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Representations of toroidal Lie algebras and Lie algebras of vector fields

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Abstract: In this talk we will discuss recent progress in the representation theory of toroidal Lie algebras, as well as several open problems in the area.

Two types of infinite-dimensional Lie algebras have numerous important applications in various areas of mathematics and physics. These are affine Kac-Moody algebras, and the Virasoro Lie algebra.

Affine Kac-Moody algebras are the central extensions of the loop algebras with added outer derivation:

$$\mathbb{C}[t,t^{-1}] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Here $\dot{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra with the Killing form $(\cdot|\cdot)$ and $d = t\frac{d}{dt}$. The Lie bracket in the affine algebra is given as follows:

$$[t^r \otimes x, t^m \otimes y] = t^{r+m} \otimes [x,y] + r\delta_{r,-m}(x|y)c,$$

$$[d, t^r \otimes x] = rt^r \otimes x, \quad c - \text{central}.$$

The Virasoro Lie algebra is also constructed as a central extension:

$$\mathbb{C}[t,t^{-1}]d \oplus \mathbb{C}c, \quad d = t\frac{d}{dt},$$

with the bracket

$$[t^r d, t^m d] = (m-r)t^{r+m}d + \frac{r^3-r}{12}\delta_{r,-m}c.$$

If we consider a change of the variable $t = e^{ix}$, we may interpret an affine Lie algebra as the universal central extension of the $\dot{\mathfrak{g}}$-valued functions on a circle with an outer derivation, and the Virasoro algebra is interpreted as the universal central extension of the Lie algebra of vector fields on a circle. Note that both Lie algebras have a natural $\mathbb{Z}$-grading.

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Most applications of these Lie algebras are obtained via their representations. The most useful representations are those with a weight decomposition with the finite dimensional weight spaces.

For the Virasoro algebra, an important classification result was obtained by Olivier Mathieu [M1] (conjectured by Victor Kac [K1]):

Theorem [M1]. An irreducible module with the finite-dimensional weight spaces for the Virasoro algebra is either:

(i) a highest weight module,

(ii) a lowest weight module, or

(iii) a subquotient of a module $V_{\alpha,\beta} = q^\alpha \mathbb{C}[q, q^{-1}]$, $\alpha, \beta \in \mathbb{C}$, with the action

$$t^{m+1} d q^s = (s + m\beta) q^{s+m}, \ s \in \alpha + \mathbb{Z},$$

and c acting as 0.

The theme of this talk is the Lie algebras of higher rank, i.e., $\mathbb{Z}^n$-graded, rather than $\mathbb{Z}$-graded. We would like to develop the representation theory for such algebras.

A $\mathbb{Z}^n$-graded analogue of the Lie algebra of vector fields on a circle, is the Witt algebra W_n. This is a Lie algebra of vector fields on the n-dimensional torus:

$$W_n = \operatorname{Der} \mathbb{C}[t_1^\pm, \ldots, t_n^\pm] = \mathop{\oplus}_{p=1}^{n} \mathbb{C}[t_1^\pm, \ldots, t_n^\pm] d_p,$$

where $d_p = t_p \frac{\partial}{\partial t_p}$. This Lie algebra is $\mathbb{Z}^n$-graded by the eigenvalues of the action of the Cartan subalgebra $< d_1, \ldots, d_n >$.

The first basic example of the weight modules for W_n with finite-dimensional weight spaces is given by tensor modules. Tensor modules have their origin in differential geometry, and describe the Lie derivative action of vector fields on tensor fields. General tensors can be constructed by taking tensor products of vectors and covectors. In a similar way, any irreducible representation of $gl_n(\mathbb{C})$ is a submodule inside a tensor product of several copies of a natural representation of $gl_n(\mathbb{C})$ and its dual. It turns out that this analogy becomes an algebraic correspondence for the Lie derivative action on tensor fields, and tensor modules are parametrized by the finite-dimensional representation of $gl_n(\mathbb{C})$.

More precisely, for a finite-dimensional $gl_n(\mathbb{C})$-module U, and for $\alpha \in \mathbb{C}^n$, we define the tensor module to be

$$q^\alpha \mathbb{C}[q_1^\pm, \ldots, q_n^\pm] \otimes U$$

with the action given as follows:

$$t^m d_j (q^s \otimes u) = s_j q^{s+m} \otimes u + \sum_{p=1}^{n} m_p q^{s+m} \otimes E_{pj} u.$$

Here and throughout the rest of the talk we will use the multi-index notations: $m = (m_1, \dots m_n) \in \mathbb{Z}^n$, $s = (s_1, \dots, s_n) \in \alpha + \mathbb{Z}^n, q^s = q_1^{s_1} \dots q_n^{s_n}, d_j = t_j \frac{\partial}{\partial t_j}$.

