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Partial actions, crossed products and partial representations [1]

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Abstract: We give a survey of algebraic results on partial representations of groups, partial actions and related concepts.
Key words: Partial actions, crossed products, partial representations .

1 Introduction

Partial actions of groups, partial representations and the notions related to them appeared in the theory of operator algebras as usefull tools of their study (see [8], [9], [10], [17], [19]). In this survey we give an account of the algebraic results on this topic. We start Section 2 by showing how partial actions of groups on abstract sets naturally appear as restrictions of usual actions, giving at the same time a motivation for the concept of partial representation.

Then we proceed by discussing the structure of partial group rings of finite groups, considering also the corresponding isomorphism problem. Section 3 deals with the structure of partial representations of arbitary groups. In Section 4 we investigate the question when a given partial action of a group on an algebra can be viewed as a restriction of a usual action and study the associativity question of crossed products defined by partial actions. In the final Section 5 we try to explore the interaction between these concepts.

By a unital ring we shall understand an associative ring with unity element. Given a unital commutative ring R, by an R-algebra we mean an associative R-algebra.

2 Partial representations and partial group rings

For an abstract set $\mathcal{X}$ consider the set $\mathcal{I}(\mathcal{X})$ of partially defined bijections of $\mathcal{X}$, that is bijections $\varphi : A \to B$, where the domain $dom(\varphi) = A$ and the range $ran(\varphi) = B$ are subsets of $\mathcal{X}$. Given two elements $\varphi, \psi \in \mathcal{I}(\mathcal{X})$ one can define their product $\varphi\psi$ by taking as the domain the largest subset of $\mathcal{X}$ for which the composition $\varphi \circ \psi$ makes sense. More precisely, we set $dom(\varphi\psi) = \varphi^{-1}(ran(\varphi) \cap dom(\psi)), ran(\varphi\psi) = \psi(ran(\varphi) \cap dom(\psi))$ and $\varphi\psi(x) = \varphi(\psi(x))$ for $x \in dom(\varphi\psi)$. Then $\mathcal{I}(\mathcal{X})$ becomes an inverse semigroup, i.e. a semigroup in which for every element φ there exists a unique φ^* such that $\varphi\varphi^*\varphi = \varphi$ and $\varphi^*\varphi\varphi^* = \varphi^*$. In fact, for $\varphi : A \to B$ the element φ^* is simply the inverse bijection $\varphi^{-1} : B \to A$ considered as an element of $\mathcal{I}(\mathcal{X})$. Clearly the symmetric group $S(\mathcal{X})$ of the bijections $\mathcal{X} \to \mathcal{X}$ is contained in $\mathcal{I}(\mathcal{X})$.

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Given an action $\beta : G \ni g \mapsto \beta_g \in S(\mathcal{Y})$ of a group G on a set $\mathcal{Y}$, usually we look at invariant subsets $\mathcal{X} \subseteq \mathcal{Y}$ when restricting β to $\mathcal{X}$. However, if $\mathcal{X} \subseteq \mathcal{Y}$ is not invariant we may take for a $g \in G$ the maximal subset $\mathcal{D}_{g^{-1}}$ of $\mathcal{X}$ whose image $\mathcal{D}_g$ with respect to β_g is still in $\mathcal{X}$. More precisely, set $D_g = \mathcal{X} \cap \beta_g(\mathcal{X})$ for each $g \in G$. Then $\mathcal{D}_{g^{-1}} = \{x \in \mathcal{X} : \beta_g(x) \in \mathcal{X}\} = \mathcal{X} \cap {\beta_g}^{-1}(\mathcal{X})$ and $\beta_g(\mathcal{D}_{g^{-1}}) = \mathcal{D}_g$.This gives the partially defined bijections $\alpha_g : D_{g^{-1}} \ni x \mapsto \beta_g(x) \in D_g$ $(g \in G)$ and we can speak about G acting on $\mathcal{X}$ by partially defined bijections or a *partial action* $\alpha = \beta|_{\mathcal{X}}$ of G on $\mathcal{X}$. The largest subset to which the composition $\alpha_h \circ \alpha_g$ can be applied is $\alpha_g^{-1}(\mathcal{D}_g \cap \mathcal{D}_{h^{-1}})$ and it is obviously contained in the domain $\mathcal{D}_{(hg)^{-1}}$ of α_{hg}, so that $\alpha_h \circ \alpha_g(x) = \alpha_{hg}(x)$ for each $x \in \alpha_g^{-1}(\mathcal{D}_g \cap \mathcal{D}_{h^{-1}})$ meaning that the function α_{gh} is an extension of the function $\alpha_g \circ \alpha_h$. This leads to the following definition.

Definition 2.1. *Let G be a group with identity element* 1 *and let $\mathcal{X}$ be a set. A partial action α of G on $\mathcal{X}$ is a collection of subsets $\mathcal{D}_g \subseteq \mathcal{X}$ $(g \in G)$ and bijections $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ such that*

(i) $\mathcal{D}_1 = \mathcal{X}$ *and* α_1 *is the identity map of* $\mathcal{X}$;
(ii) $\mathcal{D}_{(gh)^{-1}} \supseteq \alpha_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}})$;
(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ *for each* $x \in \alpha_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}})$.

Thus a partial action of G on $\mathcal{X}$ is some map $\alpha : G \to \mathcal{I}(\mathcal{X})$. The following statement explains which maps $G \to \mathcal{I}(\mathcal{X})$ are in fact partial actions.

Proposition 2.2. *[10]. Let G be a group and $\mathcal{X}$ a set. A map $\alpha : G \ni g \mapsto \alpha(g) = \alpha_g \in \mathcal{I}(\mathcal{X})$ gives a partial action of G on $\mathcal{X}$ if and only if, for all $g, h \in G$, we have*
(i) $\alpha_1 = id_{\mathcal{X}}$,
(ii) $\alpha_g \alpha_h \alpha_{h^{-1}} = \alpha_{gh} \alpha_{h^{-1}}$.
In this case α also satisfies
(iii) $\alpha_{g^{-1}} \alpha_g \alpha_h = \alpha_{g^{-1}} \alpha_{gh}$.

In other words, if $\alpha : G \to \mathcal{X}$ is a partial action then the equality $\alpha(g)\alpha(h) = \alpha(gh)$ holds when the two sides are multiplied either by $\alpha(g^{-1})$ on the left or by $\alpha(h^{-1})$ on the right. This phenomenon is "linearized" in the following definition.

Definition 2.3. *A partial representation of a group G into a unital K-algebra $\mathcal{B}$ is a map*

$$\pi : G \to \mathcal{B},$$

which sends the unit element of the group to the unity element of $\mathcal{B}$, such that for all $g, h \in G$ we have

$$\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}) \quad and \quad \pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh).$$

In particular, if $\mathcal{B}$ is the algebra of the linear transformations $\mathrm{End}(V)$ of a vector space V over a field K then we have a partial representation of G on the vector space V.

Partial representations were introduced independently by R. Exel [10], and J. C. Quigg and I. Raeburn [19], and form an effective tool in the theory of operator algebras. Their algebraic study began in [3] where the structure of partial group algebras was studied as well as their isomorphism problem. Partial group algebras are responsible for partial representations in a similar fashion to group algebras and group representations. Further investigation of partial representations and of the isomorphism problem for partial group rings was done in [7] and [5] respectively.

We see that every representation of G is a partial representation; moreover, if H is any subgroup of G and $\pi : H \longmapsto \mathrm{End}(V)$ is a partial representation of H, then the map $\tilde{\pi} : G \longmapsto \mathrm{End}(V)$ given by:

$$\tilde{\pi}(g) = \begin{cases} \pi(g), & \text{if } g \in H; \\ 0, & \text{otherwise} \end{cases}$$

defines a partial representation of G. In particular, partial representations of G can be obtained from (usual) representations of subgroups H of G in this obvious way. It is less obvious that partial representations on finite dimensional linear spaces can be constructed by tensoring (usual) representations of subgroups with certain "purely partial" representations. Precise information will be given in Section 3. We now consider some examples of "purely partial" representations.

