Resenhas IME-USP 2002, Vol. 5, No. 4, 377 - 390.

Invariant Ideals of Abelian Group Algebras Under the Action of Simple Linear Groups [1]

D. S. Passman

Abstract: Let $\mathfrak{G}$ be a group that acts on an abelian group V. Then $\mathfrak{G}$ acts on the commutative group algebra $K[V]$, and we are concerned here with classifying the $\mathfrak{G}$-stable ideals of $K[V]$. Specifically, we discuss recent work related to linear group actions. For example, we consider the case where V is a vector space over a division ring D and where $\mathfrak{G} = D^\bullet$ is the multiplicative group of D. However, for the most part, we are concerned with infinite, locally finite, quasi-simple groups $\mathfrak{G}$ of Lie type and their finite-dimensional representations. We first discuss the known results for rational representations and then we move on to describe the techniques required to deal with $\mathfrak{G}$-modules V that are not rational.

Key words: Group algebras, augmentation ideals, group actions, division rings, locally finite groups, linear groups, quasi-simple groups, groups of Lie type, rational representations, nonrational representations.

To my friend Sudarshan Sehgal, on the occasion of his almost retirement.

1. Introduction

If $\mathfrak{H}$ is a nonidentity group, then the group algebra $K[\mathfrak{H}]$ always has at least three distinct ideals, namely 0, the augmentation ideal $\omega K[\mathfrak{H}]$, and $K[\mathfrak{H}]$ itself. Thus it is natural to ask if groups exist for which the augmentation ideal is the unique nontrivial ideal. In such cases, we say that $\omega K[\mathfrak{H}]$ is *simple*. Certainly $\mathfrak{H}$ must be a simple group for this to occur and, since the finite situation is easy enough to describe, we might as well assume that $\mathfrak{H}$ is infinite simple. The first such examples, namely algebraically closed groups and universal groups, were offered in [BHPS]. From this, it appeared that such groups would be quite rare. But A. E. Zalesskiĭ has shown that, for locally finite groups, this phenomenon is really the norm. Indeed, for all locally finite infinite simple groups, the characteristic 0 group algebras $K[\mathfrak{H}]$ tend to have very few ideals. See [Z4] for a survey of this material. Additional papers of interest include [HZ3], [LP], [Z2] and [Z3].

While some work still remains to be done on the simple group case, it nevertheless makes sense to move on to the next stage of this program by considering certain abelian-by-(quasi-simple) groups. Specifically, these are the locally finite groups $\mathfrak{H}$ having a minimal normal abelian subgroup V with $\mathfrak{H}/V$ infinite simple (or perhaps just close to being simple). Note that $\mathfrak{G} = \mathfrak{H}/V$ acts as automorphisms on V, and hence on the group algebra $K[V]$. Furthermore, if I is any nonzero ideal of $K[\mathfrak{H}]$, then it is easy to see that $I \cap K[V]$ is a nonzero $\mathfrak{G}$-stable

[1] 2000 *Mathematics Subject Classification.* Primary 16S34; Secondary 20F50, 20G05

Research supported in part by NSF Grant DMS-9820271.

ideal of $K[V]$. Thus, for the most part, this second stage is concerned with classifying the $\mathfrak{G}$-stable ideals of $K[V]$. Even in concrete cases, this turns out to be a surprisingly difficult task. Fortunately, there has been some recent progress on this problem, and our goal here is to survey the results of [BE], [OPZ], [PZ1], [PZ2] and [P]. For the most part, the methods used in these papers are quite different from the usual group ring techniques.

2. Multiplicative Action of Division Rings

Recall that a field is said to be *locally finite* or *absolute* if every finite subset generates a finite subfield. In other words, F is locally finite precisely when it is a subfield of the algebraic closure of a finite field. Now suppose that W is a finite-dimensional vector space over an infinite locally finite field and let $\mathfrak{G} = \mathrm{GL}(W)$ act naturally on W. Then $\mathfrak{H} = W \rtimes \mathfrak{G}$ is an infinite locally finite group and an example of the type of second-stage group described above. In particular, it would be of interest to determine the $\mathrm{GL}(W)$-stable ideals of the group algebra $K[W]$. Notice that in this case, $\mathfrak{G}$ contains $F^\bullet$, the multiplicative group of F, and consequently every $\mathfrak{G}$-stable ideal of $K[W]$ is also $F^\bullet$-stable. Thus it is reasonable to first study the $F^\bullet$-stable ideals of $K[W]$, and we proceed to do this in a fairly general setting.

To this end, let D be a division ring of any characteristic and let V be a right D-vector space. Then $G = D^\bullet$ acts as automorphisms on V by right multiplication and consequently G acts on any group algebra $K[V]$. Note that if A is a D-subspace of V, then A is G-stable and hence the augmentation ideal $\omega K[A]\cdot K[V]$ is a G-stable ideal of $K[V]$. We start with the following elementary lemma since its proof gives an indication of both the action of G and the structure of $K[V]$.

Lemma 2.1. *Let D be a division ring and let V be a right D-vector space. If* $\operatorname{char} K \neq \operatorname{char} D$, *then any $D^\bullet$-stable ideal of $K[V]$ is semiprime.*

Proof. If $\operatorname{char} D = p > 0$, then V is an elementary abelian p-group. In particular, if $\operatorname{char} K \neq p$, then we know that $K[V]$ is a commutative von Neumann regular ring. Hence every ideal of $K[V]$ is semiprime.

On the other hand, if $\operatorname{char} D = 0$, then we must have $\operatorname{char} K = q > 0$ for some prime q. Let I be a $D^\bullet$-stable ideal of $K[V]$ and suppose by way of contradiction that $\sqrt{I} > I$. Then we can choose an element $\alpha \in \sqrt{I} \setminus I$ of minimal support size, say $n+1$. Thus $\alpha = k_0x_0 + k_1x_1 + \cdots + k_nx_n$, with $x_0, x_1, \ldots, x_n \in V$ and with $k_0, k_1, \ldots, k_n \in K \setminus 0$. Without loss of generality, we may assume that $k_0 = 1$. Since $\alpha \in \sqrt{I}$ is nilpotent modulo I, we can suppose that $\alpha^{q^s} \in I$ for some integer $s \geq 0$. Of course $\alpha^{q^s} = k_0^{q^s}x_0^{q^s} + k_1^{q^s}x_1^{q^s} + \cdots + k_n^{q^s}x_n^{q^s}$.

Now $\operatorname{char} D = 0$ so $D^\bullet \supseteq Q^\bullet$, where Q is the field of rational numbers, and hence $1/q^s \in D^\bullet$. Thus $d = 1/q^s$ acts on V by taking the unique q^sth root of each element in this uniquely divisible group, and d acts trivially on the field K. Since $\alpha^{q^s} \in I$ and I is d-stable, we see that $\beta = (\alpha^{q^s})^d \in I$ and $\beta = k_0^{q^s}x_0 + k_1^{q^s}x_1 + \cdots + k_n^{q^s}x_n$. Obviously $\operatorname{supp}\alpha = \operatorname{supp}\beta$, and note that $k_0^{q^s} = k_0$ since $k_0 = 1$. Thus $\alpha - \beta$ has support size $\leq n$, and $\alpha - \beta \equiv \alpha \mod I$. In particular, $\alpha - \beta \in \sqrt{I} \setminus I$, contradicting the minimality of n. We conclude that $\sqrt{I} = I$, as required. □

If $\operatorname{char} K = \operatorname{char} D$, then there are certainly $D^\bullet$-stable ideals of $K[V]$ that are not semiprime. In particular, the semiprime hypothesis in the following key result applies only to those situations.

