \gamma(a_1)\big)$ is a smooth sub-manifold of $\mathsf{H}^2(\mathrm{M})$ in a suitable neighborhood of γ. Hence, we obtain the following corollary:

COROLLARY 4. *With the same notation, if* γ *is a regular point of the endpoint mapping, then* γ *is a normal variational trajectory if, and only if, it is a stationary point of the restriction of the Lagrangian functional* $\mathcal{L} : \mathsf{H}^2(\mathrm{M}) \to \mathbb{R}$ *to* $\mathsf{H}^2\big(\mathrm{M}, \mathscr{C}, \gamma(a_0), \gamma(a_1)\big)$.

Remark 6. (i) The nomenclature *normal/abnormal* comes from sub-Riemannian geometry (which can be viewed as a particular case of our formulation, putting $\mathrm{V} = 0$ and f the orthogonal projection $P_{\mathscr{D}^\perp} : \mathsf{TM} \to \mathscr{D}^\perp$, where $\mathscr{D}$ is a smooth vector subbundle of TM, so that the constraint manifold is $\mathscr{D}$).

(ii) The solutions of equation (18) may lead to curves $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$ which are not regular points of the endpoint mapping. In other words, like in sub-Riemannian geometry, a curve $\gamma \in \mathsf{H}^2(\mathrm{M}, \mathscr{C})$ may be simultaneously a normal and abnormal variational trajectory.

(iii) Let $\mathscr{C} \times_{\mathrm{M}} \mathsf{S}$ be the fiber product of $\pi_{\mathscr{C}} = \tau_{\mathrm{M}}|_{\mathscr{C}} : \mathscr{C} \to \mathrm{M}$ and $\pi_{\mathsf{S}} : \mathsf{S} \to \mathrm{M}$. We have also shown in [12] that there exists an open set $\mathcal{U} \subset \mathscr{C} \times_{\mathrm{M}} \mathsf{S}$ containing $\mathscr{C} \times_{\mathrm{M}} \mathbb{O}_{\mathsf{S}}$ and a smooth vector field $\mathrm{X}_{\mathsf{H}} : \mathcal{U} \to \mathsf{T}\mathcal{U}$, which is Hamiltonian with respect to a certain symplectic form induced by f, g and the canonical symplectic form of the cotangent bundle $\mathsf{T}^*\mathrm{M}$, whose integral curves are of the form $(\dot{\gamma}, P)$, with (γ, P) a solution of equation (18). In general, however, the open set $\mathcal{U}$ cannot be taken to be the whole $\mathscr{C} \times_{\mathrm{M}} \mathsf{S}$.

§A. HAUSDORFF METRIC

Notation. In this section (X, d) will denote a metric space and $\mathfrak{K}_X$ will denote the set $\{A \subset X \mid A \neq \emptyset \text{ and } A \text{ is compact}\}$.

DEFINITION 11 (HAUSDORFF METRIC). *Consider the following maps:*

$$\begin{array}{rccc} d: & \mathfrak{K}_X \times \mathfrak{K}_X & \longrightarrow & \mathbb{R} \\ & (A,B) & \longmapsto & \sup\big\{\inf\{d(x,y) \mid y \in B\} \mid x \in A\big\} \\ D: & \mathfrak{K}_X \times \mathfrak{K}_X & \longrightarrow & \mathbb{R} \\ & (A,B) & \longmapsto & \frac{1}{2} d(A,B) + \frac{1}{2} d(B,A) \end{array} \tag{19}$$

D is called the Hausdorff metric *induced by d. This nomenclature is motivated by the following proposition:*

PROPOSITION 8. *D is a metric in* $\mathfrak{K}_X$.

