Resenhas IME-USP 2002, Vol. 5, No. 4, 293 - 304.

Dimension and Fox subgroups

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To Sudarshan K. Sehgal on the occasion of his 65th birthday.

Abstract:This article is a survey of some results on the identification of groups given by ideals in the group ring of a group over the integers and in certain cases over the integers modulo p, a prime. Dimension and Lie dimension subgroups are discussed in general and also the Fox-type subgroups within the free group ring. A new result on identifying the generalised Fox subgroups $F(n, R, m) = (1+\mathfrak{f}^n\mathfrak{r}^m)\cap F$ is also given with some applications.

Key words: Group Ring, Augmentation Ideal, Dimension Subgroup, Fox Subgroup.

Section 1 is concerened with the dimension subgroup conjectures, section 2 surveys results on the Fox problem, section 3 deals with the modular Fox problem and section 4 introduces new results on identifying the generalised Fox subgroups $F(n, R, m) = (1 + \mathfrak{f}^n\mathfrak{r}^m) \cap F$.

1 The Dimension Subgroup Conjectures:

1.1 Notation:

$\mathbb{Z}G$ is the *group ring* of the group G over the integers $\mathbb{Z}$:

$$\mathbb{Z}G = \{\sum a_g g : g \in G, a_g \in \mathbb{Z}\}.$$

The *augmentation ideal* of $\mathbb{Z}G$ is $\Delta(G) = \text{Ker } \phi(\mathbb{Z}G \to \mathbb{Z})$ where $\phi : \sum a_g g \mapsto \sum a_g$.

Define the *Lie powers* $\Delta^{(i)}(G)$ of $\Delta(G)$ inductively by: $\Delta^{(1)}(G) = \Delta(G)$, and

$$\Delta^{(m+1)}(G) = (\Delta^{(m)}(G),\ \Delta(G))\mathbb{Z}G$$

is the ideal generated by the Lie products $(x, y) = xy - yx$ with $x \in \Delta^{(m)}(G)$, $y \in \Delta(G)$.

For subsets H and K of G, the group $[H, K]$ is the subgroup of G generated by all commutators $[h, k]$ with $h \in H$ and $k \in K$. Let $\gamma_n(G)$ be the n^{th} term of the lower central series of G; $\gamma_n(G)$ is defined inductively by: $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$.

Denote by $D_n(G)$ and $D_{(n)}(G)$ the n^{th} *Dimension subgroup* and the n^{th} *Lie Dimension subgroup* respectively of G, namely,

$$D_n(G) = G \cap (1 + \Delta^n(G)),$$

$$D_{(n)}(G) = G \cap (1 + \Delta^{(n)}(G)).$$

Denoting by $[x, y]$ the group commutator $x^{-1}y^{-1}xy$, we have
$[x, y] - 1 = x^{-1}y^{-1}(xy - yx) = x^{-1}y^{-1}\{(x-1)(y-1) - (y-1)(x-1)\}$
It follows by induction that

$$\gamma_n(G) \subseteq D_{(n)}(G) \subseteq D_n(G).$$

1.2 Problems and results

The **Dimension Subgroup Conjecture** refers to the equality $D_n(G) = \gamma_n(G)$.

It has a tempestuous history and is notorious for the number of published incorrect proofs.

The **Lie Dimension Subgroup Conjecture** refers to the equality $D_{(n)}(G) = \gamma_n(G)$.

Early known results: (until about 1972)

1. $D_n(F) = \gamma_n(F)$ for a free group F . (Magnus, 1930^s)
2. The dimension conjecture is true for a group with torsion-free lower central factors. (Hall, Jennings.)
3. $D_1(G) = \gamma_1(G)$ and $D_2(G) = \gamma_2(G)$. (Exercises!)
4. $D_3(G) = \gamma_3(G)$ (Higman-Rees, Passi, Hoare, Sandling.)
5. $D_4(G) = \gamma_4(G)$ if G is of *odd* order (Passi).
6. It is only necessary to prove the Dimension conjecture for finite p-groups. (Higman)

In 1972 [20] Bob Sandling published a very nice paper in which he really got to the heart of the matter. He noticed amongst other things that the dimension conjecture is not the most natural and, perhaps, the *Lie Dimension Subgroup Conjecture*, that $D_{(n)}(G) = \gamma_n(G)$, is more appropriate.