Theorem 2.2. *Let D be an infinite division ring and let V be a finite-dimensional right D-vector space. Furthermore, let $G = D^\bullet$ act on V, by right multiplication, and hence on the group algebra $K[V]$. Then every G-stable semiprime ideal of $K[V]$ can be written in a unique manner as a finite irredundant intersection $\bigcap_{i=1}^{k} \omega K[A_i]\cdot K[V]$ of augmentation ideals, where each A_i is a D-subspace of V. As a consequence, the set of these G-stable semiprime ideals is Noetherian.*

The proof of this result is contained in a series of papers. To start with, [BE] handles $D = Q$, the field of rational numbers, with a proof using valuation-theoretic techniques reminiscent of the arguments in [Bg], but somewhat more subtle. Next, [PZ1] handles infinite locally finite fields F, building up the result from the finite field case using the usual infinite paths in trees and compactness properties. Note that the result in the finite field case differs from that given in Theorem 2.2. On the one hand, we have

Lemma 2.3. *Let F be a finite field and let V be a finite-dimensional F-vector space, viewed multiplicatively. Assume that $\operatorname{char} F \neq \operatorname{char} K$, and let $G = F^\bullet$ act on V. Then every G-stable ideal of $K[V]$ contained in $\omega K[V]$ is a finite intersection of augmentation ideals $\omega K[A]\cdot K[V]$ with A an F-subspace of V.*

On the other hand, unlike the infinite case, when F is finite there are $F^\bullet$-stable ideals of $K[V]$ not contained in $\omega K[V]$. Furthermore, the ideals contained in $\omega K[V]$ are not uniquely writable as finite irredundant intersections of augmentation ideals. Indeed, it is precisely this failure of uniqueness that causes much of the difficulty in the work of [PZ1]. Finally, [OPZ] handles arbitrary division rings via going-up and going-down type results. Specifically, it is shown that if D has an infinite central subfield satisfying the conclusion of Theorem 2.2, then the same is true of D. Furthermore, if D is an infinite subdivision ring of E and if E satisfies the conclusion of Theorem 2.2, then so does D. As an immediate consequence of this key result, we have

Corollary 2.4. *Let F be an infinite field and suppose V is a finite-dimensional F-vector space. If $\operatorname{char} F \neq \operatorname{char} K$, then $\omega K[V]$ is the unique proper $\mathrm{GL}(V)$-stable ideal of $K[V]$. In the remaining cases, when $\operatorname{char} F = \operatorname{char} K$, then $\omega K[V]$ is at least the unique proper $\mathrm{GL}(V)$-stable semiprime ideal of $K[V]$.*

As was mentioned previously, the semiprime hypothesis is definitely required in Theorem 2.2. Furthermore, the conclusion that I is a finite intersection of augmentation ideals cannot be replaced by I being a finite product of such ideals. Indeed, we have

Example 2.5. *Suppose $\operatorname{char} D = \operatorname{char} K$ and let V be any right D-vector space having a proper D-subspace B. Then $I = \omega K[B]\cdot K[V] + \omega K[V]^2$ is a $D^\bullet$-stable ideal of $K[V]$ that is neither a finite intersection nor a finite product of augmentation ideals $\omega K[A_i]\cdot K[V]$, with each A_i a D-subspace of V.*

This example and Lemma 1.1 are both contained in [P]. We close this section by briefly describing the relevant structure when V is an infinite-dimensional D-vector space. As will be apparent, there is less precise information here, and in fact the best we can do is

Theorem 2.6. *Let D be an infinite division ring and let V be a right D-vector space of arbitrary dimension. If $G = D^\bullet$ acts on V, then every G-stable semiprime ideal of $K[V]$ can be written as an intersection $\bigcap_i \omega K[A_i]\cdot K[V]$ of augmentation* *ideals, where each A_i is a D-subspace of V. In particular, every such proper ideal is contained in $\omega K[V]$.*

Again, the semiprime assumption can be dropped when $\operatorname{char} D \neq \operatorname{char} K$.

3. Rational Representations of Groups of Lie Type

Corollary 2.4 offers an example of a group $\mathrm{GL}(V)$ of Lie type acting rationally and irreducibly on the vector space V. However, our main concern here is with somewhat smaller groups like $\mathrm{SL}(V)$. Recall that a group G is said to be *quasi-simple* if $G/\mathbb{Z}(G)$ is a nonabelian simple group and if G is equal to its commutator subgroup $[G,G]$. For example, if V is a finite-dimensional vector space over an infinite field F with $\dim_F V \geq 2$, then $\mathrm{PSL}(V)$ is simple and hence $\mathrm{SL}(V)$ is quasi-simple. Such groups of Lie type do not contain subgroups that act like the multiplicative group $F^\bullet$ on V, but they do contain subgroups, the maximal tori, which act appropriately at least on certain subspaces of V. Indeed, this is the basic idea for an attack that was suggested in the paper [HZ2]. We start with some definitions and recall some old results.

Let V be an arbitrary group and let H act as automorphisms on V. Then H is said to act in a *unipotent* or *unitriangular* fashion on V if there exists a finite subnormal chain $1 = V_0 \triangleleft V_1 \triangleleft \cdots \triangleleft V_t = V$ of H-stable subgroups such that H acts trivially on each quotient V_{i+1}/V_i. The following two lemmas are slight variants of results in [RS] and [Z1], respectively.

Lemma 3.1. *Let H act in a unitriangular manner on V, and let $J \subseteq I$ be H-stable ideals of the group ring $K[V]$. If $I \neq J$, then there exists an element $\alpha \in I \setminus J$ such that H centralizes α modulo J.*

Proof. Let $1 = V_0 \triangleleft V_1 \triangleleft \cdots \triangleleft V_t = V$ describe the unitriangular action of H on V. We proceed by induction on t, the result being clear for $t = 0$ since $K[1] = K$. Assume the result holds for $t-1$, and choose $\gamma \in I \setminus J$ so that $\operatorname{supp}\gamma$ meets the minimal number, say $n+1$, of cosets of V_{t-1}. By replacing γ by γy^{-1} for some $y \in \operatorname{supp}\gamma$ if necessary, we can assume that $1 \in \operatorname{supp}\gamma$. Thus we can write $\gamma = \gamma_0 + \gamma_1 x_1 + \cdots + \gamma_n x_n$ with $0 \neq \gamma_i \in K[V_{t-1}]$ and with $1, x_1, \ldots, x_n$ in distinct cosets of V_{t-1}. Now define

$$I' = \{\alpha_0 \mid \alpha = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n \in I \text{ with } \alpha_i \in K[V_{t-1}]\}, \text{ and}$$
$$J' = \{\beta_0 \mid \beta = \beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n \in J \text{ with } \beta_i \in K[V_{t-1}]\}.$$