Proof. For all $A, B \in \mathfrak{K}_X$, we have $d(A,B) = 0$ if, and only if, $A \subset B$; therefore, $D(A,B) = 0$ if, and only if, $A = B$. The symmetry of D is obvious, and the triangular inequality follows from the fact that, for all $A, B, C \in \mathfrak{K}_X$:

$$d(A,C) \leqslant d(A,B) + d(B,C)$$

what can be proven using the triangular inequality for $d : X \times X \to \mathbb{R}$ and the definitions of inf and sup. □

Remark 7. Endowing $\mathfrak{K}_X$ with the Hausdorff metric D, the map $d : \mathfrak{K}_X \times \mathfrak{K}_X \to \mathbb{R}$ is continuous.

Notation. Until the end of this section, D will denote the Hausdorff metric in $\mathfrak{K}_X$ induced by the metric d of (X,d).

LEMMA 4. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{K}_X$, and let $A \in \mathfrak{K}_X$. Assume that there exists $n_0 \in \mathbb{N}$ and a compact set K such that $A_n \subset K$ for $n \geqslant n_0$. Then the following conditions are equivalent:*

(i) $d(A_n,A) \xrightarrow{n\to\infty} 0$

(C1) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $(\forall n \in \mathbb{N})\ x_n \in A_n$, then the limit of any convergent subsequence of $(x_n)_{n\in\mathbb{N}}$ lies in A.*

Proof. (i)⇒(C1) Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $(\forall n \in \mathbb{N})\ x_n \in A_n$, and let $(x_{n_j})_{j\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_j} \xrightarrow{j\to\infty} x \in X$. We must show that $x \in A$. In fact, for all $n_j \in \mathbb{N}$:

$$d(x,A) \leqslant d(x,x_{n_j}) + d(x_{n_j},A) \overset{x_{n_j}\in A_{n_j}}{\leqslant} d(x,x_{n_j}) + d(A_{n_j},A) \tag{20}$$

As $d(x,x_{n_j}) \xrightarrow{j\to\infty} 0$ and $d(A_{n_j},A) \xrightarrow{j\to\infty} 0$, it follows from (20) that $d(x,A) = 0$, that is, $x \in A$.

(C1)⇒(i) Suppose that (C1) holds and that $d(A_n,A) \overset{n\to\infty}{\not\longrightarrow} 0$. Then there exists $\varepsilon > 0$ and a subsequence $(A_{n_j})_{j\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that $(\forall j \in \mathbb{N})\ d(A_{n_j},A) > \varepsilon$.

Therefore, for each $j \in \mathbb{N}$, we can choose $x_{n_j} \in A_{n_j}$ such that $d(x_{n_j},A) > \varepsilon$. Taking a convergent subsequence of $(x_{n_j})_{j\in Nn}$ (there exists such a subsequence, since the sequence $(x_{n_j})_{j\in\mathbb{N}}$ lies in the compact K for $n_j > n_0$), if necessary, we can suppose that $x_{n_j} \xrightarrow{j\to\infty} x \in X$. Thus, for n_j sufficiently large:

$$d(x,A) \geqslant d(x_{n_j},A) - d(x_{n_j},x) > 0$$

and this implies that $x \notin A$, which contradicts (C1).

□

LEMMA 5. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{K}_X$, and let $A \in \mathfrak{K}_X$. Then the following conditions are equivalent:*

(i) $d(A,A_n) \overset{n\to\infty}{\longrightarrow} \mathbb{O}$

(C2) *For all $x \in A$, there exists $(x_n)_{n\in\mathbb{N}}$ sequence in X such that $(\forall n \in \mathbb{N})\ x_n \in A_n$ and $x_n \overset{n\to\infty}{\longrightarrow} x$.*

Proof. (i)⇒(C2) Given $x \in A$, for each $n \in \mathbb{N}$ take $x_n \in A_n$ such that $d(x,A_n) = d(x,x_n)$. Since $(\forall n \in \mathbb{N})\ 0 \leqslant d(x,A_n) \leqslant d(A,A_n)$, it follows from (i) that:

$$d(x,x_n) = d(x,A_n) \overset{n\to\infty}{\longrightarrow} 0$$

that is, $x_n \overset{n\to\infty}{\longrightarrow} x$, which proves (C2).