He proved among other things:

1. $D_{(n)}(G) = \gamma_n(G)$, for $n \leq 6$.
2. $D_{(n)}(G) = \gamma_n(G)$, for a metabelian group G. (G is metabelian iff $\gamma_2(\gamma_2(G)) = 1$.)
3. $$\Delta^{(n)}(G) = \Delta(\gamma_n(G))ZG + \sum \prod_j \Delta(\gamma_{n_j}(G)).ZG,$$

   where the sum is over all $n_j, n \geq n_j > 1$, for which $\sum(n_j - 1) = n - 1$

So in fact Sandling showed that the Lie Dimension Conjecture in a sense goes *twice* as far as the Dimension Conjecture and is true for metabelian groups.

Note from

$$\gamma_n(G) \subseteq D_{(n)}(G) \subseteq D_n(G).$$

that a positive solution to the Dimension Conjecture gives a positive solution to the Lie Dimension Conjecture and that a counterexample to the Lie Dimension Conjecture yields a counterexample to the Dimension conjecture.

A major breakthrough came about when Rips [19], in 1972, produced a counterexample to the Dimension Conjecture for the case $n = 4$. His example is a metabelian 2-group and is very clever.

Could it possibly be that the conjecture would be true for some values bigger than 4?

Certainly not but it wasn't until 1990 that N. D. Gupta [8] produced a metabelian 2-group $\mathcal{G}$ (depending on n) such that $D_n(\mathcal{G}) \neq \gamma_n(\mathcal{G})$ for all $n \geq 4$.

The Lie dimension subgroup conjecture refers to the equality $D_{(n)}(G) = \gamma_n(G)$. This is known to be true for $n \leq 6$ and for metabelian groups (Sandling). A counterexample must have $n > 6$ and also must be non-metabelian which makes it much more complicated.

Another complication arises from the fact that

$$\Delta^{(i)}(G) * \Delta^{(j)}(G) \subseteq \Delta^{(i+j-1)}(G)$$

so that some of the 'dimension' is lost.

However, Hurley and Sehgal [17] gave examples that disprove the Lie Dimension conjecture for all $n \geq 9$. These are also counterexamples to the Dimension Subgroup conjecture.

The examples also yield the fascinating fact for the Dimension subgroup conjecture that

$D_{4m}(G) \nsubseteq \gamma_{3m+1}(G)$ for $m \geq 2$.

The Dimension conjecture is very far off!

The following problems then remained unresolved:

1. Is the Lie dimension subgroup conjecture true for $n = 7$ or $n = 8$?

2. Are the dimension conjectures true for *odd* order groups?

3. Find the structure of $D_n(G)/\gamma_n(G)$.

In connection with 2. Gupta [7] showed that the Dimension Conjecture is true for *odd-order metabelian* groups. Remember that Sandling showed that the Lie Dimension subgroup conjecture is true for all metabelian groups.

It was then shown by N. D. Gupta and Tahara 1993 [10] that in fact the Lie Dimension conjecture is true for $n = 7, 8$. This gave the complete picture: The Dimension conjecture is true for $n \leq 3$ and is false for $n \geq 4$; the Lie Dimension conjecture is true for $n \leq 8$ and false for $n \geq 9$.

1.3 Restricted Lie Dimension subgroups

In view of these examples one might expect that $\gamma_n(G)$ is the group associated with the restricted Lie powers of $\Delta(G)$. These Lie powers are defined inductively:

$$\Delta^{[1]}(G) = \Delta(G),\ \Delta^{[m+1]}(G) = \big(\Delta^{[m]}(G), \Delta(G)\big),$$

the additive group generated by the Lie products.

