Resenhas IME-USP 2004, Vol. 6, No. 2/3, 129 - 140.

On irreducible infinite conformal algebras

Carina Boyallian and Jose I. Liberati[1]

The associative conformal algebra Cend_N and the corresponding general Lie conformal algebra gc_N are the most important examples of simple conformal algebras which are not finite (see Sect. 2.10 in [K1]). One of the most important open problems of the theory of conformal algebras is the classification of infinite subalgebras of Cend_N and of gc_N which act irreducibly on $\mathbb{C}[\partial]^N$. (For a classification of such finite algebras, in the associative case see Theorem 2.6 of the present paper, and in the (more difficult) Lie case see [CK] and [DK].)

The classical Burnside theorem states that any subalgebra of the matrix algebra $\mathrm{Mat}_N\mathbb{C}$ that acts irreducibly on $\mathbb{C}^N$ is the whole algebra $\mathrm{Mat}_N\mathbb{C}$. This is certainly not true for subalgebras of Cend_N (which is the "conformal" analogue of $\mathrm{Mat}_N\mathbb{C}$). There is a family of infinite subalgebras $\mathrm{Cend}_{N,P}$ of Cend_N, where $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$, $\det P(x) \neq 0$, that still act irreducibly on $\mathbb{C}[\partial]^N$. One of the conjectures of [K2] states that there are no other infinite irreducible subalgebras of Cend_N. This conjecture was recently proved by Kolesnikov [Ko].

In the Lie conformal case, we have a conjecture on the classification of infinite Lie conformal subalgebras of gc_N acting irreducibly on $\mathbb{C}[\partial]^N$, see Conjecture 4.4. This conjecture agrees with recent results of E. Zelmanov [Z2] and A. De Sole - V. Kac [DeK].

This is an expanded version of a talk given by the second author at the conference in Guaruja in May, 2004. It is based on a joint work with Victor G. Kac, see [BKL] for details. This is a summary of this work and an updated version with recent results by E. Zelmanov, A. De Sole and V. Kac.

The paper is organized as follows:

- Basic definitions
- Irreducible subalgebras of Cend_N and finite $\mathrm{Cend}_{N,P}$-modules
- Automorphisms, anti-automorphisms and anti-involutions of $\mathrm{Cend}_{N,P}$
- Irreducible Lie conformal algebras gc_N, $oc_{N,P}$ and $spc_{N,P}$

1. Basic definitions

An *associative conformal algebra* R is defined as a $\mathbb{C}[\partial]$-module with a $\mathbb{C}$-linear map,

$$R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R, \qquad a \otimes b \mapsto a_\lambda b$$

called the λ-product, and satisfying the axioms (a, b, $c \in R$),

[1]The authors were supported in part by Conicet, ANPCyT, Agencia Cba Ciencia, Secyt-UNC, Fundación Antorchas and Fomec (Argentina). The second author was also partially supported by FINEP.

$$(A1)_\lambda \qquad (\partial a)_\lambda b = -\lambda(a_\lambda b), \qquad a_\lambda(\partial b) = (\lambda+\partial)(a_\lambda b)$$

$$(A2)_\lambda \qquad a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$$

An associative conformal algebra is called *finite* if it has finite rank as $\mathbb{C}[\partial]$ -module. The notions of homomorphism, ideal and subalgebras of an associative conformal algebra are defined in the usual way (see [K1]).

A *module* over an associative conformal algebra R is a $\mathbb{C}[\partial]$-module M with a $\mathbb{C}$-linear map $R\otimes M \longrightarrow \mathbb{C}[\lambda]\otimes M$, denoted by $a\otimes v \mapsto a_\lambda^M v$, satisfying the properties:

$$(\partial a)_\lambda^M v = [\partial^M, a_\lambda^M]v = -\lambda(a_\lambda^M v), \quad a\in R,\, v\in M,$$
$$a_\lambda^M(b_\mu^M v) = (a_\lambda b)_{\lambda+\mu}^M v, \quad a,b\in R.$$

An R-module M is called *trivial* if $a_\lambda v = 0$ for all $a\in R$, $v\in M$ (but it may be non-trivial as a $\mathbb{C}[\partial]$-module).