(C2)⇒(i) Let $x \in A$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $(\forall n \in \mathbb{N})\ x_n \in A_n$ and $x_n \overset{n\to\infty}{\longrightarrow} x$. Then, since $(\forall n \in \mathbb{N})\ 0 \leqslant d(x,A_n) \leqslant d(x,x_n)$, it follows that:

$$d(x,A_n) \overset{n\to\infty}{\longrightarrow} 0$$

Therefore, given $\varepsilon > 0$, for each $x \in A$ there exists $n_x \in \mathbb{N}$ such that $(\forall n \geqslant n_x)\ d(x,A_n) < \frac{\varepsilon}{2}$.

Let C be the open cover $\{B_{\frac{\varepsilon}{2}}(x) \mid x \in A\}$ of the compact set A. Take a finite subcover $\{B_{\frac{\varepsilon}{2}}(x_1),\dots,B_{\frac{\varepsilon}{2}}(x_k)\}$ of C, where $x_1,\dots,x_k \in A$.

Let $N := \max\{n_{x_1},\dots,n_{x_k}\}$. Given $y \in A$, there exists $j \in \{1,\dots,k\}$ such that $y \in B_{\frac{\varepsilon}{2}}(x_j)$; therefore, for $n \geqslant N$, we have:

$$d(y,A_n) \leqslant d(y,x_j) + d(x_j,A_n) < \varepsilon$$

since $d(y,x_j) < \frac{\varepsilon}{2}$ and $d(x_j,A_n) < \frac{\varepsilon}{2}$. By the arbitrariness of the choice of $y \in A$, this shows that, for $n \geqslant N$:

$$d(A,A_n) = \sup_{y\in A} d(y,A_n) \leqslant \varepsilon$$

and, since ε was arbitrarily taken, this implies that $d(A,A_n) \overset{n\to\infty}{\longrightarrow} 0$

□

COROLLARY 5. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{K}_X$, and let $A \in \mathfrak{K}_X$. Assume that there exists $n_0 \in \mathbb{N}$ and a compact set K such that $A_n \subset K$ for $n \geqslant n_0$. Then $A_n \overset{n\to\infty}{\longrightarrow} A$ in the Hausdorff metric D if, and only if, conditions* (C1) *and* (C2) *hold.*

§B. GRASSMANN MANIFOLDS AND FIBER BUNDLES

DEFINITION 12. *Let V^n be a real vector space of dimension n, let $\langle\cdot,\cdot\rangle$ be an inner product in V, and, given $k \in \mathbb{N}$, let:*

$$Gr_k(\mathsf{V}) := \{k\text{-dimensional subspaces of } \mathsf{V}\}$$

and let also:

$$Gr(\mathsf{V}) := \bigcup_{0 \leqslant k \leqslant n} Gr_k(\mathsf{V})$$

Given an orthonormal basis $\{e_1,\dots,e_n\}$ of V, there exists a well defined action η : $\mathsf{O}(n) \times \mathsf{V} \to \mathsf{V}$, which induces the action:

$$\begin{array}{rccc} \eta: & \mathsf{O}(n) \times Gr_k(\mathsf{V}) & \longrightarrow & Gr_k(\mathsf{V}) \\ & (\sigma,[v_1,\dots,v_k]) & \longmapsto & [\sigma v_1,\dots,\sigma v_k] \end{array} \tag{21}$$

and it is obvious that η is a transitive action of $\mathsf{O}(n)$ on the set $Gr_k(\mathsf{V})$. Let $\mathsf{W} := [e_1,\dots,e_k] \in Gr_k(\mathsf{V})$. We claim that, for $1 \leqslant k \leqslant n$, the isotropy subgroup H_{W} of W is the subgroup:

$$\left\{ \left(\begin{smallmatrix} \sigma & 0 \\ 0 & \tau \end{smallmatrix}\right) \mid \sigma \in \mathsf{O}(k), \tau \in \mathsf{O}(n-k) \right\} \cong \mathsf{O}(k) \times \mathsf{O}(n-k)$$