If U is an irreducible gl_n-module then $q^\alpha \mathbb{C}[q_1^\pm, \dots, q_n^\pm] \otimes U$ is almost always an irreducible module for W_n. The exception to this are the modules in the de Rham complex of tensor modules:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n.$$

The module Ω^k of differential k-forms on a torus corresponds to an irreducible gl_n-module, but it may have proper submodules. It can be easily checked that the map d is in fact a homomorphism of W_n-modules, so its kernel and its image are submodules in Ω^k.

The Cartan subalgebra $< d_1, \dots d_n >$ acts on a tensor module diagonally, and the weight spaces clearly have a finite dimension, equal to the dimension of U. For $n = 1$, the tensor modules are just the modules $V_{\alpha,\beta}$.

Now we are going to describe another class of weight modules for the Witt algebras of rank greater than 1. It will be convenient to add a new variable t_0, and consider the Witt algebra $W_{n+1} = \mathrm{Der}\, \mathbb{C}[t_0^\pm, t_1^\pm, \dots, t_n^\pm]$. This Lie algebra is $\mathbb{Z}^{n+1}$-graded, but for the moment we will look only at its $\mathbb{Z}$-grading by degrees in t_0. Let us decompose W_{n+1} into the positive, negative and zero parts with respect to this $\mathbb{Z}$-grading:

$$W_{n+1} = W_{n+1}^- \oplus W_{n+1}^0 \oplus W_{n+1}^+.$$

The zero component is essentially the Witt algebra of rank n:

$$W_{n+1}^0 = W_n \oplus \mathbb{C}[t_1^\pm, \dots t_n^\pm] d_0.$$

Fix $\alpha \in \mathbb{C}^n$, $d \in \mathbb{C}$, and a finite-dimensional irreducible gl_n-module U. Consider a module for W_{n+1}^0:

$$q^\alpha \mathbb{C}[q_1^\pm, \dots, q_n^\pm] \otimes U,$$

which is a tensor module for W_n, and the remaining part of W_{n+1}^0 acts by shifts:

$$t^m d_0 (q^s \otimes u) = d q^{s+m} \otimes u.$$

This W_{n+1}^0-module is irreducible unless $d = 0$ and U corresponds to a module from the de Rham complex. Let T be an irreducible subquotient of this module.

To construct a W_{n+1}-module, we let W_{n+1}^+ act on T trivially, and consider the induced module

$$M(T) = \mathrm{Ind}_{W_{n+1}^0 \oplus W_{n+1}^+}^{W_{n+1}} (T) = U(W_{n+1}^-) \otimes T.$$

The module $M(T)$ has a weight decomposition, but most of its weight spaces are infinite-dimensional.

Nonetheless we can apply a more general result of Berman-Billig [BB] to obtain the following theorem:

Theorem [BB].

(i) $M(T)$ has a unique maximal submodule M^{rad} intersecting with T trivially.

(ii) The factor-module $L(T) = M(T)/M^{rad}$ is irreducible and has a weight decomposition with finite-dimensional weight spaces.

The module $L(T)$ corresponds to a choice of a special direction t_0. We can actually construct similar modules for any rational direction, by noticing that the group $GL_{n+1}(\mathbb{Z})$ acts by automorphisms on W_{n+1}. Thus we can twist $L(T)$ with an automorphism from $GL_{n+1}(\mathbb{Z})$. This will change the hyperplane in the weight space that corresponds to T to an arbitrary hyperplane with a rational normal.

Now we can formulate a conjecture for the W_{n+1}-modules:

Conjecture. An irreducible module for W_{n+1} with finite-dimensional weight spaces is either

(i) a sub-quotient of a tensor module, or

(ii) a module $L(T)$ twisted by an automorphism from $GL_{n+1}(\mathbb{Z})$.

Problem. Find the character of the W_{n+1}-module $L(T)$.

Another class of algebras that we would like to discuss here is the family of toroidal Lie algebras, which are multivariable analogues of the affine Kac-Moody algebras.