Then I' and J' are ideals of $K[V_{t-1}]$, since $V_{t-1} \triangleleft V$, and $I' \supseteq J'$. Furthermore, since H acts trivially on V/V_{t-1}, we see that I' and J' are H-stable. Note also that in the above notation, if $\alpha_0 = \beta_0$, then $\alpha - \beta$ is an element of I whose support meets at most n cosets of V_{t-1}. Thus the minimality of $n+1$ implies that $\alpha - \beta \in J$ and hence that $\alpha \in J$. In particular, it now follows that $\gamma_0 \in I' \setminus J'$. By induction, there exists an element $\alpha_0 \in I' \setminus J'$ centralized modulo J' by H, and let $\alpha = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$ be its corresponding element in I. Then $\alpha_0 \notin J'$ implies that $\alpha \notin J$. Furthermore, if $g \in H$, then $\alpha^g - \alpha$ has its 0-term in J'. Thus, by the above remarks, $\alpha^g - \alpha \in J$, and hence H centralizes α modulo J. □

Again, let V is an arbitrary group and I is an ideal of the group algebra $K[V]$. If U is a normal subgroup of V, then $(I \cap K[U]) \cdot K[V]$ is an ideal of $K[V]$, and we say that U *controls* I whenever $I = (I \cap K[U]) \cdot K[V]$. In other words, this occurs precisely when $I \cap K[U]$ contains generators for I. As is well-known, there exists a unique normal subgroup $\mathcal{C}(I)$, called the *controller* of I, with the property that $U \triangleleft V$ controls I if and only if $U \supseteq \mathcal{C}(I)$. In particular, if U_1 and U_2 control I, then so does their intersection $U_1 \cap U_2$. Furthermore, if H acts on V and stabilizes I, then H stabilizes $\mathcal{C}(I)$.

Lemma 3.2. *Let H act in a unitriangular manner on the arbitrary group V with $Z = \mathbb{C}_V(H) \triangleleft V$, and let I be an H-stable ideal of $K[V]$. If I is not controlled by Z, then there exists an element $\alpha \in K[Z] \setminus (I \cap K[Z])$ and an element $v \in V \setminus Z$ having only finitely many H-conjugates modulo Z, such that $\alpha \cdot \omega K[T] \subseteq I \cap K[Z]$, where $T = \{v^{x-1} \mid x \in \mathbb{N}_H(vZ)\}$ is a subgroup of Z. Furthermore, for any $x, y \in \mathbb{N}_H(vZ)$, we have $v^{xy-1} = v^{x-1} \cdot v^{y-1}$.*

As an immediate consequence of Lemma 3.1 and the work of the preceding section, we have

Proposition 3.3. *Let D be an infinite division ring and suppose that V is a finite-dimensional D-vector space. If $\operatorname{char} D \neq \operatorname{char} K$, then $\omega K[V]$ is the unique proper $\mathrm{GL}(V)$-stable ideal of $K[V]$. If, in addition, $D = F$ is a field and $\dim_F V \geq 2$, then $\omega K[V]$ is the unique proper $\mathrm{SL}(V)$-stable ideal of $K[V]$.*

Proof. Write $\mathfrak{G} = \mathrm{GL}(V)$ or $\mathrm{SL}(V)$ as a group of matrices and let $\mathfrak{P}$ be the subgroup of $\mathfrak{G}$ consisting of all upper triangular matrices with diagonal entries equal to 1. Then $\mathfrak{P}$ acts in a unitriangular manner on V with $\mathbb{C}_V(\mathfrak{P}) = Z \cong D^+$. Suppose $I \neq 0$ is a $\mathfrak{G}$-stable ideal of $K[V]$. Then, by Lemma 3.1 with $J = 0$ and $H = \mathfrak{P}$, there exists a nonzero element $\alpha \in I$ that is centralized by $\mathfrak{P}$. Since D is infinite, it is easy to see that $\alpha \in K[Z]$ and hence $I \cap K[Z]$ is a nonzero ideal of $K[Z]$. Next, let $\mathfrak{T}$ be the subgroup of $\mathfrak{G}$ consisting of diagonal matrices. Then $\mathfrak{T}$ normalizes Z and hence it stabilizes $I \cap K[Z]$. Furthermore, since $\dim_F V \geq 2$ when $\mathfrak{G} = \mathrm{SL}(V)$, we see that $\mathfrak{T}$ acts on Z as $D^\bullet$ acts on D^+. We therefore conclude from Theorem 2.2 that $I \cap K[Z] = \omega K[Z]$ and, since $\mathfrak{G}$ is transitive on the nonidentity elements of V, the result follows. □

To proceed further, we restrict our attention to locally finite groups. In other words, we let $\mathfrak{G}$ be a quasi-simple group of Lie type defined over an infinite locally finite field F of characteristic $p > 0$. If F is algebraically closed, then $\mathfrak{G}$ is an algebraic group and the structure of $\mathfrak{G}$ and its rational irreducible representations is given, for example, in [St]. On the other hand, if F is just an arbitrary locally finite field, then $\mathfrak{G}$ is the direct limit of groups of the same Lie type defined over finite subfields of F. Thus we can again obtain information about $\mathfrak{G}$ and its rational representations by lifting the known results on finite groups contained in [St]. In particular, $\mathfrak{G}$ has a Sylow p-subgroup $\mathfrak{P}$, playing the role of the group of upper triangular matrices in the preceding argument, and $\mathbb{N}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P} \rtimes \mathfrak{T}$ where $\mathfrak{T}$ is the analog of a maximal torus. Furthermore, if V is a finite-dimensional irreducible $\mathfrak{G}$-module, then $\mathbb{C}_V(\mathfrak{P}) = V_0$ is a one-dimensional subspace of V on which $\mathfrak{T}$ acts via the homomorphism $\eta \colon \mathfrak{T} \to \mathrm{GL}(V_0)$.

Suppose that V is a vector space over the field E of characteristic $p > 0$. Since we want V to contain no proper $\mathfrak{G}$-stable subgroup, it follows that the representation $\phi \colon \mathfrak{G} \to \mathrm{GL}(V)$ cannot be realizable over a smaller field. In particular, since $\mathfrak{G}$ is a locally finite group and $\operatorname{char} E > 0$, we know that E must be the field generated by $\chi(\mathfrak{G})$, the character values of $\mathfrak{G}$. Furthermore, as was shown in [PZ2], E is also generated by $\eta(\mathfrak{T})$, when we identify $\mathrm{GL}(V_0)$ with $E^\bullet$, and hence $E \subseteq F$. In addition, when ϕ is a rational representation of $\mathfrak{G}$, then the latter paper offers a rough description of $\eta(\mathfrak{T})$, sufficient to handle the problem at hand. For the most part, if $\mathfrak{G}$ is a group of rank n, then $\eta(\mathfrak{T}) = \{x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \mid x_1, x_2, \ldots, x_n \in F^\bullet\}$ where $a_1, a_2, \ldots, a_n$ are nonnegative integers depending upon the weight of the representation. Of course, in some of the twisted cases, $\eta(\mathfrak{T})$ also involves the known field automorphisms which define $\mathfrak{G}$. Thus, for the most part, $E = F$ and $|F^\bullet : \eta(\mathfrak{T})| < \infty$. With this, and some additional work to handle the specific field automorphisms, the arguments of Proposition 3.3 can be extended to yield the main result of [PZ2], namely

Theorem 3.4. *Let $\mathfrak{G}$ be a quasi-simple group of Lie type defined over an infinite locally finite field F of characteristic $p > 0$ and let V be a finite-dimensional vector space over a field E of the same characteristic with $\dim_E V \geq 2$. Let $\phi\colon \mathfrak{G} \to \mathrm{GL}(V)$ be a rational irreducible representation, and assume that E is generated by $\chi(\mathfrak{G})$, the character values of $\mathfrak{G}$ associated with ϕ. If K is a field of characteristic different from p, then $\omega K[V]$ is the unique proper $\mathfrak{G}$-stable ideal of the group algebra $K[V]$.*

4. Polynomial Forms

The remainder of this survey is concerned with the contribution of paper [P] to this problem. As we will see later on, the nonrational irreducible representations of groups of Lie type involve arbitrary field automorphisms. For example, if $\mathfrak{G}$ is as in the preceding discussion, then $\eta(\mathfrak{T}) = \{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \mid x_1, x_2, \ldots, x_n \in F^\bullet\}$ where each α_i is a sum of field automorphisms. Thus, it is necessary to study functions of the form $\theta\colon x \mapsto x^{\sigma_1}x^{\sigma_2}\cdots x^{\sigma_k}$ and we first work in the more general context of division rings D of finite characteristic $p > 0$. In particular, θ is a map from D to D, and we let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be endomorphisms of this ring. We begin with a rather general result.