Define the *restricted Lie Dimension subgroup* by $D_{[n]}(G) = G \cap (1 + \Delta^{[n]}(G))$. It is a nontrivial result of Gupta and Levin [9] that $\gamma_n(G) \subseteq D_{[n]}(G)$.

Thus $\gamma_n(G) \subseteq D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G) \quad \forall n.$

In [17] a modification of the Lie Dimension counterexamples is given which shows that for $n \geq 14$, $D_{[n]}(G) \neq \gamma_n(G)$.

There remained the problem of whether the restricted Lie Dimension subgroup conjecture holds for those n between 9 and 13 (inclusive). Gupta and Srivastava [6] in 1991 modified the examples to show that in fact it does not hold for these values and so the general restricted Lie dimension problem is solved.

1.4 Two-torsion

The counterexamples all involved two-torsion and the intriguing problem remained as to whether or not the Dimension subgroup conjecture holds for finite odd order groups. This however has been settled very nicely recently by N. D. Gupta [11] who showed that the dimension quotients $D_n(G)/\gamma_n(G)$ of a finite nilpotent group are 2-groups. From this follows the truth of the dimension conjecture for odd order groups.

In fact Gupta claims that the exponent of the dimension quotients divide 2 but this result has not yet appeared.

2 The Fox subgroups

Here we change our notation to work with ideals in the free group ring.

2.1 Ideals in the free group ring

Let R be a normal subgroup of the free group F. Let $\mathbb{Z}F$ be the group ring of F over the integers.

Define $\mathfrak{f} = \operatorname{Ker}(\mathbb{Z}F \to \mathbb{Z})$ and $\mathfrak{r} = \operatorname{Ker}(\mathbb{Z}F \to \mathbb{Z}(F/R))$.

These ideals were used extensively by Karl Gruenberg [2] in his *Cohomological methods in group theory* and are used to generate free resolutions of $G \cong F/R$.

Results of Gruenberg [2] (Chapter 3) on bases for these ideals and for factor ideals are particularly useful. For example the following is extensively used: "If $\mathfrak{a}$ is free (as a left ideal) on a set S then $\mathfrak{a}/\mathfrak{f}\mathfrak{a}$ is free abelian on the set $\{s + \mathfrak{f}\mathfrak{a} : s \in S\}$".

Also many of the calculations for the dimension subgroups in the last section use the free presentation of $G \cong F/R$, as the factor group of the free group F by the normal subgroup R of F, and the work is done within the free group rings of F and R.

2.2 Fox's Problem

Now define

$$F(n, R) = (1 + \mathfrak{f}^n\mathfrak{r}) \cap F$$

and call this the n^{th} *Fox subgroup of F relative to R.*

The identification of these subgroups is known as *Fox's Problem.*

The Fox problem was introduced by R.H. Fox [1] in 1953, in connection with his *free differential calculus.* The free differential calculus was devised to solve problems in knot theory but has found uses in many different areas. Indeed knot theory has recently proved useful in the disentanglement of DNA - see Sumners [24] for a discussion on this.

Solutions to Fox's problem have been given by Yunus [26], Hurley [13], and also in Gupta [5].

2.3 Statement of the Theorem

Let H be a subgroup of a group K with K/H nilpotent.

Define the *isolator*, $I_K(H)$, of H in K as follows.

$$I_K(H) = \{k \in K : \ k^m \in H \quad \text{for some non-zero integer } m\}.$$

Let $\gamma_m(G)$ denote the m^{th} term of the lower central series of a group G and write γ_m for $\gamma_m(F)$.

THEOREM

$$(1 + \mathfrak{f}^n\mathfrak{r}) \cap F = I_R(\Pi[R \cap \gamma_{i_1}, R \cap \gamma_{i_2}, \ldots, R \cap \gamma_{i_s}])$$

and the product is over all s-tuples $(i_1, i_2, \ldots, i_s)$, *with* $s \geq 2$, *and with* $(i_1 + i_2 + \cdots + i_s) - i_m \geq n$ *for all* m, $1 \leq m \leq s$.