Given a $\mathbb{C}[\partial]$-module V, a *conformal endomorphism* of V is a $\mathbb{C}$-linear map $a: V\to \mathbb{C}[\lambda]\otimes_{\mathbb{C}} V$, denoted by $a_\lambda: V\to V$, such that $[\partial, a_\lambda] = -\lambda a_\lambda$. Denote by Cend$V$ the vector space of all such maps. Cend$V$ has a $\mathbb{C}[\partial]$-module structure:

$$(\partial a)_\lambda := -\lambda a_\lambda.$$

If V is a finite $\mathbb{C}[\partial]$-module, then Cend$V$ has a canonical structure of an associative conformal algebra defined by

$$(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v), \qquad a,b\in \text{Cend } V,\ v\in V.$$

Remark. Observe that, by definition, a structure of a conformal module over an associative conformal algebra R in a finite $\mathbb{C}[\partial]$-module V is the same as a homomorphism of R to the associative conformal algebra Cend$V$.

We shall use the following notation: $\text{Cend}_N := \text{Cend}\mathbb{C}[\partial]^N$.

These is a natural isomorphism

$$\text{Cend}_N \simeq \text{Mat}_N \mathbb{C}[\partial, x]$$

and the λ-product in $\text{Mat}_N\mathbb{C}[\partial,x]$ is

$$A(\partial,x)_\lambda B(\partial,x) = A(-\lambda, x+\lambda+\partial)B(\lambda+\partial, x).$$

We shall work with this presentation of Cend_N. The λ-action of Cend_N on $\mathbb{C}[\partial]^N$ is

$$A(\partial,x)\,{}_\lambda\, v(\partial) = A(-\lambda, \lambda+\partial+\alpha)v(\lambda+\partial), \quad v(\partial)\in\mathbb{C}[\partial]^N.$$

Under the change of basis of $\mathbb{C}[\partial]^N$ by the matrix $C(\partial)$ invertible in $\mathrm{Mat}_N(\mathbb{C}[\partial])$, the symbol $A(\partial, x)$ changes by the formula:

$$A(\partial, x) \longmapsto C(\partial + x)A(\partial, x)C(x)^{-1}. \tag{1.1}$$

Observe that for any $C(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, with non-zero constant determinant, the map (1.1) gives us an automorphism of Cend_N.

It follows from the formula for λ-product that

$$\mathrm{Cend}_{P,N} := P(x + \partial)(\mathrm{Cend}_N) \quad \text{and} \quad \mathrm{Cend}_{N,P} := (\mathrm{Cend}_N)P(x),$$

with $P(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, are right and left ideals, respectively, of Cend_N. In particular, they are subalgebras of Cend_N. Another important subalgebra is

$$Cur_N := Cur\ (\mathrm{Mat}_N\mathbb{C}) = \mathbb{C}[\partial]\,(\mathrm{Mat}_N\mathbb{C}).$$

Remark. If $P(x)$ is nondegenerate, i.e., $\det P(x) \neq 0$, then

$$\mathrm{Cend}_{N,P} \simeq \mathrm{Cend}_{N,D},$$

with $D = diag(p_1(x), \cdots, p_N(x))$, where $p_i(x)$ are monic polynomials such that $p_i(x)$ divides $p_{i+1}(x)$. The $p_i(x)$ are called the elementary divisors of P. So, up to conjugation, all $\mathrm{Cend}_{N,P}$ are parameterized by the sequence of elementary divisors of P.

All left and right ideals of Cend_N were obtained by B. Bakalov. We extend the classification to $\mathrm{Cend}_{N,P}$.

Proposition 1.1. *a) All left ideals in $Cend_{N,P}$, with det $P(x) \neq 0$, are of the form $Cend_{N,QP}$, where $Q(x) \in Mat_N(\mathbb{C}[x])$.*

b) All right ideals in $Cend_{N,P}$, with det $P(x) \neq 0$, are of the form $Q(\partial + x)Cend_{N,P}$, where $Q(x) \in Mat_N(\mathbb{C}[x])$.

c) $Cend_{N,P} \simeq Cend_{P,N}$

2. Irreducible subalgebras of Cend_N and finite modules over $\mathrm{Cend}_{N,P}$

Given R an associative conformal algebra, we will establish a correspondence between the set of maximal left ideals of R and the set of irreducible R-modules. Then we will apply it to the subalgebras $\mathrm{Cend}_{N,P}$.