Proposition 4.1. *Let V be a right D-vector space and let G be a subgroup of the multiplicative group $D^\bullet$. Then G acts as automorphisms on V, by right multiplication, and hence on the group algebra $K[V]$ with $\operatorname{char} K \neq \operatorname{char} D$.*

i. If $G \cap X \neq \emptyset$ for every subgroup X of finite index in the additive group D^+, then every nontrivial G-stable ideal of $K[V]$ is contained in $\omega K[V]$.
ii. Suppose $G \cap (X + a) \neq \emptyset$ for every subgroup X of finite index in D^+ and for every element $a \in D$. If $V = D$ is one-dimensional, then $\omega K[V]$ is the unique proper G-stable ideal of $K[V]$.

Proof of the first part. By extending the field, if necessary, we can assume that K is algebraically closed. Suppose, by way of contradiction, that I is a proper G-stable ideal of $K[V]$ not contained in $\omega K[V]$. Since V is an elementary abelian p-group, there exists a finite subgroup A of V with $I \cap K[A] \not\subseteq \omega K[A]$, and hence the ideal structure of $K[A]$ implies that $e_A \in I$, where e_A is the principal idempotent of $K[A]$. Furthermore, since I is G-stable, we have $(e_A)^g \in I$ for all $g \in G$.

Recall that $K[V]$ is a commutative von Neumann regular ring, and hence so is $K[V]/I$. In particular, $K[V]/I$ is semiprimitive and consequently there exists an irreducible representation Λ of $K[V]$ with $\Lambda(I) = 0$. Since K is algebraically closed, it is easy to see that $\Lambda\colon K[V] \to K$ and that Λ is determined by a linear character $\lambda\colon V \to K^\bullet$. Furthermore, if $L = \ker\lambda$, then $|V : L| \leq p < \infty$. Now define the residual $L_A \subseteq D$ by $L_A = \{d \in D \mid Ad \subseteq L\}$, so that $L_A = \bigcap_{a\in A} L_a$

where $L_a = \{d \in D \mid ad \in L\}$. Since L_a is the kernel of the additive group homomorphism $D \to V/L$ given by $d \mapsto ad + L$, we see that L_a is a subgroup of D^+ of finite index. Since $|A| < \infty$, it follows that $|D^+ : L_A| < \infty$.

Finally, let $g \in G$ and note that $(e_A)^g = e_{A^g} = e_{Ag}$ is the principal idempotent of the subgroup $A^g = Ag$ of V. Furthermore, we know that $e_{Ag} \in I \subseteq \ker \Lambda$ and therefore the restriction of $\lambda\colon V \to K^\bullet$ to Ag cannot be the principal character. In other words, $Ag \not\subseteq \ker \lambda = L$ and, by definition, this says that $g \notin L_A$. We have therefore shown that $G \cap L_A = \emptyset$ and this contradicts the assumption that $G \cap X \neq \emptyset$ for all subgroups X of D^+ of finite index. $\square$

The proof of part (ii) is similar, but more subtle, since we have to deal with nonprincipal idempotents. Specifically, we show that if I is a nonzero G-stable ideal of $K[V]$ properly smaller than $\omega K[V]$, then there exist two distinct finite subgroups $A \subseteq B$ of V and a nonprincipal linear character $\lambda\colon V \to K^\bullet$ with $\ker \lambda = L$ such that G is disjoint from $L_A \setminus L_B$. But why must L_A be properly larger than L_B? The answer is that these residuals can in fact be equal if $\dim_D V \geq 2$. However, when $\dim_D V = 1$, then we know from Theorem 2.2 that no such ideal I can exist for $G = D^\bullet$. Thus, since the above condition can be shown to be equivalent to the existence of I, we see that $D^\bullet$ cannot be disjoint from $L_A \setminus L_B$, and therefore L_A must be properly larger than L_B, as required. At present, part (ii) does not seem to have applications to the problem at hand. But part (i) can be used effectively. We first need some definitions.

Let $\mathfrak{Z}$ be a ring, let A be an infinite left $\mathfrak{Z}$-module and let S be a finite abelian group. For convenience, let $\mathcal{I}(A)$ denote the set of all infinite $\mathfrak{Z}$-submodules of A. We say that a (not necessarily linear) function $f\colon A \to S$ is *eventually null* if every infinite submodule B of A contains an infinite submodule C with $f(C) = 0$. Obviously the zero function is eventually null and so also is any group homomorphism whose kernel is a $\mathfrak{Z}$-submodule. Indeed, in the latter situation, the finiteness of S implies that $f^{-1}(0)$ is a submodule of finite index in A. It is, of course, easy to see that a finite sum of eventually null functions is eventually null.

We are concerned with functions which are called *polynomial forms* on A. By definition, a polynomial form of degree 0 is the zero function, and for $n \geq 1$, we say that $f\colon A \to S$ is a polynomial form of degree $\leq n$ if and only if:

i. $f(a) = 0$ implies that $f(\mathfrak{Z}a) = 0$.
ii. For each $a \in A$, the function $g_a(x) = f(a+x) - f(a) - f(x)$ is a finite sum of polynomial forms of degree $\leq n-1$.

It is clear from (ii) above that the polynomial forms of degree ≤ 1 are precisely the group homomorphisms from A to S whose kernels are $\mathfrak{Z}$-submodules of A. Now suppose that R is an infinite ring, fix $r_0, r_1, \ldots, r_n \in R$ and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be endomorphisms of R. Furthermore, let $\mathfrak{Z}$ be a central subring of R stable under each σ_i, and let $A = R^+$ denote the additive subgroup of R so that A is naturally a $\mathfrak{Z}$-module. If $\mu\colon A \to S$ is a group homomorphism whose kernel is a $\mathfrak{Z}$-submodule of A, then the map $f\colon A \to S$ given by $f(x) = \mu(r_0 x^{\sigma_1} r_1 x^{\sigma_2} r_2 \cdots r_{n-1} x^{\sigma_n} r_n)$ is certainly a polynomial form on A of degree $\leq n$. The key result here is

Theorem 4.2. *Let A be an infinite $\mathfrak{Z}$-module, let S be a finite abelian group, and let $f\colon A \to S$ be a finite sum of polynomial forms. Then f is eventually null.*

It is easy to see by example that $f(A)$ need not be a subgroup of S. Furthermore, A need not have a submodule B of finite index with $f(B) = 0$. Again, let $\mathfrak{Z}$ be an arbitrary ring and let $f\colon A \to S$ be a polynomial form. Choose $f(B)$

to have minimum size over all submodules B of finite index in A. Then for any submodule C of finite index in A, we have $f(C) \supseteq f(C \cap B) = f(B)$ since $C \cap B$ is a submodule of B having finite index in A. In other words, $f(B)$ is the unique minimum value over all such C, and we call $f(B)$ the *final value* of f. In view of the preceding comments, it would be interesting to know whether the final value of a polynomial form f is necessarily a subgroup of S.