The condition $(i_1 + i_2 + \cdots + i_s) - i_m \geq n$ for all m, $1 \leq m \leq s$, is of course equivalent to $(i_1 + i_2 + \cdots + i_s) - i_t \geq n$ for $i_t = \max\{i_1, i_2, \ldots, i_s\}$. This viewpoint is important when considering the generalisation given below.

3 The modular Fox subgroups

We now look at the modular situation where we replace $\mathbb{Z}$ by $\mathbb{Z}_p$, the integers modulo a prime p.

Let R be a normal subgroup of the free group F. Let $\mathbb{Z}_pF$ be the group ring of F over the integers mod p, where p is a prime.

Define $\mathfrak{f}_p = \operatorname{Ker}(\mathbb{Z}_p F \to \mathbb{Z}_p)$ and $\mathfrak{r}_p = \operatorname{Ker}(\mathbb{Z}_p F \to \mathbb{Z}_p(F/R))$.
Then define

$$F(n, R, p) = (1 + \mathfrak{f}_p^n \mathfrak{r}_p) \cap F$$

and call this the n^{th} *p-modular Fox subgroup of F relative to R.*

When $\mathbb{Z}_p$ is replaced by $\mathbb{Z}$, the determination of $(1+\mathfrak{f}^n\mathfrak{r}) \cap F$ is the usual *Fox's problem*, which we now refer to as the *the integral Fox problem.*

For a solution to the determination of $F(n, R, p)$ see Hurley, Sehgal [15].

3.1 The solution

A p'-number is an integer relatively prime to p. Let H be a subgroup of a group K with K/H nilpotent. Define

$$I_{K,p'}(H) = \{k \in K : \ k^m \in H \quad \text{for some } p'\text{-number } m\}.$$

THEOREM:

$$(1 + \mathfrak{f}_p^n \mathfrak{r}_p) \cap F = I_{R,p'}(\Pi[R \cap \gamma_{i_1}, R \cap \gamma_{i_2}, \ldots, R \cap \gamma_{i_s}]^{p^{j_s}} \Pi (R \cap \gamma_i)^{p^{k_i}})$$

with $s \geq 2,\ j_s \geq 0,\ k_i \geq 1$; *where the first product is over all* *s-tuples* $(i_1, i_2, \ldots, i_s)$,
$s \geq 2$, *with* $p^{j_s}(i_1 + i_2 + \cdots + i_s) - i_m \geq n$ *for all* m, $1 \leq m \leq s$, *and the second product is over all* i *and* k_i *with* $(p^{k_i} - 1)\, i \geq n$.

If we let $p = 1$ in the statement of the Theorem (and then of course there is no second product as $(1-1)i$ is never greater than or equal to n) and a p'-number is interpreted as any non-zero integer, then we get a statement of the solution to the integral Fox problem. Using this interpretation, this result includes a solution of the integral Fox problem.

The cases $n = 1$:

$$(1 + \mathfrak{f}_p \mathfrak{r}_p) \cap F = R' R^p$$

for the modular case and

$$(1 + \mathfrak{f}\mathfrak{r}) \cap F = R'$$

for the integral case were well-known. The integral one is related to Magnus' representation of F/R' in a group of 2×2 matrices.

Define

$$R_n(p) = \Pi[R \cap \gamma_{i_1}, \ldots, R \cap \gamma_{i_s}]^{p^{j_s}} \Pi (R \cap \gamma_i)^{p^{k_i}}$$

with $s \geq 2,\ j_s \geq 0,\ k_i \geq 1$ and where the first product is over all s-tuples $(i_1, i_2, \ldots, i_s)$ with $p^{j_s}(i_1 + i_2 + \cdots + i_s) - i_m \geq n$ for all m, $1 \leq m \leq s$, and the second product is over all i and k_i for which $(p^{k_i} - 1)\, i \geq n$.

