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On some problems of units in integral group rings [1]

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To my friend Sudarshan Sehgal on the occosion of his 65th birthday.

Abstract: We present a survey of some recent results on problems posed by Sudarshan Sehgal.

In this paper some new results are presented on the following problems posed by Sudarshan Sehgal in [43]. The integral group ring of a group G is denoted by $\mathbb{Z}G$. Its unit group we denote by $\mathcal{U}(\mathbb{Z}G)$, its group of normalized units by $\mathcal{U}_1(\mathbb{Z}G)$ and its center by $Z(\mathcal{U}(\mathbb{Z}G))$.

> *Problem 17*: Give a presentation by generators and relations for the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ for some finite groups G.
>
> *Problem 23*: Give generators up to finite index for $\mathcal{U}(\mathbb{Z}G)$ if G is a finite group.
>
> *Problem 18*: Find a good estimate for the index $(\mathcal{U}_1(\mathbb{Z}G) : B)$, where B is the group generated by the Bass cyclic units and the bicyclic units.
>
> *Problem 19*: Is the group generated by the bicylic units torsion-free?
>
> *Problem 29*: Suppose that G is a finite nilpotent group. Does G have a normal complement N in $\mathcal{U}_1(\mathbb{Z}G)$, i.e., $\mathcal{U}_1(\mathbb{Z}G) = N \rtimes G$?
>
> *Problem 43*: Let G be a finite group. Is $\mathcal{N}_{\mathcal{U}(\mathbb{Z}G)}(G) = G\, Z(\mathcal{U}(\mathbb{Z}G))$?

In the late 1980's only for very few groups G the full unit group $\mathcal{U}(\mathbb{Z}G)$ had been described. Recall that Milies obtained one of the first such descriptions, namely for $\mathcal{U}(\mathbb{Z}D_8)$, where D_8 is the dihedral group of order 8. Next there were examples by Allen and Hobby, Kleinert, and others. Using basically only linear algebra tools, Jespers and Leal, and Jespers and Parmenter obtained a new, and more structural description in the case of the symmetric group S_3 of degree 3 and the dihedral group D_8 of order 8. It was shown that

$$\mathcal{U}(\mathbb{Z}S_3) = F_3 \rtimes (\pm S_3)$$

and

$$\mathcal{U}(\mathbb{Z}D_8) = F_3 \rtimes (\pm D_8),$$

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where F_3 is a free group of rank 3. In both cases, F_3 is generated by bicyclic units (in the former case, these generators are all the non-trivial bicyclic units, in the latter case they are any three of the four non-trivial bicyclic units, up to inverses). For all these examples we refer the reader to [43].

Recall that a bicyclic unit is a unit of the type

$$u_{g,h} = 1 + (1-g)h\widehat{g},$$

where $g, h \in G$, g has finite order, say n, and $\widehat{g} = 1 + g + \cdots + g^{n-1}$. Of course the elements $u'_{g,h} = 1 + \widehat{g}h(1-g)$ are also units.

Allen and Hobby in [1] have shown that S_3 has a normal complement with non-trivial torsion in $\mathcal{U}_1(\mathbb{Z}S_3)$. With Dooms [8] we determined all the normal complements of S_3 and D_8 in the normalized unit group of their respective integral group ring.

Proposition 1 *([8]) The group $S_3 = \langle a, b \mid a^3 = b^2 = 1,\ ba = a^2 b\rangle$ has precisely four normal complements in $\mathcal{U}_1(\mathbb{Z}S_3)$:*

1. *$\langle u_{a,b},\ u_{a,ba},\ u_{a,ba^2}\rangle$, a free group of rank 3,*
2. *$\langle au_{a,b},\ au_{a,ba},\ au_{a,ba^2}\rangle$, a free group of rank 3,*
3. *$\langle a^2u_{a,b},\ a^2u_{a,ba},\ a^2u_{a,ba^2}\rangle$, a free group of rank 3,*
4. *$\langle bu_{a,b},\ bau_{a,ba},\ ba^2u_{a,ba^2}, u_{a,b}^2,\ u_{a,ba}^2, u_{a,ba^2}^2\rangle$ a group with torsion. Actually $bu_{a,b}$, $bau_{a,ba}$ and ba^2u_{a,ba^2} are elements of order 2.*

Note that in the unit group $\mathcal{U}(\mathbb{Z}S_3)$ we have that $u'_{a,b}u_{a,ba} = a$, a periodic unit. Hence in Problem 19 one really has to restrict to bicyclic units, and thus not include its symmetric versions as well.