Lemma 2.1. *a) Let $v \in M$ and $\mu \in \mathbb{C}$, then $R_{-\partial-\mu}v$ is an R-submodule of M.*
b) Let M be a non-trivial irreducible R-module. Then there exists $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu}v \neq 0$. In particular, if M is irreducible, then $R_{-\partial-\mu}v = M$.

By this lemma, given a non-trivial irreducible R-module M we can fix $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu}v = M$ and consider the following map

$$\phi : R \to M, \qquad r \mapsto r_{-\partial-\mu}v.$$

This is onto and therefore we have that as R-modules

$$M \simeq (R/\mathrm{Ker}\ \phi)_\mu. \tag{2.1}$$

where M_μ is the μ-twisted module of M obtained by replacing ∂ by $\partial + \mu$ in the formulas for the action of R on M.

On the other hand, it is immediate that given any maximal left ideal I of R, we have that $(R/I)_\mu$ is an irreducible R-module. Therefore we have

Theorem 2.2. *Formula (2.1) defines a surjective map from the set of maximal left ideals of R to the set of equivalence classes of non-trivial irreducible R-modules.*

Using this result, we obtain

Corollary 2.3. *The $Cend_{N,P}$-module $\mathbb{C}[\partial]^N$ is irreducible if and only if $\det P(x) \neq 0$. These are all non-trivial irreducible $Cend_{N,P}$-modules up to equivalence, provided that* $\det P(x) \neq 0$.

This Corollary in the case $P(x) = I$, have been established earlier in [K2], by a completely different method (developed in [KR]). Another proof of this was also given by Retakh in [R].

A subalgebra S of Cend_N is called *irreducible* if S acts irreducibly in $\mathbb{C}[\partial]^N$.

Corollary 2.4. *The following subalgebras of $Cend_N$ are irreducible: $Cend_{N,P}$ with $\det P(x) \neq 0$, and $Cur_N := Mat_N(\mathbb{C}[\partial])$ or conjugates of it by automorphisms (1.1).*

We have the conformal analog of the Burnside Theorem, originally conjectured in [K2]:

Conjecture 2.5. *(proved by Kolesnikov* [Ko], *Feb'2004) Any irreducible subalgebra of $Cend_N$ is one of them.*

The (particular case of) classification of finite irreducible subalgebras also follows from the classification in [DK] at the Lie algebra level, see [BKL]:

Theorem 2.6. *Any finite irreducible subalgebra of $Cend_N$ is a conjugate of Cur_N.*

Now, we study representation theory of these subalgebras.

Remark. It is easy to show that every non-trivial irreducible representation of Cur_N is equivalent to the standard module $\mathbb{C}[\partial]^N$, and that every finite module over Cur_N is completely reducible.

Unfortunately, complete reducibility does not hold for Cend_N. Therefore, we have to study extensions of modules. Here we present the following:

- Classification of all extensions of $\mathrm{Cend}_{N,P}$-modules involving the standard module $\mathbb{C}[\partial]^N$ and finite dimensional trivial modules.
- Classification of all finite modules over Cend_N.

Recall the standard irreducible $\mathrm{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ with λ-action

$$a(\partial,x)P(x)_\lambda v(\partial) = a(-\lambda,\lambda+\partial+\alpha)P(\lambda+\partial)v(\lambda+\partial).$$

Consider the trivial $\mathrm{Cend}_{N,P}$-module over the finite dimensional vector space V_T, whose $\mathbb{C}[\partial]$-module structure is given by the linear operator T, that is: $\partial\cdot v = T(v)$, $v\in V_T$.

We may assume: $P(x) = diag\{p_1(x),\cdots,p_N(x)\}$ and $\det P\neq 0$.