The theorem then is that

$$(1 + \mathfrak{f}_p^n \mathfrak{r}_p) \cap F = I_{R,p'}\left(R_n(p)\right).$$

4 Back to the integral case

We now look at is the identification of

$$F(n,R,m)=(1+\mathfrak{f}^n\mathfrak{r}^m)\cap F$$

It was shown by Gruenberg [1] that

$$F(1,R,m)=(1+\mathfrak{f}\mathfrak{r}^m)\cap F=\gamma_{m+1}(R)$$

For the case $n=2$, Vermani and Razdan, [24] 1996 identified $F(2,R,m)$ for $m\leq 5$.

The case $n=2$ and for all m is due to (Hurley, Sehgal [16]) and is as follows:

$$F(2,R,m)=(1+\mathfrak{f}^2\mathfrak{r}^m)\cap F=\gamma_{m+2}(R)*\gamma_{m+1}(R\cap F')$$

Notice that in this situation there is no "isolator" part. It is deduced then that

$$\gamma_{m+1}(R)/\big(\gamma_{m+2}(R)*\gamma_{m+1}(R\cap F')\big)$$

is a free abelian group with an explicitly defined basis.

The general problem turned out to have a few twists and turns. The 'Jacobi' identity is a problem and cannot be applied as previously due to problems with bracketing.

4.1 The solution for general n and m:

Recall that γ_i means $\gamma_i(F)$. Let $C(R)$ denote the set of commutator subgroups derived using commutator bracketing from $R\cap\gamma_i$ for any $i\geq 1$. Each element of $C(R)$ has an F-weight, an R-weight and also what we call an R^m-eliminated weight. This is a technical definition given below in section 4.3 after the statement of the Theorem but it is essentially the F-weight remaining after eliminating m of the $R\cap\gamma_i$ in a *certain manner* depending on the bracketing.

For example:

$$W=[R\cap\gamma_4,R\cap\gamma_3,R\cap\gamma_3,[R\cap\gamma_4,R\cap\gamma_3]]$$

has R^2-eliminated weight 10 but

$$[R\cap\gamma_4,R\cap\gamma_3,R\cap\gamma_3,R\cap\gamma_4,R\cap\gamma_3]$$

and

$$[R\cap\gamma_4,R\cap\gamma_3,R\cap\gamma_3,R\cap\gamma_3,R\cap\gamma_4]$$

have R^2-eliminated weight 9.

Bracketing matters.

Then W will be in $(1+\mathfrak{f}^{10}\mathfrak{r}^2)\cap F$ but the other two will only be in $(1+\mathfrak{f}^9\mathfrak{r}^2)\cap F$.

Abbreviate "eliminated" to "el" and "weight" to "wt", and write R-el-wt(W) for the R^m-eliminated weight of an element $W \in C(R)$ when it is defined.

The identification then is as follows:

THEOREM:

$$(1 + \mathfrak{f}^n\mathfrak{r}^m) \cap F = I_R(\Pi\, W_j)$$

where the product is over all commutator subgroups $W_j \in C(R)$ *with* R-*wt*$(W_j) \geq (m+1)$ *and for which* R^m-*el-wt*$(W_j) \geq n$.

Notes:

1. For $m = 1$ it may be checked that each W_j can be taken to have the form $[R_{i_1}, R_{i_2}, \ldots, R_{i_s}]$ with $s \geq 2$ and $(i_1 + i_2 + \ldots + i_s) - i_k \geq n$ where i_k is the maximum of $\{i_1, i_2, \ldots, i_s\}$; as noted previously this is precisely the result for the original Fox problem stated in terms of the R^1-el-wt.