Example 2.4. $G = C_4 = \langle c;\ c^4 = 1 \rangle$

$\varphi_1 : 1 \mapsto 1, \quad c \mapsto 0, \quad c^2 \mapsto 0, \quad c^3 \mapsto 0;$

$\varphi_2 : 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ c \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c^2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ c^3 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$

$\varphi_3 : 1 \mapsto 1, \quad c \mapsto 0, \quad c^2 \mapsto \ 1, \quad c^3 \mapsto 0;$

$$\varphi_4 :\ 1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ c \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ c^2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$c^3 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The following partial representation of G is not "pure" as it envolves a non-trivial (ususal) representation of the subgroup $H = \langle c^2 \rangle : \varphi_5 : 1 \mapsto 1, \quad c \mapsto$

$0, \quad c^2 \mapsto -1, \quad c^3 \mapsto 0.$

As for group algebras, the partial group algebras $K_{\mathrm{par}}G$ can be defined by the universal property:

$$\begin{tikzcd} & KG \arrow[dr, "\tilde{\phi}"] \arrow[d, phantom, "\circlearrowright"] & \\ G \arrow[ur, "i"] \arrow[rr, "\phi"] & {} & \mathcal{A} \end{tikzcd}$$

$$\begin{tikzcd} & K_{\mathrm{par}}G \arrow[dr, "\tilde{\varphi}"] \arrow[d, phantom, "\circlearrowright"] & \\ G \arrow[ur, "i'"] \arrow[rr, "\varphi"] & {} & \mathcal{A}, \end{tikzcd}$$

which says that there exists a K-algebra $K_{\mathrm{par}}G$ and a partial representation $i' : G \to K_{\mathrm{par}}G$ such that for an arbitrary partial representation φ of G into a K-algebra $\mathcal{A}$ there is a unique homomorphism of algebras $\tilde{\varphi} : K_{\mathrm{par}}G \to \mathcal{A}$ such that $\tilde{\varphi} \circ i' = \varphi$. In order to be constructive we give the following.

Definition 2.5. *The partial group algebra $K_{\mathrm{par}}G$ of a group G over a field K is the semigroup algebra $KS(G)$, where $S(G)$ is the semigroup generated by the symbols $\{[g] : g \in G\}$ subject to relations:*
a)$[g^{-1}][g][h] = [g^{-1}][gh]$;
b)$[g][h][h^{-1}] = [gh][h^{-1}] \quad (g, h \in G)$;
c)$[1] = 1$,
where 1 *also denotes the identity element of* $S(G)$.

Taking in the above diagram $i' : G \ni g \mapsto [g] \in K_{\mathrm{par}}G$, we easily see that the semigroup algebra $KS(G)$ satisfies the universal property.

It turns out that in the case of finite groups G the partial group algebras $K_{\text{par}}G$ are direct products of matrix algebras over group algebras of subgroups of G. In order to see this, one identifies $K_{\text{par}}G$ with the groupoid algebra $K\Gamma(G)$, where $\Gamma(G)$ is a groupoid associated to G. By a groupoid we mean a small category in which every morphism is an isomorphism. In particular, a group is a groupoid with a single object. The groupoid $\Gamma(G)$ can be defined for an arbitrary group G. Its objects are the subsets $A \ni 1$ of G and the morphisms are left multiples of the A's by elements g with $g^{-1} \in A$:

$$g^{-1} \in A \quad \xrightarrow{\quad g \quad} \quad B = gA \ni g.$$

Denote this morphism by (A, g). Its inverse morphism is (gA, g^{-1}) :

$$g^{-1} \in A \quad \xleftarrow{\quad g^{-1} \quad} \quad B = gA \ni g.$$

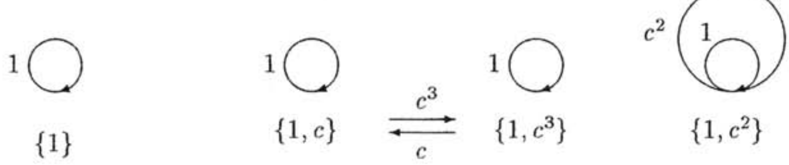
Given two morphisms:

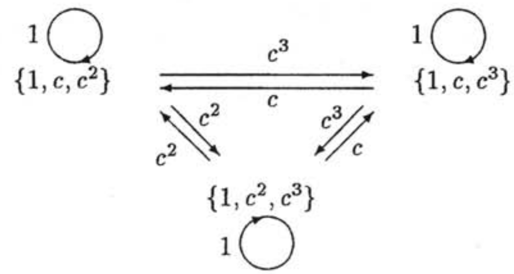
$$\gamma_1 : A \xrightarrow{\quad g \quad} B,\ \gamma_2 : C \xrightarrow{\quad h \quad} D,$$

the product $\gamma_2 \cdot \gamma_1$ exists if and only if $B = C$, and in this case it is:

$$\gamma_2 \cdot \gamma_1 : A \xrightarrow{\quad hg \quad} D.$$

It is convenient to represent $\Gamma(G)$ as a graph whose vertices are objects and whose arrows are morphisms. Then $\Gamma(G)$ is represented as a disjoint union of connected subgraphs, each of which corresponds to a connected subgroupoid of $\Gamma(G)$. By a connected groupoid we mean a groupoid in which all objects are isomorphic. Obviously G is a connected component of $\Gamma(G)$. For example, the following graph gives the connected components $\neq G$ of the groupoid associated to the cyclic group of order $4:\ G = \langle c : c^4 = 1 \rangle$.





The concept of groupoid algebra is a particular case of a more general concept of category algebra, which can be defined as follows. Let Γ be a small category and K be a field. We identify Γ with its set of morphisms. Then the *category algebra* $K\Gamma$ is a K-vector space whose basis is Γ and with the multiplication given by

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1\gamma_2, & \text{if the composite morphism } \gamma_1\gamma_2 \\ & \text{exists in } \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for the cyclic group of order 4, the groupoid algebra of the first connected component is obviously isomorphic to K. The first, second and fourth components are rigid groupoids, i. e. every object has trivial automorphism group. It is easily seen that in a connected rigid groupoid, for a fixed pair of objects there is only one isomorphism in each direction. Denote by $e_{i,j}(1)$ the elementary matrix, whose unique non-zero entry 1 is placed at the intersection of the i'th row and j'th column. Given a rigid connected groupoid Δ with a finite number n of objects, number its objects arbitrarily and assign the elementary matrix $e_{ji}(1)$ to the unique morphism which goes from the i'th object to the j'th. This map gives an isomorphism between the groupoid algebra $K\Delta$ and the full $n \times n$-matrix algebra $M_n(K)$ over K. In particular, the groupoid algebras of the second and fourth connected components in the above example are isomorphic to $M_2(K)$ and $M_3(K)$, respectively. As for the third component, its groupoid algebra is isomorphic to the group algebra of $\langle c^2 \rangle$. More generally, one can verify that in a connected groupoid Δ the automorphism groups of the objects are all isomorphic. Given a connected groupoid Δ with a finite number n of vertices, let H be the automorphism group of an object. Then there is an isomorphism $\phi_\Delta : K\Delta \cong M_n(KH)$ (see [3, Prop. 3.1]). Thus it is crucial to identify the partial group algebra with $K\Gamma(G)$.

Observe that the notions of partial K-representation, partial group K-algebra and groupoid algebra can be obviously extended to the case of an arbitrary commutative ring K.

Theorem 2.6. *[3, Theorem 2.6]. If G is a finite group and K is a commutative ring then there is a partial representation $\lambda : G \to K\Gamma(G)$ such that $K\Gamma(G)$ with λ satisfy the universal property in the definition of partial group algebra.*

As a consequence we get $K_{\mathrm{par}}G \cong K\Gamma(G)$. Now it becomes clear that if G is finite, then $K_{\mathrm{par}}G$ is a direct sum of matrix algebras over group rings of subgroups. However, we are also interested in knowing how many times a given $M_m(KH)$ shows up as a direct summand of $K_{\mathrm{par}}G$.