As before, let $\dot{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra. Consider the multiloop algebra $\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes \dot{\mathfrak{g}}$ (Lie algebra of $\dot{\mathfrak{g}}$-valued functions on a torus $\mathbb{T}^n$), and its universal central extension:

$$\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes \dot{\mathfrak{g}} \oplus \Omega^1(\mathbb{T}^n)/d\Omega^0(\mathbb{T}^n),$$

with the Lie bracket

$$[f_1(t) \otimes g_1, f_2(t) \otimes g_2] = f_1(t)f_2(t) \otimes [g_1, g_2] + (g_1|g_2)\overline{f_2(t)df_1(t)}.$$

The space $\Omega^1/d\Omega^0$ is central, and $\overline{}$ denotes the projection from Ω^1 to $\Omega^1/d\Omega^0$. Similar to the affine case, we add outer derivations:

$$\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes \dot{\mathfrak{g}} \oplus \Omega^1(\mathbb{T}^n)/d\Omega^0(\mathbb{T}^n) \oplus W_n.$$

Unlike the case of $n = 1$, the action of W_n on $\Omega^1/d\Omega^0$ is non-trivial, and is given by the Lie derivative. Thus in this bigger algebra, the space $\Omega^1/d\Omega^0$ is no longer central, and the actual center is in fact n-dimensional.

It happens that there are still additional degrees of freedom in defining the Lie bracket in this Lie algebra - instead of taking the semidirect product with W_n, we can also incorporate an $\Omega^1/d\Omega^0$ - valued 2-cocycle on W_n, defining the Lie bracket of two vector fields $\xi, \zeta \in W_n$ to be

$$[\xi, \zeta] = (\text{commutator of } \ \xi, \zeta \text{ in } \ W_n) + \tau(\xi, \zeta),$$

where $\tau \in H^2(W_n, \Omega^1/d\Omega^0)$.

The space $H^2(W_n, \Omega^1/d\Omega^0)$ is 2-dimensional. To describe its generators, we define the jacobian of a vector field $\xi = \sum\limits_i \xi_i d_i$ as a matrix $\xi^J(t)$ with $\xi^J_{ij} = d_j(\xi_i)$. The the basis of the space of 2-cocycles is given by the expressions:

$$\tau_1(\xi, \zeta) = \overline{Tr\,(\xi^J d\zeta^J)},$$

$$\tau_2(\xi, \zeta) = \overline{Tr(\xi^J) dTr(\zeta^J)}.$$

Note that the space of 2-cocycles is parametrized by the invariant symmetric bilinear forms on $gl_n(\mathbb{C})$:

$$< A, B >_1 = Tr(AB), \ \ < A, B >_2 = Tr(A)Tr(B).$$

When $n = 1$ both forms coincide, and we get the Virasoro cocycle.

The toroidal Lie algebra with the cocycle $\tau = \mu\tau_1 + \nu\tau_2$ will be denoted by $\mathfrak{g}_n(\mu, \nu)$:

$$\mathfrak{g} = \mathfrak{g}_n(\mu, \nu) = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes \dot{\mathfrak{g}} \oplus \Omega^1(\mathbb{T}^n)/d\Omega^0(\mathbb{T}^n) \oplus W_n.$$

This Lie algebra is a higher rank analogue of the Kac-Moody-Virasoro algebra.

Next we are going to discuss representations for the toroidal Lie algebra $\mathfrak{g}_n(\mu, \nu)$ with finite-dimensional weight spaces. The first class of modules we consider here is multiloop modules. These modules are constructed in the following way.

Let V be a finite-dimensional irreducible module for $\dot{\mathfrak{g}}$, U be a finite-dimensional irreducible module for $gl_n(\mathbb{C})$, and let $\alpha \in \mathbb{C}^n$. Consider the space

$$q^\alpha \mathbb{C}[q_1^{\pm}, \ldots, q_n^{\pm}] \otimes V \otimes U$$

with the action

$$t^m \otimes g(q^s \otimes v \otimes u) = q^{s+m} \otimes g(v) \otimes u,$$

W_n acts via the tensor module action and $\Omega^1/d\Omega^0$ acts trivially.

The multiloop modules have a weight decomposition with finite-dimensional weight spaces, but these modules are not particularly interesting. We are now going to define more a important class of bounded modules.