We can now apply the preceding results to some particular groups G of interest. Specifically, let D be an infinite characteristic p division ring, and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be $n \geq 1$ endomorphisms of D. Furthermore, fix a nonzero element $d \in D$ and consider the map

$$\theta \colon x \mapsto d{\cdot}x^{\sigma_1} x^{\sigma_2} \cdots x^{\sigma_n}$$

from D to D. Note that there no gain in considering more general product expressions for θ like $d_0 x^{\sigma_1} d_1 x^{\sigma_2} d_2 \cdots d_{n-1} x^{\sigma_n} d_n$ with $0 \neq d_i \in D$. Indeed, if $0 \neq d \in D$ and if σ is an endomorphism of D, then $x^\sigma d = dd^{-1}x^\sigma d = dx^{\sigma d}$. Thus each of the d_i factors above could be moved to the left at the expense of multiplying each σ_i by a suitable inner automorphism.

Let us return to the given map θ and observe that $\theta(D^\bullet) \subseteq D^\bullet$. If A is any infinite subgroup of D^+, we let $G = G(A) = \langle \theta(A^\bullet) \rangle$ be the subgroup of $D^\bullet$ generated by $\theta(A^\bullet)$. Of course, G acts as automorphisms on any right D-vector space V by right multiplication, and hence G acts as automorphisms on any group algebra $K[V]$. The following result is proved in [P].

Theorem 4.3. *Let D, V, $G = G(A)$, and K be as above with A an infinite subgroup of D^+ and with $\operatorname{char} D = p > 0$. If $\operatorname{char} K \neq p$, then all proper G-stable ideals of the group algebra $K[V]$ are contained in the augmentation ideal $\omega K[V]$. Furthermore, $\theta(A^\bullet)$ and $G(A)$ are infinite.*

Proof. By Proposition 4.1(i), it suffices to show that $G \cap X \neq \emptyset$ for all additive subgroups X of D^+ of finite index. To this end, let X be given and consider the function $f \colon A \to D^+/X$ given by $f = \mu\theta$ where $\mu \colon D^+ \to D^+/X$ is the natural epimorphism. Then f is a polynomial form with A viewed as a module over $\mathrm{GF}(p)$. Therefore, Theorem 4.2 implies that f is eventually null and consequently there exists an infinite subgroup B of A with $f(B) = 0$. By definition of f, this implies that $\theta(B^\bullet) \subseteq X$ and hence that $G \cap X \neq \emptyset$. □

5. Representations and Field Automorphisms

It remains to discuss the nonrational finite-dimensional representations of locally finite groups of Lie type. For convenience, we list the properties of the representation $\phi \colon \mathfrak{G} \to \mathrm{GL}(V)$ that are needed for the proof. As is to be expected, two possibly different fields come into play here. Indeed, F is the field of definition of the group $\mathfrak{G}$, while E is the field of character values.

Hypothesis 5.1. *Let $F \supseteq E$ be infinite locally finite fields of characteristic $p > 0$, let V be a finite-dimensional E-vector space with $\dim_E V \geq 2$, and let $\mathfrak{G}$ be a group that acts on V by way of the homomorphism $\phi \colon \mathfrak{G} \to \mathrm{GL}(V)$. Assume that*

i. $\mathfrak{P}$ is a p-subgroup of $\mathfrak{G}$ with $\mathbb{C}_V(\mathfrak{P}) = Ev_0$, and $Ev_0{\cdot}\phi(\mathfrak{G}) = V$.
ii. $\mathfrak{T}$ is a subgroup of $\mathfrak{G}$ that stabilizes the line Ev_0, and the action of $\mathfrak{T}$ on this line factors through its homomorphic image $\bar{\mathfrak{T}} = (F^\bullet)^t = F^\bullet \times F^\bullet \times \cdots \times F^\bullet$. Indeed, for each $i = 1, 2, \ldots, t$, there exists a function $\theta_i \colon F \to E \subseteq F$

given by $\theta_i(x_i) = x_i^{\sigma_{i,1}} x_i^{\sigma_{i,2}} \cdots x_i^{\sigma_{i,n_i}}$ *with*

$$v_0 \cdot \bar{\phi}(x_1, x_2, \ldots, x_t) = \theta_1(x_1)\theta_2(x_2) \cdots \theta_t(x_t) v_0$$

for all $(x_1, x_2, \ldots, x_t) \in \bar{\mathfrak{T}}$*. Here* $\bar{\phi}$ *denotes the induced action of* $\bar{\mathfrak{T}}$ *on* v_0*, each* $n_i \geq 1$*, and each* $\sigma_{i,j}$ *is a field automorphism of* F*. In addition,* E *is the linear span of the product* $\theta_1(F^\bullet)\theta_2(F^\bullet) \cdots \theta_t(F^\bullet)$*.*

iii. $\mathfrak{P}$ *is generated by one-parameter subgroups* $\mathfrak{P}_g = \{g_x \mid x \in F\}$ *such that the matrix entries of* $\phi(g_x)$ *are all* F*-linear sums of expressions of the form* $x^{\kappa_1} x^{\kappa_2} \cdots x^{\kappa_m}$*, where the* κ_i *are automorphisms of* F *and* $m \geq 0$*. Of course, these entries are contained in* E*, and* $g_x \cdot g_y = g_{x+y}$ *for all* $x, y \in F$*.*

Note that $\mathfrak{P}$ plays the role of a Sylow p-subgroup of $\mathfrak{G}$ and, because it is a p-group, it necessarily acts in a unitriangular manner on V. Thus $\mathfrak{P}$ must have nonzero fixed points in V, but the fact that these fixed points consist of just one line Ev_0 is an additional necessary assumption. Next, we see that $\mathfrak{T}$ is the analog of a maximal torus, presumably in the normalizer of $\mathfrak{P}$. In any case, we know that it acts on the line Ev_0 via homomorphisms from $F^\bullet$ to $E^\bullet$ given by products of field automorphisms. Of course, we expect $\mathfrak{P}$ to be generated by one-parameter subgroups, but here the action of these subgroups is no longer rational, but rather involves sums and products of field automorphisms. One technique for dealing with such expressions is to view products of field automorphisms as linear characters.

To this end, let $\mathfrak{F}$ be a finite subfield of F and let $\kappa_1, \kappa_2, \ldots, \kappa_m$ be $m \geq 0$ field automorphisms of F. If $\xi\colon F \to F$ is given by $\xi(x) = x^{\kappa_1} x^{\kappa_2} \cdots x^{\kappa_m}$, then certainly $\xi(\mathfrak{F}) \subseteq \mathfrak{F}$ and $\xi\colon \mathfrak{F}^\bullet \to \mathfrak{F}^\bullet$ is a multiplicative homomorphism from $\mathfrak{F}^\bullet$ to a field. In other words, ξ is a linear character of the group. For convenience, if χ and ξ are such linear characters of $\mathfrak{F}^\bullet$, then we use $[\chi, \xi] = \sum_{y \in \mathfrak{F}^\bullet} \chi(y)\xi(y^{-1})$ to denote the unnormalized character inner product. Certainly, character orthogonality implies that $[\chi, \xi] = 0$ if $\chi \neq \xi$, while $[\chi, \chi] = |\mathfrak{F}^\bullet| = |\mathfrak{F}| - 1 \equiv -1 \mod p$. Of course, the inner product $[\ ,\ ]$ extends by linearity to a function on sums of characters. Here is a sample argument.