Theorem 2.7. *a) There are no non-trivial extensions of $Cend_{N,P}$-modules of the form:*

$$0\to V_T\to E\to \mathbb{C}[\partial]^N\to 0.$$

Here and further, all the maps in these sequences are maps of $Cend_{N,P}$-modules.
b) If there exists a non-trivial extension of $Cend_{N,P}$-modules of the form

$$0\to \mathbb{C}[\partial]^N\to E\to V_T\to 0,$$

then $\det P(\alpha+c)=0$ *for some eigenvalue c of T. In this case, all torsionless extensions of $\mathbb{C}[\partial]^N$ by finite dimensional vector spaces, are parameterized by decompositions $P(x+\alpha)=R(x)S(x)$ and can be realized as follows. Consider the λ-action of $Cend_{N,P}$ on $\mathbb{C}[\partial]^N$:*

$$a(\partial,x)P(x)_\lambda v(\partial) = S(\partial)a(-\lambda,\lambda+\partial+\alpha)R(\lambda+\partial)v(\lambda+\partial).$$

Then $S(\partial)\mathbb{C}[\partial]^N$ is a submodule isomorphic to the standard module, of finite codimension in $\mathbb{C}[\partial]^N$.
c) If E is a non-trivial extension of $Cend_{N,P}$-modules of the form:

$$0\to \mathbb{C}[\partial]^N\to E\to \mathbb{C}[\partial]^N\to 0,$$

then $E=\mathbb{C}[\partial]^N\otimes\mathbb{C}^2$ as a $\mathbb{C}[\partial]$-module (with trivial action of ∂ on $\mathbb{C}^2$) and $Cend_{N,P}$ acts by

$$a(\partial,x)_\lambda(c(\partial)\otimes u) = a(-\lambda,\lambda+\partial\otimes 1+1\otimes J)c(\lambda+\partial)(1\otimes u),$$

where J is a 2×2 Jordan block matrix.

Corollary 2.8. *There are no non-trivial extensions of $Cend_N$-modules of the form:*

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0 \quad or \quad 0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0$$

Theorem 2.9. *Every finite $Cend_N$-module is isomorphic to a direct sum of its (finite dimensional) trivial torsion submodule and a free finite $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]^N \otimes T$ on which the λ-action is given by*

$$a(\partial, x)_\lambda(c(\partial) \otimes u) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes \alpha)c(\lambda + \partial)(1 \otimes u),$$

where α is an arbitrary operator on T.

3. Automorphisms and anti-automorphisms of $\mathrm{Cend}_{N,P}$

A $\mathbb{C}[\partial]$-linear map $\sigma : R \to S$ between two associative conformal algebras is called a *homomorphism* (*resp. anti-homomorphism*) if

$$\sigma(a_\lambda b) = \sigma(a)_\lambda \sigma(b) \quad (\text{resp } \sigma(a_\lambda b) = \sigma(b)_{-\lambda-\partial}\, \sigma(a)).$$

An anti-automorphism σ is an *anti-involution* if $\sigma^2 = 1$.

Theorem 3.1. *Let $P(x) \in Mat_N\mathbb{C}[x]$ with det $P(x) \neq 0$. Then all automorphisms of $Cend_{N,P}$ are those that come from $Cend_N$ by restriction. More precisely, any automorphism is of the form:*

$$a(\partial, x)P(x) \longmapsto C(\partial + x)a(\partial, x + \alpha)B(x)P(x),$$

where $\alpha \in \mathbb{C}$, and $B(x), C(x) \in Mat_N\mathbb{C}[x]$ are invertible and

$$P(x + \alpha) = B(x)P(x)C(x).$$

Theorem 3.2. *Let $P(x) \in Mat_N\mathbb{C}[x]$ with det $P(x) \neq 0$. Then we have,*
a) All non-zero homomorphisms from $Cend_{N,P}$ to $Cend_N$ are of the form:

$$a(\partial, x)P(x) \longmapsto S(\partial + x)a(\partial, x + \alpha)R(x),$$

where $\alpha \in \mathbb{C}$, and $R(x), S(x) \in Mat_N\mathbb{C}[x]$ such that

$$P(x + \alpha) = R(x)S(x).$$

(b) All non-trivial anti-homomorphisms from $Cend_{N,P}$ to $Cend_N$ are of the form:

$$a(\partial, x)P(x) \longmapsto A(\partial + x)a^t(\partial, -\partial - x + \alpha)B(x),$$

where $\alpha \in \mathbb{C}$, *and* $A(x)$ *and* $B(x)$ *are matrices in* $Mat_N\mathbb{C}[x]$ *such that*