Theorem 2.7. *Let G be a finite group, K a commutative ring and let $\mathcal{C}$ denote a full set of representatives of the conjugacy classes of subgroups of G. Then the partial group ring of G over K is of the form*

$$K_{par}G \cong \bigoplus_{\substack{H\in\mathcal{C}\\ 1\le m\le (G:H)}} c_m(H)\, M_m(KH),$$

where $c_m(H)\, M_m(KH)$ means the direct sum of $c_m(H)$ copies of $M_m(KH)$ and the coefficients $c_m(H)$ are given by the recursive formula

$$c_m(H) = \frac{1}{m}(G:N_G(H))\left(\binom{(G:H)-1}{m-1} - \sum_{\substack{H<B\le G\\ (B:H)\mid m}} \frac{m/(B:H)c_{m/(B:H)}(B)}{(G:N_G(B))}\right).$$

The above result was obtained in [3, Theorem 3.2] with an error in the recursive formula which was corrected in [5]. It is easy to observe that the error does not affect the other results of [3] (see [5]).

If $|G| = n$, the dimension of $K\Gamma(G)$ is easily computed:

$$\dim(K\Gamma(G)) = \sum_{k=0}^{n-1}(k+1)\binom{n-1}{k} = 2^{n-2}(n+1).$$

Note that the right hand side of (2) is a strictly increasing function of n. In particular, if G_1 and G_2 are finite groups such that $K\Gamma(G_1)$ is isomorphic to $K\Gamma(G_2)$, then $|G_1| = |G_2|$.

In [10] R. Exel observed that the complex partial group algebras of the two groups of order 4 are not isomorphic. This was quite surprising, since the usual complex group algebra of a finite abelian group G "remembers" only the order of G. However, Theorem 5.11 suggests that some information is hidden in the multiplicities $c_m(H)$. In fact, for a finite abelian G, looking carefully at the multiplicities of those $M_{|G|k^{-1}-1}(K)$ such that k does not factor out "too much" from $|G|$, one obtains information about the number of subgroups in G of some orders which permits to prove the next result.

Theorem 2.8. *[3, Theorem 4.4, Corollary 4.5]. Let G and H be two finite abelian groups and K be an integral domain whose characteristic does not divide $|G|$ such that the partial group algebras $K_{\mathrm{par}}G$ and $K_{\mathrm{par}}H$ are isomorphic. Then, G and H are isomorphic groups.*

The above theorem does *not* hold for noncommutative groups. To see this, consider first the following direct concequence of Theorem 2.7.

Corollary 2.9. *Let G_1 and G_2 be two finite groups. Assume that there exists an isomorphism of lattices between the lattices of subgroups of G_1 and G_2 that preserves conjugacy and such that corresponding subgroups have isomorphic group rings over a commutative ring R. Then $R_{par}G_1 \cong R_{par}G_2$.*

It turns out that the old counter-example by A. Rottländer [21] of two non-isomorphic groups whith a conjugacy and order preserving isomorphism of lattices of subgroups gives also a counter-example for our isomorphism problem.

Counter-example: Let G_1 and G_2 be the groups:

$$\begin{aligned} G_1 = \langle a, b, c \mid a^{11} = b^{11} = c^5 = 1,\ ab = ba, \\ c^{-1}ac = a^3,\ c^{-1}bc = b^9\rangle, \end{aligned}$$

$$\begin{aligned} G_2 = \langle a, b, c \mid a^{11} = b^{11} = c^5 = 1,\ ab = ba, \\ c^{-1}ac = a^3,\ c^{-1}bc = b^4\rangle. \end{aligned}$$

Then, G_1 and G_2 are non-isomorphic groups of order 605 with isomorphic partial group algebras over any algebraically closed field K of characteristic zero (see [3]).

In the modular case we have the following.

Theorem 2.10. *[5] Let K be an integral domain of characteristic $p > 0$ and let G_1, G_2 be two finite groups such that $K_{\mathrm{par}}G_1 \cong K_{\mathrm{par}}G_2$. Let S_i denote a Sylow p-subgroup of G_i, $i = 1, 2$. Then $KS_1 \cong KS_2$.*

This is used to obtain the next result.

Theorem 2.11. *[5] Let K be an integral domain of characteristic $p > 0$ and let G_1, G_2 be two finite abelian groups such that $K_{\mathrm{par}}G_1 \cong K_{\mathrm{par}}G_2$. Then $G_1 \cong G_2$.*

As for the integral case, we have.

Theorem 2.12. *[5] Let G_1 and G_2 be finite groups such that $\mathbb{Z}_{par}G_1 \cong \mathbb{Z}_{par}G_2$. Then, for every subgroup H of G_1 there exists a subgroup N of G_2 such that $\mathbb{Z}H \cong \mathbb{Z}N$. In particular, $\mathbb{Z}G_1 \cong \mathbb{Z}G_2$.*

The isomorphism problem for (usual) group rings has an exciting history (see [16], [18], [20], [22],[23], [24]). In the context of partial group rings we offer the following general question.

Problem 1 *Given an integral domain K what can we say about the groups G and H if $K_{\mathrm{par}}G \cong K_{\mathrm{par}}H$? In particular, under what circumstances G and H are isomorphic?*

It is of special interest to look at the integral partial group ring case and, in particular, to test Hertweck's recent counter-example [16]. If K is a field it seems to be unclear even how to read the commutativity of G from $K_{\mathrm{par}}G$.

We see that the structure of the lattice of subgroups of G has an important role in the structure of $K_{\mathrm{par}}G$, so that Problem 1 tries to relate the classical group ring isomorphism problem with the old question of investigating groups with isomorphic lattices of subgroups. Corollary ?? suggests a way to put these two questions together more explicitly as follows. Denote by $\mathcal{L}(G)$ the lattice of the subgroups of a group G.

Problem 2 Let G_1 and G_2 be finite groups and K be an integral domain. What can we say about G_1 and G_2 if there exists a conjugacy-preserving isomorphism of lattices $\mathcal{L}(G_1) \cong \mathcal{L}(G_2)$ such that corresponding subgroups have isomorphic group rings over K? In particular, is it true that $G_1 \cong G_2$ in the case $K = \mathbb{Z}$?

It follows from the above given counter-example that G_1 and G_2 may be non-isomorphic if K is a large field.

3 The structure of partial representations

As we have mentioned already the notion of partial representation appeared in the theory of operator algebras. They introduction was motivated by the desire to study algeras generated by partial isometries on a Hilbert space. Among the C^*-algebras successfully studied using partial representations are the so-called Cuntz-Krieger algebras (see [11] and [13]) introduced in [2]. The partial representations considered in operator algebra theory are partial representations of groups by bounded operators on a Hilbert space and their definition involves adjoint operators. Thus in abstract algebraic context the definition required an adjustment; however, up to equivalence the two definitions coincide as was shown in [3, Proposition 2.3].

The proofs of theorems 2.6 and 2.7 give a recipe for obtaining the irreducible partial representations of a finite group on a finite dimensional vector space from the usual ones and permit us to draw conclusions about their structure. However, the algebras $K_{\mathrm{par}}G$ and $K\Gamma(G)$ are not isomorphic if G is infinite, simply because $K_{\mathrm{par}}G$ is a unital algebra and $K\Gamma(G)$ is not. Nevertheless, it turns out that it is possible to get general information about the structure of partial representations even for infinite G. This is done in [7] (see also[6]). The main ingredient of Theorem 2.6 is the partial representation $\lambda : G \to K\Gamma(G)$. It is defined by the formula $\lambda(g) = \sum_{A \ni 1, g^{-1}} (A, g)$. If G is infinite, this sum becomes infinite. However, if we restrict ourself to partial representations on finite dimensional vector spaces, it is enough to look only at those connected components Δ of $\Gamma(G)$ which have a finite number of objects, so fixing such Δ, we let A run only over the objects of Δ (see Theorem 3.1 below).

We proceed with some obvious definitions. Given a partial K-representation $\pi : G \to \mathrm{End}(V)$, V can be considered as a *partial G-space*, that is, a vector space over K with a product $G \times V \to V$ satisfying for all $g, t \in G$ and $x \in V$ the conditions:

(i) $ex = x$;

(ii) $g^{-1}(g(tx)) = g^{-1}((gt)x)$;

(iii) $g(t(t^{-1}x)) = (gt)(t^{-1}x)$.