As a matter of fact, it is clear that $\sigma \in \mathsf{O}(k) \times \mathsf{O}(n-k)$ like above leaves W fixed. On the other hand, assume that $\sigma \in \mathsf{O}(n)$ is such that $\sigma \cdot \mathsf{W} = \mathsf{W}$. Then it follows that:

(i) $(\forall j \in \{1,\dots,k\})\, \sigma \cdot e_j = \sum_{i=1}^n \sigma^i_j e_i \in [e_1,\dots,e_k]$, *so that $\sigma^i_j = 0$ for $i \in \{k+1,\dots,n\}$;*

(ii) *as we also have $\sigma \cdot \mathsf{W}^\perp = \mathsf{W}^\perp$, it follows that $\sigma^i_j = 0$ for $i \in \{1,\dots,k\}$ and $j \in \{k+1,\dots,n\}$, and this concludes the proof of the assertion.*

Since $\mathsf{O}(k) \times \mathsf{O}(n-k)$ is a closed subgroup of $\mathsf{O}(n)$, the quotient $\mathsf{O}(n)/[\mathsf{O}(k) \times \mathsf{O}(n-k)]$ is a homogeneous manifold, and we transport this manifold structure to $Gr_k(\mathsf{V})$ through the bijection:

$$\begin{array}{rccc} \widetilde{\eta}: & \mathsf{O}(n)/H_{\mathsf{W}} & \longrightarrow & Gr_k(\mathsf{V}) \\ & [\sigma] & \longmapsto & \sigma \cdot \mathsf{W} \end{array}$$

$Gr_k(\mathsf{V})$, endowed with this manifold structure, is called the Grassmannian manifold of k-planes of V *, and we topologize $Gr(\mathsf{V})$ as the topological sum of the spaces $Gr_k(\mathsf{V})$, $0 \leqslant k \leqslant n$.*

PROPOSITION 9. *The manifold structure of $Gr_k(\mathsf{V})$ is independent of the orthonormal basis $\mathcal{E} = (e_1,\dots,e_n)$ initially chosen.*

PROPOSITION 10. *Each $Gr_k(\mathsf{V})$, $0 \leqslant k \leqslant n$, is path connected, so that these spaces are the connected components of $Gr(\mathsf{V})$.*

DEFINITION 13. *Denote by* $B_1(\mathbb{O})$ *the closed ball of radius 1 in the euclidean space* $(\mathsf{V},\langle\cdot,\cdot\rangle)$*, and by D the Hausdorff metric of* $\mathfrak{K}_{B_1(\mathbb{O})}$. *Let* $Gr_k(B_1(\mathbb{O}))$ *be the metric subspace of* $(\mathfrak{K}_{B_1(\mathbb{O})},D)$ *given by:*

$$Gr_k(B_1(\mathbb{O})) := \{\mathsf{W}\cap B_1(\mathbb{O}) \mid \mathsf{W}\in Gr_k(\mathsf{V})\}$$

PROPOSITION 11. *The map:*

$$\begin{array}{rccc} N: & Gr_k(\mathsf{V}) & \longrightarrow & Gr_k(B_1(\mathbb{O})) \\ & \mathsf{W} & \longmapsto & \mathsf{W}\cap B_1(\mathbb{O}) \end{array}$$

is a homeomorphism. In other words, the topology of $Gr_k(\mathsf{V})$ *can be defined by the Hausdorff metric D of* $\mathfrak{K}_{B_1(\mathbb{O})}$.