Proposition 2 *([8]) The group $D_8 = \langle a, b \mid a^4 = b^2 = 1,\ ba = a^3 b\rangle$ has precisely eight normal complements in $\mathcal{U}_1(\mathbb{Z}D_8)$:*

$$\begin{array}{lllllll}
N_1 & = & \langle b_1,\ b_2,\ b_4\rangle, & & N_2 & = & \langle b_1a^3,\ b_2a,\ b_4a\rangle,\\
N_3 & = & \langle b_1a,\ b_2a^3,\ b_4a^3\rangle, & & N_4 & = & \langle b_1a^2,\ b_2a^2,\ b_4a^2\rangle,\\
N_5 & = & \langle b_1a,\ b_2a^2,\ b_4a^2\rangle, & & N_6 & = & \langle b_1a^2,\ b_2,\ b_4\rangle,\\
N_7 & = & \langle b_1a^3,\ b_2a^3,\ b_4a^3\rangle, & & N_8 & = & \langle b_1a,\ b_2a,\ b_4a\rangle,
\end{array}$$

where $b_1 = u_{a,b}$, $b_2 = u_{a,ab}$, $b_3 = u_{a,a^2b}$ and $b_4 = u_{a,a^3b}$ are all the non-trivial bicyclic units of $\mathbb{Z}D_8$ (up to inverses).

All the groups N_i, $i = 1, \cdots, 8$, are free of rank three.

Actually there are only four finite groups G so that the unit group $\mathcal{U}(\mathbb{Z}G)$ has a free subgroup of finite index [16]: S_3, D_8, the quaternion group Q_{12} of order 12, and the group $P = \langle a, b \mid a^4 = b^4 = 1, aba^{-1}b^{-1} = a^2\rangle$. In a series of papers with Leal and del Río this result was extended in such a way that we characterized when precisely the most naive generalization of Higman's result holds for the unit group of the integral group of a finite group (so this is related to Problems 17 and 23). In order to state more precisely the final result obtained [22, 25, 26, 29], we use the following notation. By $\mathbb{H}(K)$ we denote a classical quaternion algebra over a field K and $\left(\frac{a,b}{K}\right)$ denotes a generalized quaternion algebra over K. The cyclic group of order n is denoted C_n.

Theorem 3 *([26]) Let G be a finite group. The following conditions are equivalent:*

1. *$\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index that is a (finite) direct product of free products of (finitely many finitely generated) abelian groups.*
2. *$\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index that is a (finite) direct product of (finitely generated) free groups.*
3. *Every non-abelian simple quotient of $\mathbb{Q}G$ is isomorphic to either $M_2(\mathbb{Q})$, $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ or $\mathbb{H}(K)$, with $K = \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$.*
4. *G either is abelian or isomorphic to $H \times C_2^k$, where H is one of the following groups:*
   - (a) $\langle x, y \mid x^4 = y^4 = [x^2, y] = [x, y^2] = [x, [x, y]] = [y, [x, y]] = 1\rangle$,
   - (b) $\langle x, y_1, \cdots, y_n \mid x^4 = y_i^2 = [y_i, y_j] = [x^2, y_i] = [[x, y_i], y_j] = [[x, y_i], x] = 1\rangle$,
   - (c) $\langle x, y_1, \cdots, y_n \mid x^4 = y_i^4 = y_i^2[x, y_i] = [y_i, y_j] = [x^2, y_i] = [y_i^2, x] = 1\rangle$,
   - (d) $\langle x, y_1, \cdots, y_n \mid x^2 = y_i^2 = [y_i, y_j] = [[x, y_i], y_j] = [x, y_i]^2 = 1\rangle$,
   - (e) $\langle x, y_1, \cdots, y_n \mid x^2 = y_i^4 = y_i^2[x, y_i] = [y_i, y_j] = [[x, y_i], x] = 1\rangle$,
   - (f) $\langle x, y_1, \cdots, y_n \mid x^4 = y_i^4 = x^2y_1^2 = y_i^2[x, y_i] = [y_i, y_j] = [y_i^2, x] = 1\rangle$,
   - (g) $\langle x, y_1, \cdots, y_n \mid x^4 = x^2y_i^4 = y_i^2[x, y_i] = [y_i, y_j] = 1\rangle$,
   - (h) *$Z \rtimes \langle x\rangle$ where Z is an elementary abelian 3-group, x has order 2 or 4 and $z^x = x^{-1}zx = z^{-1}$ for every $z \in Z$,*
   - (i) *$Z \rtimes H$ where Z is an elementary abelian 3-group, $H = \langle x, y\rangle \cong Q_8$ and $z^x = x^{-1}zx = z^{-1} = y^{-1}zy = z^y$ for every $z \in Z$.*

A consequence of this theorem is a characterization of when the unit group contains a subgroup of finite index that is a free product of abelian groups.