If V is finite dimensional then taking a basis in V we obtain the corresponding partial matrix representation $\pi : G \to M_n(K)$, where $n = \dim_K(V)$. Two partial matrix representations $\pi_1 : G \to M_n(K)$ and $\pi_2 : G \to M_n(K)$ are *equivalent* if there exists an invertible matrix $C \in GL_n(K)$ such that $C^{-1}\pi_1(g)C = \pi_2(g)$ for all $g \in G$. It follows that two partial representations $\pi_i : G \to \mathrm{End}(V_i)$, $i \in \{1, 2\}$, are *equivalent* if there exists a K-vector space isomorphism $\varphi : V_1 \to V_2$ such that $\varphi\pi_1(g) = \pi_2(g)\varphi$ for all $g \in G$.

We say that $\pi : G \to \mathrm{End}(V)$ is *reducible*, if V contains a proper invariant partial G-subspace $V' \subsetneq V$. Otherwise, V is called *irreducible*. The spaces V' and V/V' give rise to partial representations $\pi' : G \to \mathrm{End}(V')$ and $\pi'' : G \to \mathrm{End}(V/V')$. In matrix language, $\pi : G \to M_n(K)$ is *reducible* if there exist an invertible matrix $C \in M_n(K)$ and partial matrix representations π', π'' such that

$$C^{-1}\pi(g)C = \begin{pmatrix} \pi'(g) & * \\ 0 & \pi''(g) \end{pmatrix}$$

for all $g \in G$. If in the above formula we can choose $C \in M_n(K)$ such that the star is the zero matrix, then π is called *decomposable*. This means that the corresponding partial G-space is a direct sum of two proper partial G-subspaces. The representation π is called *completely reducible* if its partial G-space is a direct sum of irreducible partial G-subspaces.

If the characteristic of K does not divide the order of a finite group H then by Maschke's theorem KH is semisimple and consequently $M_n(KH)$ is also semisimple. Hence, by Theorem 2.7 if the characteristic of K does not divide the order of a finite group G, every partial K-representation of G is completely reducible.

The next fact is crucial in the study of finite degree partial representations of arbitrary groups. Denote by O_Δ the set of objects of a groupoid Δ.

Theorem 3.1. *[7] Let G be a group. For every connected component Δ of $\Gamma(G)$ with a finite number of objects the map $\lambda_\Delta : G \to K\Delta$, defined by*

$$\lambda_\Delta(g) = \sum_{\substack{A \in O_\Delta \\ A \ni g^{-1}}} (A, g),$$

is a partial representation of G into $K\Delta$. Moreover, for every irreducible finite degree partial K-representation $\pi : G \to \mathrm{End}(V)$ there exist a unique connected component Δ of $\Gamma(G)$ with a finite number of vertices and a unique irreducible representation $\tilde{\pi} : K\Delta \to \mathrm{End}(V)$ such that $\tilde{\pi} \circ \lambda_\Delta = \pi$.

The above remains true if one replaces the term "irreducible" by "indecomposable". Thus we get a one-to-one correspondence between the irreducible (indecomposable) finite degree partial representations of G and the irreducible (indecomposable) representations of the groupoid algebras of those connected components of $\Gamma(G)$ which have a finite number of objects.

In other words, every irreducible (indecomposable) finite degree partial representation π of a group G is of the form $\pi = \sigma \circ \phi_\Delta \circ \lambda_\Delta(g)$, where $\phi_\Delta : K\Delta \cong M_n(KH)$ for some connected component Δ with a finite number of objects, H is the automorphism group of an object of Δ and σ is an irreducible (indecomposable) K-representation of $M_n(KH)$. We see that the map $\phi_\Delta \circ \lambda_\Delta(g)$: $G \to M_n(KH)$ is responsible for the "purely partial part" of π. We shall fix $\phi_\Delta : K\Delta \cong M_n(KH)$ in some natural way and the partial representations of form $\phi_\Delta \circ \lambda_\Delta(g) : G \to M_n(KH)$ shall be called the *elementary partial representations* of G. Together with the irreducible (indecomposable) representations of subgroups of G they form the elementary blocks from which the irreducible (indecomposable) partial representations are constructed.

A natural way of fixing an isomorphism $\phi_\Delta : K\Delta \cong M_n(KH)$ has been indicated already when Δ is a connected rigid groupoid ($H = 1$) with a finite number of objects. For the general (non-necessarily rigid) case it is similar. More precisely, suppose that Δ is a connected component of $\Gamma(G)$ with a finite number of objects. This means that Δ is obtained starting with a set $1 \in A \subseteq G$ such that A is a union of finite number n of right cosets of the stabilizer $H = St(A) = \{h \in G : hA = A\}$. The stabilizer H is, of course, the automorphism group of the object A of Δ. Fix n elements $(A, g_1),\dots, (A, g_n)$ of Δ such that $g_1A, \dots, g_nA$ give all the objects of Δ (we obviously may suppose that $g_1 = 1$). Then for an arbitrary element $(g_iA, g) \in \Delta$ we have that $gg_iA = g_jA$ for some $j = j(i)$ and $g_j^{-1}gg_i \in H$. Thus $g = g_jhg_i^{-1}$ for some $h \in H$. Then ϕ_Δ maps (g_iA, g) into $e_{j,i}(h)$, where $e_{i,j}(h)$ denotes the elementary matrix whose unique non-zero entry is $h \in H$, which is placed at the intersection of the i'th row and the j'th column.

The structure of irreducible or indecomposable partial representations can be described as tensor products of corresponding modules. More precisely, let H be a subgroup of G and let V be a free right KH-module of finite rank. Observe that for free KH-modules the "finite rank" is a well defined number. If $\varphi : G \to \mathrm{End}(V_{KH})$ is a partial KH-representation of a finite group G then V becomes a $K_{\mathrm{par}}G$-KH-bimodule and each $K_{\mathrm{par}}G$-KH-bimodule, which is KH-free of finite rank, gives rise to a partial KH-representation of G on V_{KH} in such a way that equivalent partial representations correspond to isomorphic $K_{\mathrm{par}}G$-KH-bimodules.

It follows from Theorem 3.1 that each irreducible (indecomposable) finite dimensional partial G-space V can be considered as an irreducible (indecomposable) left $K\Delta$-module where Δ is a connected component of $\Gamma(G)$ with a finite number of vertices.

Theorem 3.2. *[7] Let $\pi : G \to \mathrm{End}(V)$ be an irreducible (respectively, indecomposable) finite degree partial K-representation of G, Δ the connected component of $\Gamma(G)$ related to π and ${}_{K\Delta}V$ the left $K\Delta$-module corresponding to π. Then ${}_{K\Delta}V \cong {}_{K\Delta}W \otimes_{KH} U$, where $H \leqslant G$ is the automorphism group of an object of Δ, U is an irreducible (respectively, indecomposable) left KH-module and W is the $K\Delta$-KH-bimodule corresponding to an elementary partial representation of G.*

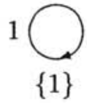
What can we say about the elementary partial representations? For a particular finite group we have a procedure to obtain them in a finite number of steps, although not much can be said about their structure in general. We only know that they are "monomial over H," i.e. for an elementary $K\Delta$-KH-bimodule W there is a free KH-basis such that for every $g \in G$ each row and each column of the matrix corresponding to g contains at most one non-zero entry, which is an element of H (observe that zero rows and zero columns are allowed). The description of the elementary representations of arbitrary finite groups seems to be a wild task. We shall see in Section 5 that this is a subproblem of the description of the elementary G-gradings of the matrix algebra $M_n(KH)$ for arbitrary n and arbitrary subgroups H of a group G. However, things become somewhat better if G is abelian. In [7] the decription of $n \times n$-elementary partial representations is given with $n \leq 4$ for an arbitrary (non-necessarily finite) abelian group G. The list of such partial representations with $n \leq 3$ is already available in [6]. It is not clear yet if the $n \times n$-elementary partial representations of arbitrary abelian groups can be understood for general n.