Proof. It is clear that N is a bijection. Therefore, since $Gr_k(\mathsf{V})\cong \mathsf{O}(k)\times\mathsf{O}(n-k)$ is compact and $Gr_k(B_1(\mathbb{O}))$ is Hausdorff, it is sufficient to show that N is continuous. But this is equivalent to show that the map:

$$\begin{array}{rccc} N\circ\widetilde{\eta}\circ\pi: & \mathsf{O}(n) & \longrightarrow & Gr_k(B_1(\mathbb{O})) \\ & \sigma & \longmapsto & \sigma\cdot\mathsf{W}\cap B_1(\mathbb{O}) \end{array}$$

is continuous, since

$$\begin{array}{rccc} \widetilde{\eta}\circ\pi: & \mathsf{O}(n) & \longrightarrow & Gr_k(\mathsf{V}) \\ & \sigma & \longmapsto & \sigma\cdot\mathsf{W} \end{array}$$

is a quotient map.

Indeed, given a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathsf{O}(n)$ such that $\sigma_n \overset{n\to\infty}{\longrightarrow} \sigma\in\mathsf{O}(n)$, we assert that $\sigma_n\cdot\mathsf{W}\cap B_1(\mathbb{O}) \overset{n\to\infty}{\longrightarrow} \sigma\cdot\mathsf{W}\cap B_1(\mathbb{O})$. By Corollary 5, we must verify conditions (C1) and (C2).

(C1) Let $A_n := \sigma_n\cdot\mathsf{W}\cap B_1(\mathbb{O})$ and $A := \sigma\cdot\mathsf{W}\cap B_1(\mathbb{O})$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $B_1(\mathbb{O})$ such that $(\forall n\in\mathbb{N})\ x_n\in A_n$, and let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_n$ such that $x_{n_k} \overset{n\to\infty}{\longrightarrow} x\in B_1(\mathbb{O})$. We must verify that $x\in A$. As a matter of fact, let $(y_k)_{k\in\mathbb{N}}$ be the sequence in W defined by $y_k := \sigma_{n_k}^{-1}\cdot x_{n_k}$. Then, by continuity we have $y_k = \sigma_{n_k}^{-1}\cdot x_{n_k} \overset{n\to\infty}{\longrightarrow} \sigma^{-1}\cdot x\in\mathsf{W}$, since W is closed in V. But this implies $x=\sigma\cdot(\sigma^{-1}\cdot x)\in\sigma\cdot\mathsf{W}$, so that $x\in\sigma\cdot\mathsf{W}\cap B_1(\mathbb{O}) = A$ and (C1) is verified.

(C2) Given $x\in A$, take $y:=\sigma^{-1}\cdot x\in\mathsf{W}\cap B_1(\mathbb{O})$ and let $(x_n)_{n\in\mathbb{N}}$ be the sequence in $B_1(\mathbb{O})$ defined by $x_n := \sigma_n\cdot y$. Then we have:

(i) $(\forall n\in\mathbb{N})\ x_n\in A_n = \sigma_n\cdot\mathsf{W}\cap B_1(\mathbb{O})$.

(ii) $x_n = \sigma_n\cdot y \overset{n\to\infty}{\longrightarrow} \sigma\cdot y = x$

so that (C2) is verified.

□

COROLLARY 6. *$(Gr_k(B_1(\mathbb{O})),D)$ is a compact metric space.*

PROPOSITION 12. *Let $(\mathsf{V}^n,\langle\cdot,\cdot\rangle)$, $(\mathsf{W}^m,\langle\cdot,\cdot\rangle)$ be inner product spaces, and let $f:\mathsf{V}\to\mathsf{W}$ be a linear isomorphism (so that $n\leqslant m$). Given $k\in\mathbb{N}$, $1\leqslant k\leqslant n$, let:*

$$\begin{array}{rccc} Gr_k(f): & Gr_k(\mathsf{V}) & \longrightarrow & Gr_k(\mathsf{W}) \\ & [v_1,\dots,v_k] & \longmapsto & [f\cdot v_1,\dots,f\cdot v_k] \end{array}$$