Lemma 5.2. *Let* F *be an infinite locally finite field and let* $\Psi(x)\colon F \to F$ *be a map that can be written as a finite* F*-linear combination of functions of the form* $\xi(x) = x^{\kappa_1} x^{\kappa_2} \cdots x^{\kappa_m}$*, where each* κ_i *is a field automorphism and* $m \geq 0$*. Assume, in addition, that* $\Psi(0) = 0$*.*

i. *If* $\Psi(F)$ *is finite, then* $\Psi(F) = 0$*.*

ii. *If* $\Psi(x)$ *is an additive map with* $\Psi(F) \neq 0$*, then* F *is a finite sum of* F*-translates of its additive subgroup* $\Psi(F)$*, that is* $F = \sum_{k=1}^{n} b_k \Psi(F)$ *for suitable field elements* $b_1, b_2, \ldots, b_n$*.*

Proof. Say $\Psi(x) = \sum_{i=1}^{t} a_i \xi_i(x)$ with $a_i \in F$. By combining terms if necessary, we can clearly assume that, as functions, the $\xi_i(x)$ are all distinct. In particular, there is at most one $\xi_i(x)$ given by the empty product, and then, since $\Psi(0) = 0$, this term cannot appear in $\Psi(x)$. In other words, $\xi_i(0) = 0$ for all i and hence, since these functions are distinct, then differ on nonzero elements. It follows that there exists a finite subfield $\mathfrak{F} \subseteq F$ such that the various $\xi_i(x)$ give rise to distinct linear characters $\xi_i\colon \mathfrak{F}^\bullet \to \mathfrak{F}^\bullet$.

For each $y \in \mathfrak{F}^\bullet$, define $\Psi_y(x) = \Psi(y^{-1}x)$ for all $x \in F$. Then

$$\Psi_y(x) = \sum_{i=1}^{t} a_i\xi_i(y^{-1}x) = \sum_{i=1}^{t} a_i\xi_i(y^{-1})\xi_i(x).$$

Thus, for any fixed subscript j, we have

$$\sum_{y\in\mathfrak{F}^\bullet} \xi_j(y)\Psi_y(x) = \sum_{i=1}^{t}\sum_{y\in\mathfrak{F}^\bullet} \xi_j(y)\xi_i(y^{-1})\cdot a_i\xi_i(x) = \sum_{i=1}^{t}[\xi_j,\xi_i]\cdot a_i\xi_i(x)$$

where $[\xi_j,\xi_i]$ denotes the unnormalized character inner product. In particular, character orthogonality yields

$$(*) \qquad \sum_{y\in\mathfrak{F}^\bullet} \xi_j(y)\Psi_y(x) = -a_j\xi_j(x) \qquad \text{for all } x \in F.$$

(i) If $\Psi(F)$ is finite, then each $\Psi_y(F)$ is finite and hence, by the above, $a_j\xi_j(F)$ is finite. But $\xi_j(F)$ is infinite, by Theorem 4.3, and hence we must have $a_j = 0$ for all j. In other words, $\Psi(F) = 0$.

(ii) Now suppose that $\Psi(x)$ is additive with $\Psi(F) \neq 0$, and let the subscript j be chosen with $a_j \neq 0$. Since each $\Psi_y(x)$ is clearly also an additive function, it follows from $(*)$ that $a_j\xi_j(x)$ is additive and hence so is $\xi_j(x)$. But $\xi_j(x)$ is given to be a multiplicative function, so it is an endomorphism of the locally finite field F, and hence an automorphism. In particular, $\xi_j(F) = F$, so $(*)$ yields

$$\sum_{y\in\mathfrak{F}^\bullet} \xi_j(y)\Psi(F) = \sum_{y\in\mathfrak{F}^\bullet} \xi_j(y)\Psi_y(F) \supseteq -a_j\xi_j(F) = F,$$

as required. □

The following is, in some sense, the main result of paper [P]. The argument here is quite different from that of Theorem 3.4.

Theorem 5.3. *Let $F \supseteq E$, V and $\mathfrak{G}$ satisfy Hypothesis 5.1, and let K be a field of characteristic different from p. Then the augmentation ideal $\omega K[V]$ is the unique proper $\mathfrak{G}$-stable ideal of the group algebra $K[V]$.*

Outline of the proof. Use the notation of Hypothesis 5.1, set $Z = Ev_0 \cong E^+$, and let I be a proper $\mathfrak{G}$-stable ideal of $K[V]$. We first note that I is not controlled by Z. Indeed, it follows from parts (i) and (ii) of the hypothesis that Z contains no nonidentity $\mathfrak{G}$-stable subgroup (see, for example, the last paragraph of the proof of Lemma 6.1). In particular, if $Z \supseteq \mathcal{C}(I)$ then $\mathcal{C}(I)$, being $\mathfrak{G}$-stable, must be the identity subgroup and consequently $I = 0$ or $K[V]$, a contradiction.

Next, since $\mathfrak{P}$ is a p-group, it acts in a unitriangular manner on V and hence, by Lemma 3.2, there exists an element $\alpha \in K[Z]\setminus(I \cap K[Z])$ and an element $v \in V\setminus Z$ having only finitely many $\mathfrak{P}$-conjugates modulo Z, such that $\alpha\cdot\omega K[T] \subseteq I \cap K[Z]$. Here $T = \{v^{x-1} \mid x \in \mathbb{N}_{\mathfrak{P}}(vZ)\}$ is a subgroup of Z and, for all $x, y \in \mathbb{N}_{\mathfrak{P}}(vZ)$, we have $v^{xy-1} = v^{x-1}\cdot v^{y-1}$. We study the configuration $\alpha\cdot\omega K[T] \subseteq I \cap K[Z]$.

The goal now is to show that T is actually a large subgroup of $Z \cong E^+$. To this end, choose a one-parameter subgroup $\mathfrak{P}_g$ of $\mathfrak{P}$ that does not centralize v. Since v has just finitely many $\mathfrak{P}_g$-conjugates modulo Z, part (iii) of the hypothesis and Lemma 5.2(i) easily imply that $\mathfrak{P}_g$ centralizes v modulo Z and hence $\mathfrak{P}_g$

normalizes the coset vZ. Furthermore, the various linearity conditions along with Lemma 5.2(ii) show that $T'a_1 + T'a_2 + \cdots + T'a_k = F^+$, where $T' = \{v^{x-1} \mid x \in \mathfrak{P}_g\}$ and where $a_1, a_2, \ldots, a_k$ are suitable elements of F. But $T' \subseteq T \subseteq E \subseteq F$, so we conclude that $Tb_1 + Tb_2 + \cdots + Tb_k = E^+$ for suitable elements $b_1, b_2, \ldots, b_k \in E$. Indeed, since E is the linear span of the product $\theta_1(F^\bullet)\theta_2(F^\bullet)\cdots\theta_t(F^\bullet)$, we can assume that each b_i is the v_0-eigenvalue of an element $y_i \in \mathfrak{T}$.