The procedure of obtaining the elementary partial representations of a given finite group is as follows: take a subset $A \subseteq G$ with $A \ni 1$; multiplying A from left by the elements g with $g^{-1} \in A$ yields some other subsets: $A_1 = A, A_2, \ldots, A_n$ with $A_1 \ni 1$, the subsets $A_1, \ldots A_n$ are the objects of a connected subgroupoid Δ of $\Gamma(G)$; fix elements g_i with $A_i = g_i A$; this determines the isomorphism $\phi_\Delta : K\Delta \cong M_n(KH)$, given by $(g_i A, g) \mapsto e_{j,i}(h)$, where $H \ni h = g_j^{-1} g g_i$; the map $\phi_\Delta \circ \lambda_\Delta(g)$, where λ_Δ is defined in Theorem 3.1, is the elementary partial representation which corresponds to Δ. Since G is finite we obtain all elementary partial representations in a finite number of steps. Observe that taking $A = G$ results in a connected component with the single object G. Thus the group ring KG is a direct summand of $K_{\mathrm{par}}G$. The elementary partial representation corresponding to this component is the trivial map $G \to G$.

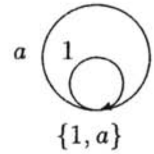
In Section 2 we gave a list of partial representations of the cyclic group C_4 of order 4. Looking at the connected componets of $\Gamma(C_4)$, which were also given in Section 2, the reader can easily check that we have in fact listed, for arbitrary fields K, all irreducible partial representations of C_4 which are not the usual representations. All of them except φ_5 are elementary. Together with the trivial map $G \to G$ they form the complete list of the elementary partial representations of C_4.

We give one more example: the connected components $\neq G$ and the irreducible non-usual partial representations of the Klein four-group over a field K. Among them $\varphi_1, \varphi_2, \varphi_4, \varphi_6, \varphi_8$ are elementary and together with the trivial map $G \to G$ form the list of the elementary partial representations of the Klein four-group.

Example 3.3. $G = \langle a;\ a^2 = 1\rangle \times \langle b;\ b^2 = 1\rangle$

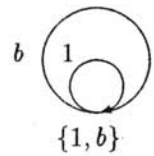


$\varphi_1: \quad 1 \mapsto 1, \quad a \mapsto 0, \quad b \mapsto 0, \quad ab \mapsto 0$



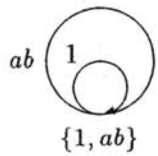
$\varphi_2: \quad 1 \mapsto 1, \quad a \mapsto 1, \quad b \mapsto 0, \quad ab \mapsto 0$

$\varphi_3: \quad 1 \mapsto 1, \quad a \mapsto -1, \quad b \mapsto 0, \quad ab \mapsto 0$



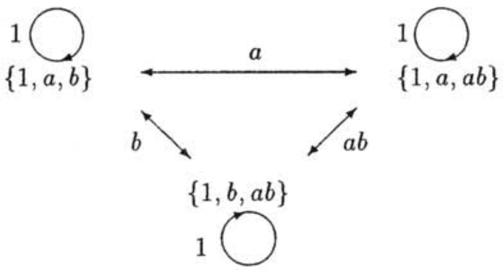
$\varphi_4: \quad 1 \mapsto 1, \quad a \mapsto 0, \quad b \mapsto 1, \quad ab \mapsto 0$

$\varphi_5: \quad 1 \mapsto 1, \quad a \mapsto 0, \quad b \mapsto -1, \quad ab \mapsto 0$



$\varphi_6: \quad 1 \mapsto 1, \quad a \mapsto 0, \quad b \mapsto 0, \quad ab \mapsto 1$

$\varphi_7: \quad 1 \mapsto 1, \quad a \mapsto 0, \quad b \mapsto 0, \quad ab \mapsto -1$



$$\varphi_8 : 1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$ab \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

4 Partial actions and crossed products

In order to define a partial action α of a group G on an (associative) non necessarily unital (non-unital) K-algebra $\mathcal{A}$ we suppose in Definition 2.1 that each $\mathcal{D}_g$ ($g \in G$) is an ideal of $\mathcal{A}$ and that every map $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ is an isomorphism of algebras (in the category of non-unital algebras). In what follows by an *algebra* we shall mean an associative non-unital algebra.

Together with the notion of partial actions a generalization of the concept of crossed product appeared in the theory of operator algebras (see [8], [9], [11], [17]). For simplicity we assume that the twisting is trivial, so we give the definition in the context of corresponding skew group rings.

Definition 4.1. *Given a partial action α of a group G on an algebra $\mathcal{A}$, the skew group ring $\mathcal{A} *_\alpha G$ corresponding to α is the set of all finite formal sums $\{\sum_{g\in G} a_g\delta_g : a_g \in \mathcal{D}_g\}$, where δ_g are symbols. Addition is defined by the obvious way, and multiplication is determined by $(a_g\delta_g)\cdot(b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}$.*

Obviously, $\mathcal{A} \ni a \mapsto a\delta_1 \in \mathcal{A} *_\alpha G$ is an embedding which permits us to identify $\mathcal{A}$ with $\mathcal{A}\delta_1$. The first question which naturally arises is whether or not $\mathcal{A} *_\alpha G$ is associative. The associativity of this construction has been proved in [9] in the context of C^*-algebras using the existence of aproximate units. It is known that each closed ideal in a C^*-algebra is an idempotent ideal, i.e. satisfies the equality $\mathcal{I}^2 = \mathcal{I}$ (see [14, Theorem V.9.2]). It turns out that $\mathcal{A} *_\alpha G$ is not always associative; however, it is associative if $\mathcal{A}$ is an algebra whose ideals are idempotent (Theorem 4.5).

Let K be a field, $\mathcal{A}$ an associative K-algebra with unity element and $\mathcal{I}$ an ideal of $\mathcal{A}$. Take an element $x \in \mathcal{A}$ and consider the left and right multiplications of $\mathcal{I}$ by x: $L_x : \mathcal{I} \ni a \mapsto xa \in \mathcal{I}$, $\quad R_x : \mathcal{I} \ni a \mapsto ax \in \mathcal{I}$. Then $L = L_x$ and $R = R_x$ are linear transformations of $\mathcal{I}$ such that the following properties are satisfied for all $a, b \in \mathcal{I}$:

(i) $L(ab) = L(a)b$;
(ii) $R(ab) = aR(b)$;

(iii) $R(a)b = aL(b)$.

These properties are obviuos consequences of the associativity of $\mathcal{A}$.

Definition 4.2. *The algebra of multipliers (see e.g. [15, 3.12.2]) of an algebra $\mathcal{I}$ is the set $M(\mathcal{I})$ of all ordered pairs (L, R), where L and R are linear transformations of $\mathcal{I}$ which satisfy the properties (i) - (iii). For $(L, R), (L', R') \in M(\mathcal{I})$ and $\alpha \in K$ the operations are given by $\alpha(L, R) = (\alpha L, \alpha R)$, $(L, R) + (L', R') = (L+L', R+R')$, $(L, R)(L', R') = (L \circ L', R' \circ R)$. We say that L is a left multiplier and R is a right multiplier of $\mathcal{I}$.*

It is immediately verified that $M(\mathcal{I})$ is an associative algebra with unity element (L_1, R_1), where L_1 and R_1 are identity maps (which in the case of an ideal $\mathcal{I}$ in a unital algebra $\mathcal{A}$ can be considered as multiplications by the unity element of $\mathcal{A}$ from left and right, respectively). Define the map $\phi : \mathcal{I} \to M(\mathcal{I})$ by putting $\phi(x) = (L_x, R_x)$, $x \in \mathcal{I}$. This is a homomorphism of algebras since it is K-linear and, moreover, $L_{xy} = L_x \circ L_y, R_{xy} = R_y \circ R_x$, which gives $\phi(xy) = (L_x \circ L_y, R_y \circ R_x) = \phi(x)\phi(y)$. The kernel of ϕ is the intersection of the left annihilator of $\mathcal{I}$ in $\mathcal{I}$ with its right annihilator in $\mathcal{I}$.