Corollary 4 *([26]) The following conditions are equivalent for a finite group G:*

1. *$\mathcal{U}(\mathbb{Z}G)$ contains a subgroup of finite index that is a free product of abelian groups.*

2. *$\mathcal{U}(\mathbb{Z}G)$ contains a subgroup of finite index that is either abelian or non-abelian free.*

3. *$\mathbb{Q}G$ is a direct product of fields, division rings of the form $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ or $\mathbb{H}(K)$, with $K = \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$, and at most one copy of $M_2(\mathbb{Q})$.*

4. *One of the following conditions holds:*
   (a) *$G = Q_8 \times C_2^n$ (in this case $\mathcal{U}(\mathbb{Z}G)$ is finite),*
   (b) *G is abelian,*
   (c) *G is one of the following groups: D_6, D_8, Q_{12} or P (in this case $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index that is non-abelian free).*

Ritter and Sehgal [38, 39, 40], and Jespers and Leal [23, 24] proved that, for "most" finite groups G, the group generated by the Bass cyclic units (denoted B_{Bass}) and bicyclic units of both types (denoted B_{bic}) is of finite index in $\mathcal{U}(\mathbb{Z}G)$ (this answers Problem 23 for most finite groups). The exceptions occur when $\mathbb{Q}G$ has a simple component of exceptional type, such as a non-commutative division ring (which is not a totally definite quaternion algebra) or $M_2(\mathbb{Q})$. Note that there exist finite groups for which $\langle B_{Bass}, B_{bic}\rangle$ is not of finite index in $\mathcal{U}(\mathbb{Z}G)$. Among the groups G of order 16, only D_{16} and $D_8 \times C_2$ have the property that $\langle B_{Bass}, B_{bic}\rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$. For more information and references on these results we refer the reader to the survey paper [17].

In case G is a finite group, again with some restrictions on the simple components of $\mathbb{Q}G$ (these are all satisfied if G is a nilpotent group of odd order), or the symmetric group S_n of degree n, with Leal [21] we determined an upper bound for the index of $\langle B_{Bass}, B_{bic}\rangle$ in $\mathcal{U}(\mathbb{Z}G)$. This upper bound is of the form $|G|^{27|G|}k$, where k is a number determined by the central units (note that k is not needed in case of S_n). The method for obtaining this upper bound is to construct congruence subgroups in the unit group. This result gives a reasonable answer to Problem 18 for the class of groups under consideration.

Concerning Problem 19, recently Olivieri and del Río showed that in general the answer is negative.

Proposition 5 *([35]) Let n be a positive integer n and let B be the group generated by the bicyclic units in $\mathbb{Z}S_n$. Then*

$$B \cap S_n = \begin{cases} \{1\} & \text{if } n \leq 3 \\ \langle (1\,2)(1\,3),\ (1\,3)(2\,4)\rangle & \text{if } n = 4 \\ A_n \text{ or } S_n & \text{if } n \geq 5 \end{cases}$$

The first part follows from the description of $\mathcal{U}(\mathbb{Z}S_3)$, and the case $n \geq 5$ follows at once from the case $n = 4$. The proof of the latter makes use of the description of the unit group of $\mathcal{U}(\mathbb{Z}S_4)$. Shortly after, using computer calculations, Hertweck [9] showed that there is an element of A_4 that is a product of 7 distinct bicyclic units (and no shorter product will do).

So at present the aim of determining the structure of the full unit group $\mathcal{U}(\mathbb{Z}G)$ or of the group generated by all the bicyclic units is beyond reach. So one could pose the following question.

> *Problem* : Let b_1 and b_2 be two bicyclic units in the integral group ring of a finite group G. What is the structure of the group $\langle b_1, b_2\rangle$?

In this context Marciniak and Sehgal [32] proved the following result that gives an explicit construction of a free group in the unit group. By $*$ we denote the $\mathbb{Z}$-linear involution on $\mathbb{Z}G$ defined by $g^* = g^{-1}$, for $g \in G$.

Theorem 6 *([32]) Let G be any group and let $g \in G$ be an element of finite order that does not generate a normal subgroup in G. Then the group*

$$\langle u_{a,b},\ u^*_{a,b}\rangle$$

is free of rank two.