2. Both $\gamma_{m+1}(R \cap \gamma_n)$ and $\gamma_{n+m}(R)$ are in the product; these are the "extremities" so that if any W_j is in the product with R-wt$(W_j) \geq n + m$ then $W_j \subseteq \gamma_{n+m}(R)$ (and may be absorbed into $\gamma_{n+m}R$); if R-wt$(W_j) = m + 1$ (exactly) then $W_j \subseteq \gamma_{m+1}(R \cap \gamma_n)$.

3. For the case $n = 2$ it is known - see [16] - that precisely these extremities come up and that the "isolator part" is omitted.

4.2 When can the "isolator part" be omitted?

It would be interesting to know for which groups $G = F/R$ the "isolator part" of the formula can be omitted, i.e. for which groups is

$$(1 + \mathfrak{f}^n\mathfrak{r}^m) \cap F = \Pi\, W_j$$

where W_j is as in the statement of the Theorem.

C.K. Gupta and N.D. Gupta [3] identifies $F(n, F', 1)$, i.e. the case $m = 1$ and $R = F'$, the commutator subgroup of F, and there is no "isolator" part in the formula.

What is likely is that the isolator part may be omitted when G has torsion-free lower central factors. This may be verified for the free polynilpotent groups and gives rise to some interesting varieties.

See for example Hanna Neumann [18] for background to ***varieties, fully invariant subgroups, free polynilpotent groups*** and ***residual properties.***

Note that if R is fully invariant in F then so is $(1 + \mathfrak{f}^n\mathfrak{r}^m) \cap F$ and hence $F/(1 + \mathfrak{f}^n\mathfrak{r}^m) \cap F$ is a free group in a variety. Call such a variety a ***Fox variety.*** Substituting a free polynilpotent group for R will thus give rise to a Fox variety and some interesting ones arise from the Theorem.

Define $R = \gamma_1\gamma_2 \ldots \gamma_{n+1}$ inductively to be $\gamma_1(\gamma_2 \ldots \gamma_{n+1})$ and this is the term of the ***polycentral series of*** F relative to the sequence $\{\gamma_1, \gamma_2, \ldots, \gamma_{n+1}\}$ – recall that γ_n means $\gamma_n(F)$. Then in this case, $F/F(n, R, m)$ is a Fox variety and so is residually(torsion-free nilpotent) and residually(a finite p-group) for all primes p, since by [14] it inherits these residual properties of F/R.

For example if we take $R = \gamma_3(F)$ and $m = 2$ then the theorem gives that
$F(n, R, 2) = \gamma_3\gamma_3$ for $n \leq 3$,
$F(4, R, 2) = \gamma_3\gamma_4 * \gamma_4\gamma_3$,
$F(5, R, 2) = \gamma_3\gamma_5 * \gamma_4\gamma_3$,
$F(6, R, 2) = \gamma_3\gamma_6 * \gamma_4\gamma_3$.

This would imply for example that $F/(\gamma_3\gamma_4 * \gamma_4\gamma_3)$, $F/(\gamma_3\gamma_5 * \gamma_4\gamma_3)$ and $F/(\gamma_3\gamma_6 * \gamma_4\gamma_3)$, are all residually(torsion-free-nilpotent) and residually(a finite p-group) for all primes p. These are the free groups in the varieties "(nilpotent-of-class-2 by nilpotent-of-class-t) $\cap$ (nilpotent-of-class-3 by nilpotent-of-class-2)" for $t = 3, 4, 5$.

The cases $n \geq 7$ become more complicated so for example $F(7, \gamma_3, 2) = \gamma_3\gamma_7 * [[\gamma_4, \gamma_3], [\gamma_4, \gamma_3]] * \gamma_5\gamma_3$.

More generally let $R = \gamma_t$, a general term of the central series, and then from the theorem it follows that $F(t + i, \gamma_t, m) = \gamma_{m+1}\gamma_{t+i} * \gamma_{m+2}\gamma_t$ for $1 \leq i \leq t$ and hence for these t, $F/(\gamma_{m+1}\gamma_{t+i} * \gamma_{m+2}\gamma_t)$, the free group in the variety "(nilpotent-of-class-m by nilpotent-of-class-$(t+i-1)$) $\cap$ (nilpotent-of-class-$(m+1)$ by nilpotent-of-class-$(t-1)$)", is residually(torsion-free-nilpotent) and residually(a finite p-group) for all primes p.