Proposition 4.3. *[4] The following statements hold:*
(i) $\phi(\mathcal{I})$ is an ideal of $M(\mathcal{I})$.
(ii) $\phi : \mathcal{I} \to \mathcal{M}(\mathcal{I})$ is an isomorphism if and only if $\mathcal{I}$ is a unital algebra.
(iii) $(R' \circ L)(ab) = (L \circ R')(ab)$ for arbitrary $(L, R), (L', R') \in M(\mathcal{I})$ and $a, b \in \mathcal{I}$.
(iv) If $\mathcal{I}$ is an idempotent algebra, that is $\mathcal{I}^2 = \mathcal{I}$, then $(R' \circ L)(a) = (L \circ R')(a)$ for all $a \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal in an algebra $\mathcal{A}$ then one may consider the homomorphism ψ : $\mathcal{A} \ni a \mapsto (L_a, R_a) \in M(\mathcal{I})$, whose kernel is the intersection of the left annihilator of $\mathcal{I}$ in $\mathcal{A}$ with its right annihilator in $\mathcal{A}$.

Let $\pi : \mathcal{I} \to \mathcal{J}$ be an isomorphism of K-algebras. Then it is easy to see that for $(L, R) \in M(\mathcal{I})$ the pair $(\pi \circ L \circ \pi^{-1}, \pi \circ R \circ \pi^{-1})$ is an element of $M(\mathcal{J})$ and we obviously have the following:

Proposition 4.4. *[4] The map $\bar{\pi} : M(\mathcal{I}) \to M(\mathcal{J})$, defined by*

$$\bar{\pi}(L, R) = (\pi \circ L \circ \pi^{-1}, \pi \circ R \circ \pi^{-1}),$$

is an isomorphism of K-algebras.

It turns out that for an (associative) algebra $\mathcal{A}$, a group G and a partial action α of G on $\mathcal{A}$ the associativity of the skew group ring $\mathcal{A} *_\alpha G$ is equivalent to the condition

$$(\alpha_g \circ R_c \circ \alpha_{g^{-1}}) \circ L_a = L_a \circ (\alpha_g \circ R_c \circ \alpha_{g^{-1}}) \tag{1}$$

being valid on $\mathcal{D}_g$ for every $g \in G$ and all $a, c \in \mathcal{A}$ (see [4]). Consider R_c as a right multiplier of $\mathcal{D}_{g^{-1}}$ and L_a as a left multiplier of $\mathcal{D}_g$. By Proposition 5.11, $\alpha_g \circ R_c \circ \alpha_{g^{-1}}$ is a right multiplier of $\mathcal{D}_g$. Hence, if all ideals $\mathcal{D}_g$ $(g \in G)$ are idempotent, then (1) follows from (iv) of Proposition 4.3. Thus, we have the following:

Theorem 4.5. *[4] If $\mathcal{A}$ is an (associative) algebra and α is a partial action of a group G on $\mathcal{A}$ such that each $\mathcal{D}_g$ $(g \in G)$ is an idempotent ideal, then the skew group ring $\mathcal{A} *_\alpha G$ is associative.*

Definition 4.6. *We say that a K-algebra $\mathcal{A}$ is strongly associative if for any group G and an arbitrary partial action α of G on $\mathcal{A}$ the skew group ring $\mathcal{A} *_\alpha G$ is associative.*

As a consequence of the above theorem we have:

Corollary 4.7. *An (associative) algebra $\mathcal{A}$ whose ideals are idempotent is strongly associative.*

Next we give an easy example which shows that $\mathcal{A} *_\alpha G$ is not associative in general.

Example 4.8. Let $\mathcal{A}$ be a four-dimensional K-vector space with basis $\{1, t, u, v\}$. Define the multiplication on $\mathcal{A}$ by setting $u^2 = v^2 = uv = vu = tu = ut = t^2 = 0, tv = vt = u$ and $1a = a1 = a$ for each $a \in \mathcal{A}$. Then $\mathcal{A}$ is an associative K-algebra with unity. Let $G = \langle g : g^2 = 1 \rangle$ and $\mathcal{I}$ be the ideal of A generated by v (it is the subspace generated by u and v). Consider the partial action α of G on $\mathcal{A}$ given by $\mathcal{D}_g = \mathcal{I}$, $\alpha_g : u \mapsto v, v \mapsto u$ (by definition $\mathcal{D}_1 = \mathcal{A}$ and α_1 is the identity map of $\mathcal{A}$). Then the skew group ring $\mathcal{A} *_\alpha G$ is not associative. More precisely, taking $x = t\delta_1 + u\delta_g$ we have that $(xx)x = 0$ and $x(xx) = u\delta_g$, so that $\mathcal{A}$ does not even have associative powers.

It is easily seen that in the category of finite dimensional unital algebras the algebras whose ideals are idempotent are exactly the semisimple algebras. An important class of non-semisimple algebras is formed by the group algebras KG of finite groups G with $charK$ dividing the order of G. It is easily seen that if $charK = 2$ then the algebra of the above example is isomorphic to the group algebra of the Klein four-group over K. On the other hand, it can be verified that the group algebra of the cyclic group of order 4 over a field of characteristic 2 is strongly associative. Thus it would be interesting to characterize the strongly associative group algebras. Another classical example of non-semisimple algebras is given by the algebra $T(n, K)$ of upper-triangular $n \times n$-matrices over K.

Proposition 4.9. *[4] The algebra $\mathcal{A} = T(n, K)$ is strongly associative if and only if $n \leq 2$.*

Natural examples of partial actions can be obtained by restricting (global) actions on non-necessarily invariant subsets (ideals in our case). More precisely, suppose that a group G acts on an algebra $\mathcal{B}$ by automorphisms $\beta_g : \mathcal{B} \to \mathcal{B}$ and let $\mathcal{A}$ be an ideal of $\mathcal{B}$. Set $\mathcal{D}_g = \mathcal{A} \cap \beta_g(\mathcal{A})$ and let α_g be the restriction of β_g to $\mathcal{D}_{g^{-1}}$. Then it is easily verified that we have a partial action $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g : g \in G\}$ of G on $\mathcal{A}$. We shall say that α is a *restriction* of β on $\mathcal{A}$. We want to know circumstances under which a given partial action can be obtained "up to equivalence" as the restriction of a (global) action. If $\mathcal{B}_1$ is the subalgebra of $\mathcal{B}$ generated by $\cup_{g\in G}\beta_g(\mathcal{A})$, it may happen that $\mathcal{B} \neq \mathcal{B}_1$ and, since $\mathcal{B}_1$ is invariant with respect to β, we see that α can be obtained as a restriction of an action of G on $\mathcal{B}_1$ which is a proper subalgebra of $\mathcal{B}$. Thus for uniqueness purposes it is reasonable to require that $\mathcal{B} = \mathcal{B}_1$ and in this case we say that α is an *admissible restriction* of β. The notion of the *equivalence* of partial actions we define as follows:

Definition 4.10. *We say that a partial action $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g : g \in G\}$ of a group G on an algebra $\mathcal{A}$ is equivalent to the partial action $\alpha' = \{\alpha'_g : \mathcal{D}'_{g^{-1}} \to \mathcal{D}'_g : g \in G\}$ of G on an algebra $\mathcal{A}'$ if there exists an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that for every $g \in G$ the following two conditions hold:*
(i) $\varphi(\mathcal{D}_g) = \mathcal{D}'_g$;
(ii) $\alpha' \circ \varphi(x) = \varphi \circ \alpha(x)$ for all $x \in \mathcal{D}_{g^{-1}}$.

We deal with *enveloping actions*:

Definition 4.11. *An action β of a group G on an algebra $\mathcal{B}$ is said to be an enveloping action for the partial action α of G on an algebra $\mathcal{A}$ if α is equivalent to an admissible restriction of β to an ideal of $\mathcal{B}$.*

In other words, β is an enveloping action for α if there exists an algebra isomorphism φ of $\mathcal{A}$ onto an ideal of $\mathcal{B}$ such that for all $g \in G$ the following two properties are satisfied:
(i') $\varphi(\mathcal{D}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$;
(ii') $\varphi \circ \alpha_g(x) = \beta_g \circ \varphi(x)$ for each $x \in \mathcal{D}_{g^{-1}}$.