$$P^t(-x+\alpha) = B(x)A(x).$$

(c) The conformal algebra $Cend_{N,P}$ *has an anti-automorphism (i.e. it is isomorphic to its opposite conformal algebra) if and only if the matrices* $P^t(-x+\alpha)$ *and* $P(x)$ *have the same elementary divisors for some* $\alpha \in \mathbb{C}$. *In this case, all anti-automorphisms of* $Cend_{N,P}$ *are of the form:*

$$a(\partial, x)P(x) \longmapsto Y(\partial + x)a^t(\partial, -\partial - x + \alpha)W(x)P(x),$$

where $Y(x)$ *and* $W(x)$ *are invertible matrices in* $Mat_N\mathbb{C}[x]$ *such that*

$$P^t(-x+\alpha) = W(x)P(x)Y(x).$$

(d) The conformal algebra $Cend_{N,P}$ *has an anti-involution if and only if there exist an invertible in* $Mat_N\mathbb{C}[x]$ *matrix* $J(x)$ *such that*

$$J^t(-x+\alpha)P^t(-x+\alpha) = \epsilon P(x)J(x) \tag{3.1}$$

for $\epsilon = 1$ *or* -1. *In this case all anti-involutions are given by*

$$\sigma_{P,J,\epsilon,\alpha}(a(\partial, x)P(x)) = \varepsilon J(\partial + x)a^t(\partial, -\partial - x + \alpha)J^t(-x+\alpha)^{-1}P(x)$$

where $J(x)$ *is an invertible in* $Mat_N\mathbb{C}[x]$ *matrix satisfying (3.1).*

Corollary 3.3. *Let* $P(x)$, $Q(x)$ $\in Mat_N\mathbb{C}[x]$ *be two non-degenerate matrices. Then* $Cend_{N,P}$ *is isomorphic to* $Cend_{N,Q}$ *if and only if there exist* $\alpha \in \mathbb{C}$ *such that* $Q(x)$ *and* $P(x+\alpha)$ *have the same elementary divisors.*

Two anti-involutions σ, τ of an associative conformal algebra R are called *conjugate* if $\sigma = \varphi \circ \tau \circ \varphi^{-1}$ for some automorphism φ of R.

Theorem 3.4. *Any anti-involution of* $Cend_N$ *is, up to conjugation by an automorphism of* $Cend_N$*:*

$$a(\partial, x) \mapsto a^*(\partial, -\partial - x),$$

where * *is the adjoint with respect to a non-degenerate symmetric or skew- symmetric bilinear form over* $\mathbb{C}$.

In [BKL], we also found a characterization of equivalent anti-involutions in $\mathrm{Cend}_{N,P}$ and a relation of anti-involutions for different P.

4. Lie conformal algebras gc_N, $oc_{N,P}$ and $spc_{N,P}$

A *Lie conformal algebra* R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes R \longrightarrow \mathbb{C}[\lambda]\otimes R$, $a\otimes b\mapsto [a_\lambda b]$, called the λ-bracket, satisfying the following axioms $(a,b,c\in R)$,

$(C1)_\lambda \qquad [(\partial a)_\lambda b] = -\lambda[a_\lambda b], \qquad [a_\lambda(\partial b)] = (\lambda+\partial)[a_\lambda b]$

$(C2)_\lambda \qquad [a_\lambda b] = -[a_{-\partial-\lambda} b]$

$(C3)_\lambda \qquad [a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]$.

A *module* M over a conformal algebra R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes M \longrightarrow \mathbb{C}[\lambda]\otimes M$, $a\otimes v\mapsto a_\lambda v$, satisfying the following axioms $(a,\, b\in R)$, $v\in M$,

$(M1)_\lambda \qquad (\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v,$

$(M2)_\lambda \qquad [a^M_\lambda, b^M_\mu]v = [a_\lambda b]^M_{\lambda+\mu} v.$

In general, given any associative conformal algebra R with λ-product $a_\lambda b$, the λ-bracket defined by

$$[a_\lambda b] := a_\lambda b - b_{-\partial-\lambda} a$$

makes R a Lie conformal algebra.