Let us consider an $(n+1)$-toroidal Lie algebra $\mathfrak{g}_{n+1}(\mu,\nu)$. As before, we decompose this algebra into the positive, negative and zero parts with respect to the $\mathbb{Z}$-grading by degrees in t_0. The zero component is essentially a toroidal Lie algebra of rank n:

$$\mathfrak{g}^0_{n+1}(\mu,\nu) = \mathfrak{g}_n(\mu,\nu) \oplus \mathbb{C}[t_1^\pm,\ldots,t_n^\pm](t_0^{-1}dt_0) \oplus \mathbb{C}[t_1^\pm,\ldots,t_n^\pm]d_0.$$

Take the module T to be a multiloop module for $\mathfrak{g}_n(\mu,\nu)$ as above, and extend the action to $\mathfrak{g}^0_{n+1}(\mu,\nu)$ in the following way:

$$t^m t_0^{-1} dt_0 (q^s \otimes v \otimes u) = c q^{s+m} \otimes v \otimes u,$$

$$t^m d_0 (q^s \otimes v \otimes u) = d q^{s+m} \otimes v \otimes u,$$

for some fixed constants $c, d \in \mathbb{C}$.

Then we let $\mathfrak{g}^+_{n+1}(\mu,\nu)$ act on T trivially and define $M(T)$ as an induced module:

$$M(T) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^0 \oplus \mathfrak{g}^+}(T) = U(\mathfrak{g}^-) \otimes T.$$

Again by the same result of [BB], $M(T)$ has a unique maximal submodule M^{rad} intersecting T trivially, and the module

$$L(T) = M(T)/M^{rad}$$

has finite-dimensional weight spaces.

Note that the module $L(T)$ is irreducible, with a possible exception when U is de Rham, V is trivial and $c = d = 0$.

Problem. Find the character of $L(T)$.

We solve the above problem with the following theorem:

Theorem. Let $c \neq 0, c \neq -h^\vee, 1 - c\mu \neq -n, 1 - c\mu - cn\nu \neq 0$. Then the irreducible $\mathfrak{g}_{n+1}(\mu,\nu)$-module $L(T)$ has the following structure:

$$L(T) \cong q^\alpha \mathbb{C}[q_1^\pm,\ldots,q_n^\pm] \otimes L_{aff(\dot{\mathfrak{g}})}(V,c) \otimes \mathbb{C}[u_{pj}, v_{pj}|^{p=1,\ldots,n}_{j=1,2,3,\ldots}]$$

$$\otimes L_{aff(gl_n)}(U, 1 - c\mu, n(1 - c\mu - cn\nu)) \otimes L_{Vir}(h, c_{Vir}).$$

Here $h^\vee$ is the dual Coxeter number for $\dot{\mathfrak{g}}$, $L_{aff(\dot{\mathfrak{g}})}(V,c)$ is an irreducible highest weight module for the affinization of $\dot{\mathfrak{g}}$ of central charge c, and the $\dot{\mathfrak{g}}$-submodule generated by the highest weight vector being V. The irreducible module for the affinization of gl_n $L_{aff(gl_n)}(U, c_1, c_2)$ is defined in a similar way, with the only difference is that the affine gl_n has two central charges - one for affine sl_n, and the second for the infinite-dimensional Heisenberg subalgebra that arise as an affinization of the scalar matrices. The last component $L_{Vir}(h, c_{Vir})$ is a highest weight irreducible module for the Virasoro algebra. Its central charge is given by the formula:

$$c_{Vir} = 12(\mu+\nu)c - \frac{c\dim\dot{\mathfrak{g}}}{c+h^\vee} - 2n - \frac{(1-c\mu)(n^2-1)}{1-c\mu+n} - 1 + \frac{3(2\nu-1)^2 n}{1-c\mu-cn\nu}.$$

The only essential restriction in this theorem is that $c \neq 0$. All other cases can be treated in a similar way, but the answer has to be written in a different form. We are omitting these details here.

The action of the toroidal Lie algebra can be given rather explicitly by means of the vertex operators. Since the characters of all factors in the tensor product above are well-known, we immediately obtain the character of $L(T)$.

Let us outline the proof of this theorem. We use the machinery of the vertex Lie algebras which allows to associate with a vertex Lie algebra $\mathcal{L}$ a universal enveloping vertex algebra $V_{\mathcal{L}}$.

Definition [DLM]. A Lie algebra $\mathcal{L}$ is called a vertex Lie algebra if it has a spanning set $\{x_k^\alpha | k \in \mathbb{Z}, \alpha \in J\}$, and a linear map $D : \mathcal{L} \to \mathcal{L}$, such that $Dx_k^\alpha = -kx_{k-1}^\alpha$, and the formal fields $x^\alpha(z) = \sum\limits_k x_k^\alpha z^{-k-1}$ satisfy the property that

$$[x^\alpha(z_1), x^\beta(z_2)] = \sum_{j\geq 0, k\geq 0, \gamma\in J}^{\text{finite}} a_{jk\gamma} \left(\left(\frac{\partial}{\partial z_2}\right)^k x^\gamma(z_2) \right) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^j \delta\left(\frac{z_2}{z_1}\right) \right]$$

where $\delta\left(\frac{z_2}{z_1}\right) = \sum\limits_{i\in\mathbb{Z}} \left(\frac{z_2}{z_1}\right)^i$.