Then *$Gr_k(f)$ is a smooth embedding.*

DEFINITION 14. *Let $k\in\mathbb{N}^*$ and let M be a finite dimensional Hausdorff second countable differentiable manifold. Let also $\pi_\xi:\xi\to\mathrm{M}$ be a finite dimensional smooth vector bundle over M with rank $n\geqslant k$, and let us define:*

$$Gr_k(\xi):=\bigcup_{p\in\mathrm{M}} Gr_k(\xi_p)$$

where $\xi_p:=\pi_\xi^{-1}(p)$ is the fiber of ξ over $p\in\mathrm{M}$.

Let $\pi_{Gr_k(\xi)}:Gr_k(\xi)\to\mathrm{M}$ be the obvious projection. We will define a manifold structure in the set $Gr_k(\xi)$ in such a way that $\pi_{Gr_k(\xi)}:Gr_k(\xi)\to\mathrm{M}$ be a locally trivializable differentiable fiber bundle over M. In order to do that, let $\{(\mathcal{U}_\alpha,\varphi_\alpha),\ \alpha\in A\}$ be a vector bundle atlas of $\pi_\xi:\xi\to\mathrm{M}$.

For each $\alpha\in A$ let us define:

$$\begin{array}{rccc} Gr_k(\varphi_\alpha): & \pi^{-1}_{Gr_k(\xi)}(\mathcal{U}_\alpha) & \longrightarrow & \mathcal{U}_\alpha\times Gr_k(\mathbb{R}^n) \\ & \tau_p & \longmapsto & (p,Gr_k(\varphi_p)\cdot\tau_p) \end{array}$$

where φ_p is the restriction $\varphi_\alpha|_{\xi_p}:\xi_p\to\mathbb{R}^n$.

PROPOSITION 13. *Using the notation of the above definition, the collection $\{(\mathcal{U}_\alpha,Gr_k(\varphi_\alpha)),\ \alpha\in A\}$ is a smooth fiber bundle atlas in $Gr_k(\xi)$, so that $\pi_{Gr_k(\xi)}:Gr_k(\xi)\to\mathrm{M}$ is a differentiable fiber bundle over M.*

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REFERENCES

[1] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, 1965.

[2] D. G. EBIN AND J. MARSDEN, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math., 92 (1970), pp. 102–163.

[3] J. EELLS, *A setting for global analysis*, Bull. Amer. Math. Soc., 72 (1966), pp. 751–807.

[4] E. HEBEY, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Institute of Mathematical Sciences, 1999.

[5] I. KUPKA AND W. M. OLIVA, *The nonholonomic mechanics*, Journal of Differential Equations, 169 (2001).

[6] S. LANG, *Fundamentals of Differential Geometry*, Springer-Verlag, 1999.

[7] I. H. MADSEN AND J. TORNEHAVE, *From Calculus to Cohomology : De Rham Cohomology and Characteristic Classes*, Cambridge University Press, 1997.

[8] R. S. PALAIS, *Foundations of Global Non-linear Analysis*, W. A. Benjamin, Inc., 1968.

[9] P. PICCIONE AND D. TAUSK, *Lagrangian and Hamiltonian formalism for constrained variational problems*. To appear in *The Royal Society of Edinburgh Proceedings A (Mathematics)*. (see also LANL math.OC/0004148), 2000.

[10] P. PICCIONE AND D. V. TAUSK, *On the Banach differential structure for sets of maps on non-compact domains*, Journal of Nonlinear Analysis, 46 (2001), pp. 245–265.

[11] R. C. RIDELL, *A note on Palais' axioms for section functors*, Proc. Amer. Math. Soc., 25 (1970), pp. 808–810.

[12] G. TERRA AND M. H. KOBAYASHI, *On the variational mechanics with non-linear constraints*, To appear, (2002).

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