Other interesting Fox varieties can be obtained by substituting some more terms of the polycentral series for R and applying the theorem.

4.3 Technical definition:

Let $C(R)$ denote the set of commutator subgroups derived using commutator bracketing from $R \cap \gamma_i, i \geq 1$. For the purpose of assigning weights to the elements of $C(R)$ we consider $R \cap \gamma_i = R_i$ as simply an *expression*.

Define the F-weight of $R \cap \gamma_i$ to be i and its R-weight to be 1.

Suppose P, Q are commutator subgroups in $C(R)$ and that F-wt(P), F-wt(Q), R-wt(P), and R-wt(Q) have been defined. Then define:

F-wt$([P, Q]) = F$-wt $(P) + F$-wt(Q) and R-wt$([P, Q]) = R$-wt$(P) + R$-wt(Q).

With $W \in C(R)$ and when R-wt$(W) \geq m$, define the R^m-el-wt of W, written R^m-el-wt(W), as follows. (This weight should be considered as a weight in F rather than a weight in R and is not related to the R-weight.)

R^1-el-wt$(R \cap \gamma_i)$ is defined to be 0.

If now $P, Q \in C(R)$ then define R^1-el-wt$([P, Q])$ to be
min $\{F\text{-wt}(P) + R^1\text{-el-wt}(Q), F\text{-wt}(Q) + R^1\text{-el-wt}(P)\}$.

For example the R^1-el-wt of $[R_{i_1}, R_{i_2}, \ldots, R_{i_t}]$ is $\sum_{j=i}^{t} i_j - \max\{i_1, i_2, \ldots, i_t\}$. Compare this with the solution to Fox's problem given above.

Suppose now $P, Q \in C(R)$, R-wt$([P, Q]) \geq u$ and that the R^t-el-wt(P) and R^t-el-wt(Q) have been defined for $t < u$.

First we define the R^u-el-wt$([P, Q])$ in the case when R-wt$(P) \geq R$-wt(Q) and then define R^s-el-wt$([P, Q]) = R^s$-el-wt$([Q, P])$ when R-wt$(P) < R$-wt(Q)

Suppose then R-wt$(P) = t$ and R-wt$(Q) = s$ and $t \geq s$.

(i) If $s \geq u$, then R^u-el-wt($[P, Q]$) is defined to be the $\min\{F\text{-wt}(P) + R^u\text{-el-wt}(Q), F\text{-wt}(Q) + R^u\text{-el-wt}(P)\}$

(ii) If $t \geq u > s$, then R^u-el-wt($[P, Q]$) is defined to be $\min\{R^{u-s}\text{-el-wt}(P), F\text{-wt}(Q) + R^s\text{-el-wt}(P)\}$

(iii) If $u > t \geq s$, then R^s-el-wt($[P, Q]$) is defined to be $\min\{R^{u-s}\text{-el-wt}(P), R^{u-t}\text{-el-wt}(Q)\}$

Note that for (iii), $s + t \geq u$ so that $0 < u - s \leq t$ and $0 < u - t \leq s$.

End of technical definition.

4.4 Problems

(i) Identify in general

$$(1 + \mathfrak{f}^n \mathfrak{r}^m \mathfrak{f}^t) \cap F$$

(ii) Generalise the modular case.

Problem (i) is likely to be very difficult. When $n = 1, t = 1$ and for general m, Stöhr [22] proves that $(1 + \mathfrak{f}\mathfrak{r}^m \mathfrak{f}) \cap F = I_R([\gamma_{m+1}(R), F])$. Here though the 'isolator part' may not be removed even for $m = 1$ and $R = F'$ (for this see C.K. Gupta [4]).

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