Let $\mathcal{L}^+$ be a subspace in $\mathcal{L}$ spanned by $\{x_k^\alpha | k \geq 0, \alpha \in J\}$, and $\mathcal{L}^-$ be a subspace spanned by $\{x_k^\alpha | k < 0, \alpha \in J\}$. Then both $\mathcal{L}^+$ and $\mathcal{L}^-$ are subalgebras in $\mathcal{L}$ and

$$\mathcal{L} = \mathcal{L}^- \oplus \mathcal{L}^+.$$

Consider a 1-dimensional trivial module $\mathbb{C}1$ for $\mathcal{L}^+$ and define

$$V_{\mathcal{L}} = \mathrm{Ind}_{\mathcal{L}^+}^{\mathcal{L}}(\mathbb{C}1).$$

Theorem [DLM]. $V_{\mathcal{L}}$ has a structure of a vertex algebra.

The main feature of the vertex algebra is the existence of the state-field correspondence map

$$Y: V_{\mathcal{L}} \to End(V_{\mathcal{L}})[[z, z^{-1}]]$$

(see [K2] for the axioms and properties of the vertex algebras).

It turns out that the toroidal Lie algebras $\mathfrak{g} = \mathfrak{g}_{n+1}(\mu, \nu)$ are vertex Lie algebras, though it requires some work to find the appropriate spanning set. Moreover, for U, V trivial 1-dimensional, $\alpha = 0$ and $d = \frac{c}{2}(\mu + \nu)$, the corresponding module $L(T_0)$, $T_0 = \mathbb{C}[q_1^{\pm}, \ldots q_n^{\pm}]$, also has a structure of a vertex algebra, and in fact it is a factor-algebra of $V_{\mathfrak{g}}$. All other modules $L(T)$ are modules for the vertex algebra $L(T_0)$. Once we find the structure for $L(T_0)$, the result for the rest of the modules will follow immediately.

To find the structure of $L(T_0)$, we consider a projection

$$V_{\mathfrak{g}} \to L(T_0).$$

This is a homomorphism of $\mathbb{Z}^{n+1}$-graded algebras. The technique developed in [BB] allows us to calculate explicitly any given component of the kernel of this map. Of course, this calculation is feasible only for the components of low degrees. However, by applying the state-field correspondence map Y, we transform an element of the kernel into a relation between the fields in $L(T_0)$.

It turns out that we only need to describe the components of the kernel in degrees one and two, in order to obtain enough relations between the fields in $L(T_0)$, so that we get a complete description of the structure of $L(T_0)$. The result for the remaining modules $L(T)$ follows immediately from the principle of preservation of identities (see [L] for details).

In conclusion, we would like to connect this result for the toroidal Lie with the representation theory of the Witt algebra, as well as graded infinite-dimensional modules for the finite-dimensional simple Lie algebras.

If in the above result we specialize $\mu = \nu = 0$, and omit the multiloop algebra, we will get that

$$\widetilde{L}(T) = q^{\alpha}\mathbb{C}[q_1^{\pm}, \ldots q_n^{\pm}] \otimes \mathbb{C}[u_{pj}, v_{pj}|_{j=1,2,3,\ldots}^{p=1,\ldots,n}] \otimes L_{aff(gl_n)}(U, 1, n) \otimes L_{Vir}(h, 2n)$$

is a module for W_{n+1} with finite-dimensional weight spaces. This module, however, is not irreducible as a W_{n+1} module in general.

Problem. Find the irreducible quotient of the W_{n+1}-module $\widetilde{L}(T)$.

Conjecture. For a generic T, the W_{n+1}-module $\widetilde{L}(T)$ is irreducible.

These modules may also be helpful in studying graded modules of induced type for the finite-dimensional simple Lie algebras (see e.g., [Fu], [Fe], [M2]). Indeed, since sl_{n+2} is a subalgebra in W_{n+1}, we obtain that $\widetilde{L}(T)$ is a module for sl_{n+2} with the finite-dimensional weight spaces.

Problem. Find the irreducible quotient of $\widetilde{L}(T)$ as an sl_{n+2}-module.

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