Let V be a finite $\mathbb{C}[\partial]$-module. The λ-bracket on $\mathrm{Cend}\, V$, makes it a Lie conformal algebra denoted by $\mathrm{gc}\, V$ and called the *general conformal algebra* (see [DK]).

For any positive integer N, we define

$$\mathrm{gc}_N := \mathrm{gc}\,\mathbb{C}[\partial]^N = Mat_N\mathbb{C}[\partial, x],$$

and the λ-bracket:

$$[A(\partial,x)_\lambda B(\partial,x)] = A(-\lambda, x+\lambda+\partial)B(\lambda+\partial,x) - B(\lambda+\partial,-\lambda+x)A(-\lambda,x).$$

Recall that, any anti-involution in Cend_N is, up to conjugation

$$\sigma_*(A(\partial,x)) = A^*(\partial,-\partial-x),$$

where $*$ stands for the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over $\mathbb{C}$. These anti-involutions give us two important subalgebras of gc_N: the set of $-\sigma_*$ fixed points is the *orthogonal conformal algebra* oc_N (resp. the *symplectic conformal algebra* spc_N), in the symmetric (resp. skew-symmetric) case.

- *Description in terms of conformal bilinear forms:*

The conformal subalgebras oc_N and spc_N, as well as the anti-involutions given by Section 3, can be described in terms of conformal bilinear forms. Let V be a $\mathbb{C}[\partial]$-module. A *conformal bilinear form* on V is a $\mathbb{C}$-bilinear map $\langle \ , \ \rangle_\lambda : V \times V \to \mathbb{C}[\lambda]$ such that

$$\langle \partial v, w\rangle_\lambda = -\lambda\langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda, \quad \text{for all } v, w \in V.$$

The conformal bilinear form is *non-degenerate* if $\langle v, w\rangle_\lambda = 0$ for all $w \in V$, implies $v = 0$. The conformal bilinear form is *symmetric* (resp. *skew-symmetric*) if $\langle v, w\rangle_\lambda = \epsilon\langle w, v\rangle_{-\lambda}$ for all $v, w \in V$, with $\epsilon = 1$ (resp. $\epsilon = -1$).

Given a conformal bilinear form on a $\mathbb{C}[\partial]$-module V, we have a homomorphism of $\mathbb{C}[\partial]$-modules, $L : V \to V^*$, $v \mapsto L_v$, given as usual by

$$(L_v)_\lambda w = \langle v, w\rangle_\lambda, \quad v \in V. \tag{4.1}$$

where V^* is the conformal dual of V.

Let V be a free finite rank $\mathbb{C}[\partial]$-module and fix $\beta = \{e_1, \cdots, e_N\}$ a $\mathbb{C}[\partial]$-basis of V. Then *the matrix of* $\langle \ , \ \rangle_\lambda$ *with respect to* β is defined as $P_{i,j}(\lambda) = \langle e_i, e_j\rangle_\lambda$. Hence, identifying V with $\mathbb{C}[\partial]^N$, we have

$$\langle v(\partial), w(\partial)\rangle_\lambda = v^t(-\lambda)P(\lambda)w(\lambda). \tag{4.2}$$

Observe that $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ (resp. $\epsilon = -1$) if the conformal bilinear form is symmetric (resp. skewsymmetric). We also have that $\operatorname{Im} L = P(-\partial)V^*$, where L is defined in (4.1). Indeed, given $v(\partial) \in V$, consider $g_\lambda \in V^*$ defined by $g_\lambda(w(\partial)) = v^t(-\lambda)w(\lambda)$, then by (4.2)

$$(L_{v(\partial)})_\lambda w(\partial) = v^t(-\lambda)P(\lambda)w(\lambda) = g_\lambda(P(\partial)w(\partial)) = (P(-\partial)g)_\lambda(w(\partial)),$$

where in the last equality we are identifying V^* with $\mathbb{C}[\partial]^N$ in the natural way, that is $f \in V^*$ corresponds to $(f_{-\partial}e_1, \cdots, f_{-\partial}e_N) \in \mathbb{C}[\partial]^N$. Therefore, if the conformal bilinear form is non-degenerate, then L gives an isomorphism between V and $P(-\partial)V^*$, with $\det P \neq 0$.