In joint work with del Río and Ruiz [27] we dealt with this problem for some dihedral groups. But first we recall a result of Salwa [41] (its proof is based on the methods used in the proof of the result of Marciniak and Sehgal).

Theorem 7 *([27]) Let K be a subfield of $\mathbb{C}$ and let R be a finite dimensional K-algebra. Suppose $a, b \in R$ are such that $a^2 = b^2 = 0$. Then the following properties hold:*

1. *(Salwa [41]) if ab is not nilpotent then there exists a positive integer m so that*
$$\langle 1 + a,\ (1 + b)^m\rangle$$
is a free group of rank two.

2. *if ab is nilpotent, then $\langle 1 + a,\ 1 + b\rangle$ is a nilpotent group.*

In case there is a trace function tr on some simple epimorphic image $\rho(R)$ of R so that $|\mathrm{tr}(\rho(ab))| \geq 4$ then, in part one of the theorem, one can take $m = 1$. So in this situation one controls m.

Proposition 8 *([27]) Let D_{2n} denote the dihedral group of order $2n$. If $x, y, g, h \in D_{2n}$ are such that $\langle y, h\rangle \subseteq \langle x, g\rangle$ then*

$$\langle u_{x,g},\ u_{y,h}\rangle$$

is a free group of rank two.

In case p is a prime number, then for any bicyclic units b_1, b_2 in $\mathbb{Z}D_{2p}$ the group

$$\langle b_1, b_2 \rangle$$

is either torsion-free abelian or free of rank two.

Crucial in the proof of the above result is to show that for any positive integers i, j, n with $1 \leq i, j < n$, $n \geq 3$ and $(n, i, j) \neq (6, 3, 1)$ there exists an integer k so that

$$\left| \sin\left(\frac{2\pi k}{n}\right) \sin\left(\frac{\pi i k}{n}\right) \sin^2\left(\frac{\pi j k}{n}\right) \right| \geq \frac{1}{4}.$$

This inequality will prove that some trace is at least 4.

It is well known that $\mathbb{Q}D_{2p} = \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}(\xi_p + \xi_p^{-1}))$, where ξ_p is a primitive p-th root of unity. Hence the result actually shows that some 2-by-2 matrices over $\mathbb{Q}(\xi_p + \xi_p^{-1})$ generate a free group of rank 2. As mentioned in [27], the group $\langle b_1, b_2 \rangle = \langle 1 + a, 1 + b \rangle$ (for some a, b with $a^2 = b^2 = 0$) is free of rank 2 if the group

$$\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \rangle$$

with $2\lambda = \mathrm{tr}(\rho(ab)) \in \mathbb{C}$ is free. Here ρ denotes some homomorphism of the $\mathbb{C}$-algebra generated by a and b onto $M_2(\mathbb{C})$. A λ for which this property holds is called a free point. A well known result of Sanov says that any λ with $|\lambda| > 2$ is a free point. Further it is also known that the free real points are dense in the interval $[-2, 2]$. It seems to be unknown whether $\sqrt{3}$ is a free point (see for example [4]). The latter also seems to be the main difficulty to extend the result to arbitrary dihedral groups.

Apart from free subgroups of the unit group, other special subgroups are very important as well. Indeed, since Hertweck's counter example to the isomorphism problem [13] we know that the normalizer $\mathcal{N}_{\mathcal{U}(\mathbb{Z}G)}(G)$ of G in $\mathcal{U}(\mathbb{Z}G)$ plays a very crucial role. Recall that Hertweck actually first constructed a counter example to the normalizer problem, that is, there exists a finite group G so that $\mathcal{N}_{\mathcal{U}(\mathbb{Z}G)}(G) \neq G\, Z(\mathcal{U}(\mathbb{Z}G))$.

Earlier it was already shown by Mazur in [34], that, for groups that are a direct product $A \times N$, where N is a finite group and A is the infinite cyclic group, there is a strong relationship between the isomorphism and the normalizer problem. Later Jespers and Juriaans [19] extended this to arbitrary finitely generated groups A. Hence answering a question of Mazur.

Theorem 9 *[19] Let N be a finite group and A a finitely generated torsion-free abelian group. Then the isomorphism problem holds for $A \times N$ if and only if both the isomorphism problem and the normalizer problem hold for N.*

The proof of this result shows that also the center $Z(\mathcal{U}(\mathbb{Z}G))$, the hyper center $\widetilde{Z}(\mathcal{U}(\mathbb{Z}G))$ and the finite conjugacy center $\Delta(\mathcal{U}(\mathbb{Z}G))$ of the unit group $\mathcal{U}(\mathbb{Z}G)$ are very important subgroups.