At this point, the action of $\mathfrak{T}$ comes into play in a completely different manner. Unfortunately, this argument is really quite technical, so we cannot discuss it here in full detail. Roughly, we assume that K is algebraically closed and we consider those irreducible representations $\Lambda\colon K[Z] \to K$ with $\Lambda(I \cap K[Z]) = 0$ and $\Lambda(\alpha) \neq 0$. There must, of course, be representations of this sort, and we show, with a good deal of work, that any such Λ is necessarily the principal representation of $K[Z]$. In particular, since $\Lambda(\alpha) \neq 0$, we now know that $\alpha \notin \omega K[Z]$.

Finally, let $\beta = \alpha^{\phi(y_1)}\alpha^{\phi(y_2)}\cdots\alpha^{\phi(y_k)}$ and note that $\beta \notin \omega K[Z]$ since the augmentation ideal is a prime ideal of $K[Z]$. Furthermore, $\alpha^{\phi(y_i)}\cdot\omega K[T^{\phi(y_i)}] \subseteq I \cap K[Z]$ and hence $\beta\cdot\omega K[Z] \subseteq I \cap K[Z]$, since $Z \cong E^+ = \sum_i Tb_i = \sum_i T^{\phi(y_i)}$. Now if $J = \{\gamma \in K[Z] \mid \gamma\cdot\omega K[Z] \subseteq I \cap K[Z]\}$, then J is certainly a $\mathfrak{T}$-stable ideal of $K[Z]$, and $J \nsubseteq \omega K[Z]$ since $\beta \in J$. Thus, Theorem 4.3 implies that $J = K[Z]$ and, in particular, that $1 \in J$. Consequently $\omega K[Z] \subseteq I \cap K[Z] \subseteq I$ and, since I is $\mathfrak{G}$-stable and $Z\phi(\mathfrak{G}) = V$, we conclude that $\omega K[V] \subseteq I$, as required. □

6. Nonrational Representations of Groups of Lie Type

At this point, it is necessary to apply a number of known results on the representation theory of groups of Lie type.

Lemma 6.1. *Let $\mathfrak{G}$ be a quasi-simple group of Lie type defined over an infinite locally finite field F of characteristic $p > 0$, and let W be a finite-dimensional F-vector space on which $\mathfrak{G}$ acts both nontrivially and absolutely irreducibly. Then there exists a subfield E of F of finite index and a $\mathfrak{G}$-stable E-subspace V of W such that*

i. V has no proper $\mathfrak{G}$-stable subgroup.
ii. $W = F \otimes_E V$.
iii. $F \supseteq E$, V and $\mathfrak{G}$ satisfy Hypothesis 5.1.

Proof. By results of [BT] and [HZ1], we have $W = W_1^{\sigma_1} \otimes_F W_2^{\sigma_2} \otimes_F \cdots \otimes_F W_k^{\sigma_k}$, where each W_i is a rationally irreducible $F[\mathfrak{G}]$-module and where each σ_i is a field automorphism. Now let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{G}$ and let $\mathfrak{T}$ be a maximal torus of $\mathfrak{G}$ contained in $\mathbb{N}_{\mathfrak{G}}(\mathfrak{P})$. As we have already mentioned, $\mathfrak{P}$ centralizes a unique line Fw_0 in W, and therefore $w_0 = w_1^{\sigma_1} \otimes w_2^{\sigma_2} \otimes \cdots \otimes w_k^{\sigma_k}$, where each w_i with $i \geq 1$ is a highest weight vector in W_i. Thus, by the description of the action of $\mathfrak{T}$ on each W_i given in [PZ2], we see that $\mathfrak{T}$ acts on Fw_0 by way of the functions $\theta_1, \theta_2, \ldots, \theta_t\colon F \to F$ of Hypothesis 5.1(ii). Furthermore, if E is the subfield of F spanned by the product $\theta_1(F^\bullet)\theta_2(F^\bullet)\cdots\theta_t(F^\bullet)$, then it follows fairly easily that $(F : E) < \infty$ and that E is the linear span of the product $\theta_1(E^\bullet)\theta_2(E^\bullet)\cdots\theta_t(E^\bullet)$.

Next, results of [PZ2] imply that E is equal to the field $\mathrm{GF}(p)[\chi]$ generated by all values $\chi(\mathfrak{G})$ of the group character $\chi\colon \mathfrak{G} \to F$ corresponding to ϕ. Thus, since $\mathfrak{G}$ is locally finite and $\operatorname{char} F = p > 0$, the representation associated with W is actually realizeable over E. In other words, there exists a $\mathfrak{G}$-stable E-subspace $V \subseteq W$ with $W = F \otimes_E V$. This proves (ii), and of course $\mathfrak{G}$ must act nontrivially and irreducibly on ${}_EV$ since it acts nontrivially and irreducibly on ${}_FW$.

Since $\mathfrak{P}$ is a p-group, it acts in a unitriangular manner on V and hence $\mathbb{C}_V(\mathfrak{P}) \neq 0$. Indeed, since $F \otimes_E \mathbb{C}_V(\mathfrak{P}) \subseteq \mathbb{C}_W(\mathfrak{P})$, it follows that $\mathbb{C}_V(\mathfrak{P})$ is a line Ev_0 in V and, without loss of generality, we can assume that $w_0 = v_0 \in V$. With this, we now understand the action of $\mathfrak{T}$ on Ev_0. Furthermore, as can be seen for example in [St], $\mathfrak{P}$ is generated by one-parameter subgroups $\mathfrak{P}_g$ determined by root vectors, and each such subgroup acts on each W_i via polynomial maps. Of course, in the twisted cases, the defining field automorphisms also come into play. Thus, since $F \otimes_E V = W = W_1^{\sigma_1} \otimes_F W_2^{\sigma_2} \otimes_F \cdots \otimes_F W_k^{\sigma_k}$, we can now conclude that $F \supseteq E$, V and $\mathfrak{G}$ satisfy Hypothesis 5.1.

Finally, suppose U is a nonzero $\mathfrak{G}$-stable subgroup of V. Then $0 \neq \mathbb{C}_U(\mathfrak{P}) \subseteq \mathbb{C}_V(\mathfrak{P}) = Ev_0$. Furthermore, $\mathfrak{T}$ acts on $\mathbb{C}_U(\mathfrak{P})$ and, since E is the linear span of the product $\theta_1(E^\bullet)\theta_2(E^\bullet)\cdots\theta_t(E^\bullet)$, it follows that $\mathbb{C}_U(\mathfrak{P}) = Ev_0$. As a consequence, we have $U \supseteq Ev_0 \cdot \phi(\mathfrak{G}) = V$, and the proof is complete. □

It is now a simple matter to bring all these ingredients together. Indeed, with just a bit more work on the representations of $\mathfrak{G}$, the preceding lemma and Theorem 5.3 combine to yield

Theorem 6.2. *Let $\mathfrak{G}$ be a quasi-simple group of Lie type defined over an infinite locally finite field F of characteristic $p > 0$, and let V be a finite-dimensional vector space over a characteristic p field E. Assume that $\mathfrak{G}$ acts nontrivially on V by way of the representation $\phi\colon \mathfrak{G} \to \mathrm{GL}(V)$, and that V contains no proper $\mathfrak{G}$-stable subgroup. If K is a field of characteristic different from p, then $\omega K[V]$ is the unique proper $\mathfrak{G}$-stable ideal of the group algebra $K[V]$.*

It follows easily from the celebrated result of [Be], [Bo], [HS] and [T] that any quasi-simple, infinite, locally finite linear group is a group of Lie type defined over an infinite locally finite field F of the same characteristic p. Thus, the preceding theorem yields

Corollary 6.3. *Let V be a finite-dimensional vector space over a field E of characteristic $p > 0$ and let $\mathfrak{G}$ be an infinite locally finite subgroup of $\mathrm{GL}(V)$. Assume that $\mathfrak{G}$ is quasi-simple and that it stabilizes no proper subgroup of V. If K is a field of characteristic different from p, then the augmentation ideal $\omega K[V]$ is the unique proper $\mathfrak{G}$-stable ideal of the group algebra $K[V]$.*

Finally, we return to the original problem of studying ideals in group algebras of locally finite abelian-by-simple groups. Recall that if V is a normal abelian subgroup of $\mathfrak{H}$, then $\mathfrak{H}/V$ acts on V by conjugation.