We have the following result:

Proposition 4.1. *(a) Let* $\langle \ , \ \rangle_\lambda$ *be a non-degenerate symmetric or skew-symmetric conformal bilinear form on* $\mathbb{C}[\partial]^N$*, and denote by* $P(\lambda)$ *the matrix of* $\langle \ , \ \rangle_\lambda$ *with respect to the standard basis of* $\mathbb{C}[\partial]^N$ *over* $\mathbb{C}[\partial]$*. Then the map* $aP \mapsto (aP)^*$ *from* $Cend_{N,P}$ *to* $Cend_N$ *defined by*

$$\langle a_\mu v, w\rangle_\lambda = \langle v, a^*_\mu w\rangle_{\lambda-\mu}.$$

is the anti-involution of $Cend_{N,P}$ *given by*

$$(a(\partial, x)P(x))^* = \epsilon a^t(\partial, -\partial - x)P(x), \tag{4.3}$$

where $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ or -1, depending on whether the conformal bilinear form is symmetric or skew-symmetric.
(b) Consider the Lie conformal subalgebra of gc_N defined by

$$\begin{aligned} g_* &= \{a \in Cend_{N,P} \,:\, a^* = -a\} \\ &= \{a \in Cend_{N,P} \,:\, \langle a_\mu v, w\rangle_\lambda + \langle v, a_\mu w\rangle_{\lambda-\mu} = 0, \quad \text{for all } v, w \in \mathbb{C}[\partial]^N\}, \end{aligned}$$

where * is defined by (4.3). Then under the pairing (4.1) we have $\mathbb{C}[\partial]^N \simeq P(-\partial)(\mathbb{C}[\partial]^N)^*$ as g_*-modules.

Observe that oc_N (resp. spc_N), can be described as the subalgebra g_* of gc_N in Proposition 4.1(b), with respect to the conformal bilinear form

$$\langle p(\partial)v, q(\partial)w\rangle_\lambda = p(-\lambda)q(\lambda)\,(v,w) \qquad \text{for all } v, w \in \mathbb{C}^N,$$

where $(\cdot,\cdot)$ is a non-degenerate symmetric (resp. skew-symmetric) bilinear form on $\mathbb{C}^N$. For general P, see (6.16) in [BKL].

Observe that $gc_{N,P} := gc_N P(x)$ is a conformal subalgebra of gc_N, for any $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$.
A matrix $Q(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ will be called *hermitian* (resp. *skew-hermitian*) if

$$Q^t(-x) = \varepsilon Q(x) \qquad \text{with } \varepsilon = 1 \quad (\text{ resp. } \varepsilon = -1).$$

Up to conjugacy, it suffices to consider the anti-involutions

$$\sigma_{P,\varepsilon}(a(\partial, x)P(x)) = \varepsilon a^t(\partial, -\partial - x)P(x)$$

where P is non-degenerate hermitian or skew-hermitian, depending on whether $\varepsilon = 1$ or -1.

Notation (P non-degenerate):

$$\begin{aligned} oc_{N,P} :=& \{a \in \mathrm{Cend}_{N,P} \;:\; \sigma_{P,1}(a) = -a\} \qquad \text{if } P \text{ hermitian} \\ spc_{N,P} :=& \{a \in \mathrm{Cend}_{N,P} \;:\; \sigma_{P,-1}(a) = -a\} \quad \text{if } P \text{ skew-hermitian.} \end{aligned}$$

Remark. a) These subalgebras can be obtained in a more invariant form using conformal bilinear forms.
b) In the special case $N = 1$ and $P(x) = x$, the involution $\sigma_{x,-1}$ is the conformal version of the involution used by S. Bloch [B] in connection with certain values of ζ-function.

Proposition 4.2. *The subalgebras $gc_{N,P}$, $oc_{N,P}$ and $spc_{N,P}$ with* $\det P(x) \neq 0$ *are simple and act irreducibly on* $\mathbb{C}[\partial]^N$.

Two matrices a and b in $\mathrm{Mat}_N\mathbb{C}[x]$ are called *congruent* if $b = c^*ac$ for some invertible in $\mathrm{Mat}_N\mathbb{C}[x]$ matrix c, where $c(x)^* := c(-x)^t$.