With Juriaans, de Miranda and Rogerio [20] we proved the following results.

Theorem 10 *([20])) Let G be an arbitrary group. If $u \in \mathcal{N}_{\mathcal{U}(\mathbb{Z}G)}(G)$ then there exist $g \in G$, and a finite normal subgroup N of G so that*

$$u = gw$$

for some $w \in \mathbb{Z}N$. Moreover, w induces on G an automorphism φ of order a divisor of $2|N|$. If N is of odd order, then φ is inner on G.

This theorem in some sense reduces the problem of determining the normalizer $\mathcal{U}_{\mathcal{U}(\mathbb{Z}G)}(G)$ of an arbitrary group to that of a finite group. Apart from Hertweck's counter examples, there are several other important results on the normalizer problem for finite groups, such as those of Coleman, Krempa (see [43]), Marciniak and Jackowski [31], Hertweck [10, 11, 12] and Kimmerle [14, 28]. The above theorem and the results for finite groups then yield several applications, such as,

Corollary 11 *([20]) Let G be a group. Then $\mathcal{N}_{\mathcal{U}(\mathbb{Z}G)}(G) = G\ Z(\mathcal{U}(\mathbb{Z}G))$ provided that G belongs to one of the following classes:*

1. *finite conjugacy groups without non-trivial 2-torsion,*
2. *periodic groups with normal Sylow 2-subgroup,*
3. *locally nilpotent groups.*

For applications on the center of the unit group we refer the reader to [19]. We finish with some very recent results (joint with Juriaans) on the finite conjugacy center of the unit group. The first one solves a problem posed by Artamonov and Bovdi [3] (see also [5, 6]). Recall that a (not necessarily finite) group G is said to be a Q^*-group if G has an abelian subgroup A of index 2 so that there exists an element $a \in A$ of order 4 and such that for all $h \in A$ and all $g \in G \setminus A$, $g^2 = a^2$ and $g^{-1}hg = h^{-1}$. Such an element a is called a distinguished element of G. These groups also appeared in a paper by Williamson [44] who showed that periodic Q^*-groups are exactly those periodic groups G containing a non-central element of $\Delta(\mathcal{U}(\mathbb{Z}G))$. Parmenter in [36] showed that a weaker conjugation condition also characterizes these groups.

Theorem 12 *([18]) Let G be a finite group. Then, $\Delta(\mathcal{U}(\mathbb{Z}G))$ is central if and only if G is not a Q^*-group. Moreover, if G is a Q^*-group which is not a Hamiltonian 2-group then $\Delta(\mathcal{U}(\mathbb{Z}G)) = Z(\mathcal{U}(\mathbb{Z}G))\ \langle a \rangle$, where a is a distinguished element of G.*

Recall that the unit group $\mathcal{U}(\mathbb{Z}G)$ of a finite Hamiltonian 2-group G is trivial.

The description of the second center of $\mathcal{U}(\mathbb{Z}G)$ obtained by Arora and Passi [2], and Li and Parmenter [30], yields the following application.

Corollary 13 *([18]) If G is a finite group then* $\Delta(\mathcal{U}(\mathbb{Z}G)) = \widetilde{Z}(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G))$*, the second center of* $\mathcal{U}(\mathbb{Z}G)$.

Also for the finite conjugacy center there is a "reduction" from arbitrary groups to finite groups. By $\mathrm{supp}(u)$ we denote the support of an element $u \in \mathbb{Z}G$.

Theorem 14 *([18]) Let G be an arbitrary group. If $u \in \Delta(\mathcal{U}(\mathbb{Z}G))$ then $H = \langle supp(u) \rangle$ is a finite conjugacy group and*

$$u \in \mathcal{U}(\mathbb{Z}N)\, H$$

for some finite normal subgroup N of H.

This result allows one to extend Corollary 13 to arbitrary periodic groups.

Corollary 15 *([18]) Let G be an arbitrary periodic group. Then $\Delta(\mathcal{U}(\mathbb{Z}G)) = Z(\mathcal{U}(\mathbb{Z}G))$ if and only if G is not a Q^*-group. If G is a Q^*-group, but not a Hamiltonian 2-group, then $\Delta(\mathcal{U}(\mathbb{Z}G)) = Z(\mathcal{U}(\mathbb{Z}G))\, \langle a \rangle = Z_2(\mathcal{U}(\mathbb{Z}G))$, where a is a distinguished element of G.*

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