Corollary 6.4. *Let V be a finite-dimensional vector space over a field E of characteristic $p > 0$, and let V be a minimal normal abelian subgroup of the locally finite group $\mathfrak{H}$. Assume that $\mathfrak{H}/V$ is an infinite quasi-simple group that acts faithfully as an E-linear group on V. If K is a field of characteristic different from p and if I is a nonzero ideal of $K[\mathfrak{H}]$, then $I \supseteq \omega K[V] \cdot K[\mathfrak{H}]$ and hence I is the complete inverse image in $K[\mathfrak{H}]$ of an ideal of $K[\mathfrak{H}/V]$.*

Proof. Let $0 \neq I \triangleleft K[\mathfrak{H}]$ and suppose, by way of contradiction, that $I \cap K[V] = 0$. Note that V acts in a unitriangular manner on $\mathfrak{H}$ since it centralizes both V and $\mathfrak{H}/V$. Thus since $\mathbb{C}_{\mathfrak{H}}(V) = V \triangleleft \mathfrak{H}$ and since V does not control I, it follows from Lemma 3.2 that these exists $0 \neq \alpha \in K[V]$ and $h \in \mathfrak{H} \setminus V$ with $\alpha \cdot \omega K[T] = 0$ and with $T = [h, V]$, the commutator group determined by the action of h on V. But T is a nonzero E-subspace of V, so T is infinite, and hence $\alpha \cdot \omega K[T] = 0$ implies that $\alpha = 0$, a contradiction.

We now know that $I \cap K[V]$ is a nonzero $\mathfrak{H}/V$-stable ideal of $K[V]$. Furthermore, since V is a minimal normal subgroup of $\mathfrak{H}$, it is clear that $\mathfrak{G} = \mathfrak{H}/V$ stabilizes no proper subgroup of V. Consequently, Corollary 6.3 implies that $I \cap K[V] \supseteq \omega K[V]$, so $I \supseteq \omega K[V] \cdot K[\mathfrak{H}]$ and the result follows. □

In particular, any information on the lattice of ideals of $K[\mathfrak{H}/V]$ carries over immediately to information on the lattice of ideals of $K[\mathfrak{H}]$.

We remark, in closing, that there is still much to be done on variants of this particular question. Two problems which immediately spring to mind are: (1) Can the results on general polynomial forms or even on the special polynomial forms we consider be improved so that Proposition 4.1(ii) becomes applicable? It might help if one can prove that the final value of a polynomial form $f: A \to S$ is necessarily a subgroup of S. Of course, a positive solution here would lead to a more direct proof of Theorem 5.3. (2) Can Theorem 6.2 be extended to handle modules V that are completely reducible rather than just simple? In this case, one would hope for an answer analogous to that given in Theorem 2.2. At first glance, both of these problems appear to be quite difficult, but one never knows.

References

[Be] V. V. Belyaev, *Locally finite Chevalley groups* (in Russian), in "Studies in Group Theory", Urals Scientific Centre of the Academy of Sciences of USSR, Sverdlovsk, 1984, pp. 39–50.

[Bg] G. M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. AMS **157** (1971), 459–469.

[BHPS] K. Bonvallet, B. Hartley, D. S. Passman and M. K. Smith, *Group rings with simple augmentation ideals*, Proc. AMS **56** (1976), 79–82.

[BT] A. Borel and J. Tits, *Homomorphisms "abstraits" de groupes algebriques simples*, Ann. Math. **97** (1973), 499–571.

[Bo] A. V. Borovik, *Periodic linear groups of odd characteristic*, Soviet Math. Dokl. **26** (1982), 484–486.

[BE] C. J. B. Brookes and D. M. Evans, *Augmentation modules for affine groups*, Math. Proc. Cambridge Philos. Soc. **130** (2001), 287–294.

[HS] B. Hartley and G. Shute, *Monomorphisms and direct limits of finite groups of Lie type*, Quart. J. Math. Oxford (2) **35** (1984), 49–71.

[HZ1] B. Hartley and A. E. Zalesskiĭ, *On simple periodic linear groups - dense subgroups, permutation representations and induced modules*, Israel J. Math. **82** (1993), 299–327.

[HZ2] B. Hartley and A. E. Zalesskiĭ, *Group rings of periodic linear groups*, unpublished note (1995).

[HZ3] B. Hartley and A. E. Zalesskiĭ, *Confined subgroups of simple locally finite groups and ideals of their group rings*, J. London Math. Soc. (2) **55** (1997), 210–230.

[LP] F. Leinen and O. Puglisi, *Ideals in group algebras of simple locally finite groups of 1-type*, to appear.

[OPZ] J. M. Osterburg, D. S. Passman, and A. E. Zalesskiĭ, *Invariant ideals of abelian group algebras under the multiplicative action of a field, II*, Proc. AMS **130** (2002), 951–957.

[P] D. S. Passman, *Invariant ideals and polynomial forms*, Trans. AMS, **354** (2002), 3379-3408.

[PZ1] D. S. Passman and A. E. Zalesskiĭ, *Invariant ideals of abelian group algebras under the multiplicative action of a field, I*, Proc. AMS **130** (2002), 939–949.

[PZ2] D. S. Passman and A. E. Zalesskiĭ, *Invariant ideals of abelian group algebras and representations of groups of Lie type*. Trans. AMS **353** (2001), 2971–2982.

[RS] J. E. Roseblade and P. F. Smith, *A note on hypercentral group rings*, J. London Math. Soc. (2) **13** (1976), 183–190.

[St] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, 1968.

[T] S. Thomas, *The classification of simple linear groups*, Archiv der Mathematik **41** (1983), 103–116.

[Z1] A. E. Zalesskiĭ, *Intersection theorems in group rings* (in Russian), No. 395-74, VINITI, Moscow, 1974.

[Z2] A. E. Zalesskiĭ, *Group rings of inductive limits of alternating groups*, Leningrad Math. J. **2** (1990), 1287–1303.

[Z3] A. E. Zalesskiĭ, *A simplicity condition for the augmentation ideal of the modular group algebra of a locally finite group*, Ukrainian Math. J. **43** (1991), 1021–1024.

[Z4] A. E. Zalesskiĭ, *Group rings of simple locally finite groups*, in "Proceedings of the Istanbul NATO ASI Conference on Finite and Locally Finite Groups", Kluwer, Dordrecht, 1995, pp. 219–246.

D. S. Passman
Department of Mathematics, University of
Wisconsin, Madison, Wisconsin 53706
E-mail address: passman@math.wisc.edu