Proposition 4.3. *(a) The subalgebras $oc_{N,P}$ and $oc_{N,Q}$ (resp. $spc_{N,P}$ and $spc_{N,Q}$) are conjugated by an automorphism of $Cend_N$ if and only if P and Q are congruent hermitian (resp. skew-hermitian) matrices.*
(b) The subalgebras $oc_{N,P}$ and $spc_{N,Q}$ are not conjugated by any automorphism of $Cend_N$.

A classification of finite irreducible subalgebras of gc_N was given by D'Andrea-Kac. It is natural to propose:

Conjecture 4.4. *Any infinite Lie conformal subalgebra of gc_N acting irreducibly on $\mathbb{C}[\partial]^N$ is conjugate by an automorphism of $Cend_N$ to one of the following subalgebras:*

(a) $gc_{N,P}$, where $\det P \neq 0$,
(b) $oc_{N,P}$, where $\det P \neq 0$ *and* $P(-x) = P^t(x)$,
(c) $spc_{N,P}$, where $\det P \neq 0$ *and* $P(-x) = -P^t(x)$.

This conjecture agrees with the results of the E. Zelamov [Z1]-[Z2] and A. De Sole-V. Kac [DeK]. It is proved in [DeK] that every infinite irreducible Lie conformal subalgebra of gc_N which is sl_2-module (with respect to certain Virasoro-like element of gc_N) is of type $oc_{N,P}$. On the other hand, E. Zelmanov shows that every simple irreducible Lie conformal subalgebra of gc_N of infinite type that contains $\mathrm{Cur}(sl_2)$, is isomorphic to either $gc_{N,P}$ or $oc_{N,P}$.

Acknowledgment. The authors were supported in part by Conicet, ANPCyT, Agencia Cba Ciencia, Secyt-UNC, Fundación Antorchas and Fomec (Argentina). Special thanks go to I. Shestakov and V. Futorny for the hospitality during the conference. We would like to thanks P. Kolesnikov and E. Zelmanov for useful correspondance.

References

[BKL] C. Boyallian, V. Kac and J. Liberati, *On the classification of subalgebras of $Cend_n$ and gc_n*, J. Algebra **260** (2003), 32-63.

[B] S. Bloch, *Zeta values and differential operators on the circle*, J. Algebra **182** (1996), 476-500.

[CK] S. Cheng and V. Kac, *Conformal modules*, Asian J. Math. **1** (1997), no1, 181-193.

[DK] A. D'Andrea and V. Kac, *Structure theory of finite conformal algebras*, Selecta Math. **4** (1998), no. 3, 377-418.

[DeK] A. De Sole and V. Kac, *Subalgebras of gc_N and Jacobi polynomials*, preprint math-ph/0112028.

[K1] V. Kac, *Vertex algebras for beginners. Second edition*, American Mathematical Society, 1998.

[K2] V. Kac, *Formal distributions algebras and conformal algebras*, in Proc. XIIth International Congress of Mathematical Physics (ICMP '97)(Brisbane), 80-97, Internat. Press, Cambridge 1999; preprint math.QA/9709027.

[Ko] P. Kolesnikov, *Associative conformal algebras with finite faithful representation*, arXiv: math.QA/0402330, 2004.

[KR] V. Kac and A. Radul, *Quasifinite highest weight modules over the Lie algebra of differential operators on the circle*, Comm. Math. Phys. **157** (1993), 429-457.

[R] A. Retakh, *Unital associative pseudoalgebras and their representations*, preprint math. QA/0109110.

[Z1] E. Zelmanov, *On the structure of conformal algebras*, Combinatorial and computational algebra (Hong Kong, 1999), 139–153, Contemp. Math., 264, Amer. Math. Soc., Providence, RI, 2000.

[Z2] E. Zelmanov, *Idempotents in conformal algebras*, Proc. of Third Internat. Alg. Conf. (Y. Fong et al, eds) (2003), 257-266.

Carina Boyallian
Ciem - FAMAF
Universidad Nacional de Córdoba
(5000) Córdoba
Argentina
boyallia@mate.uncor.edu

Jose I. Liberati
Ciem - FAMAF
Universidad Nacional de Córdoba
(5000) Córdoba
Argentina
liberati@mate.uncor.edu