Thus it is natural to ask whether or not a given partial action possesses an enveloping action.

With respect to the associativity question we observe the following:

Proposition 4.12. *If β is an action of a group G on an algebra $\mathcal{B}$, which is enveloping for the partial action α of G on an algebra $\mathcal{A}$, then the skew group ring $\mathcal{A} *_\alpha G$ has an embedding into $\mathcal{B} *_\beta G$. In particular, $\mathcal{A} *_\alpha G$ is associative.*

Proof. Obvious. □

Thus it already follows from Example 4.8 (or Proposition 4.9) that not every partial action admits an enveloping action. The general answer to this problem is given in the following.

Theorem 4.13. *[4] Let $\mathcal{A}$ be a unital algebra. Then a partial action α of a group G on $\mathcal{A}$ admits an enveloping action β if and only if each ideal $\mathcal{D}_g$ $(g \in G)$ is a unital algebra. Moreover, β, if it exists, is unique up to equivalence.*

5 Relating partial representations with partial actions via crossed products

In this section we use crossed products to relate partial actions with partial representations of groups. First, given a partial action one obtains a partial representation in the following way.

Lemma 5.1. *[4] Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \ (g \in G)\}$ be a partial action of G on an algebra $\mathcal{A}$ such that each $\mathcal{D}_g$ $(g \in G)$ is a unital algebra with unity 1_g. Then the map $\pi_\alpha : G \ni g \mapsto 1_g \delta_g \in \mathcal{A} *_\alpha G$ is a partial representation.*

We shall use the notion of the *equivalence* of partial representations in a general setting.

Definition 5.2. *Two partial representations $\pi : G \to \mathcal{B}$ and $\pi' : G \to \mathcal{B}'$ are equivalent if there is an isomorphism $\varphi : \mathcal{B}' \to \mathcal{B}$ such that*

$$\pi(g) = \varphi(\pi'(g))$$

for all $g \in G$.

Remark 5.3. It is easily verified that the map $\alpha \mapsto \pi_\alpha$ sends equivalent partial actions into equivalent partial representations.

Next, given a partial representation $\pi : G \to \mathcal{B}$ of a group G into a unital K-algebra $\mathcal{B}$, we construct a partial action. Note first that by (2), (3) of [3] the elements $\varepsilon_g = \pi(g)\pi(g^{-1})$ $(g \in G)$ are commuting idempotents such that

$$\pi(g)\varepsilon_h = \varepsilon_{gh}\pi(g), \quad \varepsilon_h \pi(g) = \pi(g)\varepsilon_{g^{-1}h}. \tag{2}$$

Let $\mathcal{A}$ be the subalgebra of $\mathcal{B}$ generated by all the ε_g $(g \in G)$ and for a fixed $g \in G$ set $\mathcal{D}_g = \varepsilon_g \mathcal{A}$.

Lemma 5.4. *[4] The maps $\alpha_g^\pi : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ $(g \in G)$, defined by $\alpha_g^\pi(a) = \pi(g)a\pi(g^{-1})$ $(a \in \mathcal{D}_{g^{-1}})$, are isomorphisms of K-algebras, which determine a partial action α^π of G on $\mathcal{A}$.*

Remark 5.5. As in the previous case it is readily seen that $\pi \mapsto \alpha^\pi$ also preserves the equivalence relations.

Thus we have two maps: one from the equivalence classes of partial actions to the equivalence classes of partial representations and another one from the classes of partial representations back to the classes of partial actions. We want to know what happens if we compose these two maps.

Proposition 5.6. *[4] Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g\ (g \in G)\}$ be a partial action of G on an algebra $\mathcal{A}$ such that each $\mathcal{D}_g$ $(g \in G)$ is a unital algebra with unity 1_g. Let further $\mathcal{A}'$ be the subalgebra of $\mathcal{A} *_\alpha G$ generated by all $1_g\delta_1$ $(g \in G)$. Then the map $\varphi_\alpha : \mathcal{A}' \ni 1_g\delta_1 \mapsto 1_g \in \mathcal{A}$ is a monomorphism such that $\varphi_\alpha \circ \alpha_g^{\pi_\alpha} = \alpha_g \circ \varphi_\alpha$ for each $g \in G$. In particular, if $\mathcal{A}$ is generated by the elements 1_g $(g \in G)$, then the partial actions α^{π_α} and α are equivalent.*

Composing in the opposite way we have the following.

Proposition 5.7. *[4] Let $\pi : G \to \mathcal{B}$ be a partial representation and suppose that the subalgebra $\mathcal{A} \subseteq \mathcal{B}$ and the partial action α^π of G on $\mathcal{A}$ are as in Lemma 5.4. Then the map $\phi_\pi : \mathcal{A} *_{\alpha^\pi} G \to \mathcal{B}$, defined by $\phi_\pi(\sum_{g\in G} a_g\delta_g) = \sum_{g\in G} a_g\pi(g)$, is a homomorphism of K-algebras such that $\phi_\pi \circ \pi_{\alpha^\pi} = \pi$. In particular, if ϕ_π is an isomorphism, then the partial representations π and π_{α^π} are equivalent.*

Group algebras form a particular case of crossed products. Our first application of the above facts shows that the partial group algebras are also crossed products but not in a completely obvious way. The proof becomes short at this stage so we include it here.

Theorem 5.8. *[4] The partial representation $\tilde{\pi} : G \ni g \mapsto [g] \in K_{\mathrm{par}}G$ results in the isomorphism $\phi_{\tilde{\pi}} : \mathcal{A} *_{\alpha^{\tilde{\pi}}} G \cong K_{\mathrm{par}}G$.*

Proof. By Proposition 5.7, $\phi_{\tilde{\pi}} : \mathcal{A} *_{\alpha^{\tilde{\pi}}} G \to K_{\mathrm{par}}G$ is a homomorphism and we shall show that it has an inverse. We remind the reader that the subalgebra $\mathcal{A} \subseteq K_{\mathrm{par}}G$ is generated by the elements $\tilde{\varepsilon}_g = \tilde{\pi}(g)\tilde{\pi}(g^{-1}) = [g][g^{-1}]$ and that $\mathcal{D}_g$ is spanned by the elements of form $\tilde{\varepsilon}_g \cdot \tilde{\varepsilon}_{h_1} \cdot \ldots \cdot \tilde{\varepsilon}_{h_s}$. By the universal property of partial group rings, the partial representation $\pi_{\alpha^{\tilde{\pi}}} : G \ni g \mapsto \tilde{\varepsilon}_g\delta_g \in \mathcal{A} *_{\alpha^{\tilde{\pi}}} G$ gives a homomorphism $\psi : K_{\mathrm{par}}G \to \mathcal{A} *_{\alpha^{\tilde{\pi}}} G$ such that $\psi([g]) = \tilde{\varepsilon}_g\delta_g$.

Obviously, $\phi_{\tilde{\pi}} \circ \psi[g] = \phi_{\tilde{\pi}}(\tilde{\varepsilon}_g\delta_g) = \tilde{\varepsilon}_g[g] = [g][g^{-1}][g] = [g]$ for each $g \in G$, so that $\phi_{\tilde{\pi}} \circ \psi$ is the identity map. On the other hand, since $\tilde{\varepsilon}_g\delta_g\tilde{\varepsilon}_{g^{-1}}\delta_{g^{-1}} = \tilde{\varepsilon}_g\delta_1$, we have

$$
\begin{aligned}
&\psi \circ \phi_{\tilde{\pi}}((\tilde{\varepsilon}_g \cdot \tilde{\varepsilon}_{h_1} \cdot \ldots \cdot \tilde{\varepsilon}_{h_s})\delta_g) = \psi((\tilde{\varepsilon}_g \cdot \tilde{\varepsilon}_{h_1} \cdot \ldots \cdot \tilde{\varepsilon}_{h_s})[g]) = \\
&= \psi([g][g^{-1}][h_1][h_1^{-1}]\ldots[h_s][h_s^{-1}][g]) = \\
&= (\tilde{\varepsilon}_g\delta_g\tilde{\varepsilon}_{g^{-1}}\delta_{g^{-1}})(\tilde{\varepsilon}_{h_1}\delta_{h_1}\tilde{\varepsilon}_{h_1^{-1}}\delta_{h_1^{-1}})\ldots(\tilde{\varepsilon}_{h_s}\delta_{h_s}\tilde{\varepsilon}_{h_s^{-1}}\delta_{h_s^{-1}})\tilde{\varepsilon}_g\delta_g = \\
&= \tilde{\varepsilon}_g\delta_1\tilde{\varepsilon}_{h_1}\delta_1\ldots\tilde{\varepsilon}_{h_s}\delta_1\tilde{\varepsilon}_g\delta_g = \tilde{\varepsilon}_g^2\tilde{\varepsilon}_{h_1}\ldots\tilde{\varepsilon}_{h_s}\delta_g = \tilde{\varepsilon}_g\tilde{\varepsilon}_{h_1}\ldots\tilde{\varepsilon}_{h_s}\delta_g.
\end{aligned}
$$

Thus $\psi \circ \phi_{\bar{\pi}}$ is also the identity map and consequently ψ is the inverse of $\phi_{\bar{\pi}}$. □

The next fact helps to identify algebras as crossed products.

Proposition 5.9. *[4] Let $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \ (g \in G)\}$ be a partial action of G on an algebra $\mathcal{A}$ such that for each $g \in G$ the right annihilator of $\mathcal{D}_g$ in $\mathcal{D}_g$ is zero. Suppose that $\varphi : \mathcal{A} *_\alpha G \to \mathcal{B}$ is a homomorphism of algebras whose restriction on $\mathcal{A}$ is injective. If there exists a K-linear transformation $E : \mathcal{B} \to \mathcal{B}$ such that for every $g \in G$, $a \in \mathcal{D}_g$*

$$E(\varphi(a\delta_g)) = \begin{cases} \varphi(a\delta_g) & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

then φ is injective.

We apply the above results for matrix partial representations. By the *equivalence class* of an elementary partial representation we mean the class of all partial representations of G which are equivalent in the sense of Definition 5.2 to the given elementary partial representation.

A straightforward verification shows that an elementary partial representation $\pi : G \to M_n(KH)$ gives rise to a certain partial action of G on the diagonal subalgebra $diag(K, \ldots, K)$ of the full matrix algebra $M_n(K)$. This diagonal algebra is obviously isomorphic to K^n, the n'th direct power of K, so we speak about partial actions on K^n. Given a partial action α of G on $diag(K, \ldots, K) \cong K^n$ we see that each $\mathcal{D}_g$ is an algebra with unity 1_g which is a sum of some minimal idempotents $e_{i,i}(1)$. Set $A_i(\alpha) = \{g \in G : (1_g)_{i,i} \neq 0\}$, where $(1_g)_{i,i}$ denotes the i'th diagonal entry of 1_g. Coming back to our elementary partial representation $\pi : G \to M_n(KH)$, one verifies that $St(A_1(\alpha^\pi)) = H$ and $(A_1(\alpha^\pi) : H) = n$. Proposition 5.7 gives us the map $\phi_\pi : \mathcal{A} *_{\alpha^\pi} G \to M_n(KH)$. Using Proposition 5.9 we are able to show that ϕ_π is a monomorphism, and since it is easy to check that it is also an epimorphism, it follows from Proposition 5.7 that the partial representations π and π_{α^π} are equivalent. It turns out that starting with a partial action $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \ (g \in G)\}$ of G on K^n such that $St(A_1(\alpha)) = H$ and the index of H in $A_1(\alpha)$ is n, it is possible to find an elementary partial representation of G equivalent to π_α. We also show that the partial actions α and α^{π_α} are equivalent and thus we come to the following result.

Theorem 5.10. *[4] For a fixed $n > 0$ and a fixed subgroup H of a group G the maps $\pi \mapsto \alpha^\pi$ and $\alpha \mapsto \pi_\alpha$ establish a one-to-one correspondence between the equivalence classes of the elementary partial representations $\pi : G \to M_n(KH)$ and the equivalence classes of the partial actions α of G on K^n with $St(A_1(\alpha)) = H$ and $(A_1(\alpha) : H) = n$.*

The next two facts follow from the proof of the above result.

Corollary 5.11. *[4] For each elementary partial representation $\pi : G \to M_n(KH)$ the map $\phi_\pi : K^n *_{\alpha^\pi} G \to M_n(KH)$, given by $\sum_{g\in G} a_g\delta_g \mapsto \sum_{g\in G} a_g\pi(g)$, is an isomorphism.*

Corollary 5.12. *[4] The partial action α^π which corresponds to an elementary partial representation $\pi : G \to M_n(KH)$ acts transitively on the minimal idempotents of $diag(K,\dots,K)$, i.e. for every $1 \le i,j \le n$ there exists an element $g \in G$ such that $e_{i,i}(1) \in \mathcal{D}_{g^{-1}}, e_{j,j}(1) \in \mathcal{D}_g$ and $\alpha_g(e_{i,i}(1)) = e_{j,j}(1)$.*

We see that the full matrix algebra $M_n(KH)$ can be viewed as a crossed product $K^n *_\alpha G$ in many ways. In particular, we have the following.

Corollary 5.13. *For each positive integer n and an arbitrary group G of order $n+1$ there is a partial action α of G on K^n such that $M_n(K) \cong K^n *_\alpha G$.*

Proof. Pick an element $1 \neq g \in G$ and take $A = G\setminus\{g\}$. Then $St(A) = 1$ and A determines an elemetary partial representation $\pi : G \to M_n(K)$, which gives rise to the isomorphism $\phi_\pi : K^n *_{\alpha^\pi} G \to M_n(K)$. □

A crossed product structure on $M_n(KH)$ gives a grading of this K-algebra by G. It is easily seen that this grading is elementary. More precisely, we recall that a grading on the K-algebra $M_n(K)$ by a group G is called *elementary* if each elementary matrix $e_{i,j}(1)$ is homogeneous. It is known (see [1]) that each elementary grading on $M_n(K)$ by a group G is determined by an n-tuple $(g_1 = 1, g_2, \dots, g_n)$ of non-necessarily distinct elements of G in such a way that the homogeneous degree $deg(e_{i,j}(1))$ of $e_{i,j}(1)$ is ${g_i}^{-1}g_j$. Conversely, in this manner each n-tuple $(g_1 = 1, g_2, \dots, g_n)$ determines an elementary grading on $M_n(K)$. It turns out that if in $(g_1 = 1, g_2, \dots, g_n)$ the g_i's are pairwise distinct and $St(\{g_1, g_2, \dots, g_n\}) = 1$, then the corresponding elementary grading of $M_n(K)$ necessarily comes from a crossed product structure $K^n *_{\alpha^\pi} G \cong M_n(K)$. More generally, we say that a grading of the K-algebra $M_n(KH)$ by a group G is *elementary* if for each $h \in H, i,j \in \{1,\dots,n\}$ the elementary matrix $e_{i,j}(h)$ is homogeneous. Given a subset $1 \in A$ of G with $St(A) = H$ one defines an elementary grading of $M_n(KH)$ by the equality $deg(e_{i,j}(h)) = {g_i}^{-1}hg_j$ $(h \in H, i,j \in \{1,\dots,n\})$, where $A = Hg_1 \cup Hg_2 \cup \dots \cup Hg_n$, $g_1 = 1$ and $Hg_i \neq Hg_j$ for $1 \le i \neq j \le n$. Changing the g_i's to ${g_i}^{-1}$'s in the definition of the elementary partial representations, we easily obtain the following.

Corollary 5.14. *[4] For the elementary grading of $M_n(KH)$ by a group G, determined by a subset $1 \in A \subseteq G$ with $H = St(A)$, and for the elementary partial representation*
$\pi : G \to M_n(KH)$, constructed from $A_1 = A, A_2 = {g_2}^{-1}A, \dots, A_n = {g_n}^{-1}A$, the map $\phi_\pi : K^n *_{\alpha^\pi} G \to M_n(KH)$ is an isomorphism of graded K-algebras.

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