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Dimension formulas for the free nonassociative algebra[1]

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Abstract: The free nonassociative algebra contains two subspaces closed under both the commutator and the associator: the Akivis elements and the primitive elements. Every Akivis element is primitive, but there are primitive elements which are not Akivis. Using a theorem of Shestakov, we obtain a recursive formula for the dimension of the Akivis elements. Using a theorem of Shestakov and Umirbaev, we present a closed formula for the dimension of the primitive elements. These results generalize the Witt dimension formula for the Lie elements in the free associative algebra.

1 Introduction

There is a well-developed theory relating free associative algebras and free Lie algebras. One of the most important results is Witt's dimension formula, which gives the dimension of the subspace of Lie polynomials of any degree in the free associative algebra on any number of generators. (By Friedrichs' criterion, the Lie elements are exactly the elements which are primitive in the sense of Hopf algebras.) Within the last few years Shestakov and Umirbaev [5, 6] have proved some important theorems on free nonassociative algebras which have made it possible to generalize this result. In this talk I will review the associative theory and give a proof of the Witt dimension formula. I will then introduce the nonassociative theory in which Akivis algebras play the role of Lie algebras. In the nonassociative case, the Akivis elements and the primitive elements do not coincide. Every Akivis element is primitive, but there are primitive elements which are not Akivis [6]. I will give examples of Akivis and primitive elements in one variable. Using a theorem of Shestakov [5], we derive a recursive formula for the dimension of the Akivis elements. Using a theorem of Shestakov and Umirbaev [6], we present a closed formula for the dimension of the primitive elements.

2 Free associative algebras

Let X be a nonempty countable set. We regard the elements of X as letters in an alphabet. We form all words in these letters. A word of length n is an ordered n-tuple of elements of X. For example, if $X = \{a, b\}$, then here are the words of

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length $1 \leq n \leq 3$:

$$
\begin{aligned}
&a,\ b \\
&aa,\ ab,\ ba,\ bb \\
&aaa,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ bbb
\end{aligned}
$$

We write X^n for the set of all words of length n. If $|X| = r$ then there are r^n words of length n. We write X^* for the set of all words of finite length:

$$X^* = \bigcup_{n \geq 1} X^n.$$

We can multiply two words $v, w \in X^*$ by juxtaposition. For example, the product of aab and bab is just $aabbab$.

Let A be the vector space with basis X^* over some field $\mathbb{F}$. We extend the product in X^* to A by distributivity:

$$\Big(\sum_{i \in I} \alpha_i v_i\Big)\Big(\sum_{j \in J} \beta_j w_j\Big) = \sum_{i \in I, j \in J} \alpha_i \beta_j v_i w_j.$$

This makes A into an associative algebra (without a unit element). If $|X| = r$, say $X = \{a_1, \ldots, a_r\}$, then we call A the **free (non-unital) associative algebra** over $\mathbb{F}$ on r (free) generators.

We write A_n for the subspace of A spanned by the words of length n. That is, A_n is the subspace with basis X^n. Then we have the direct sum decomposition

$$A = \bigoplus_{n \geq 1} A_n,$$

and the algebra A is graded in the sense that

$$A_m A_n \subseteq A_{m+n}.$$

3 Free Lie algebras

On the free associative algebra A we can define a new operation, called the **commutator** or **Lie bracket**, by

$$[f, g] = fg - gf, \text{ for any } f, g \in A.$$

This operation satisfies the identities

$$
\begin{aligned}
&[f, f] = 0 \text{ (anticommutative)}, \\
&[[f, g], h] + [[g, h], f] + [[h, f], g] = 0 \text{ (Jacobi)}.
\end{aligned}
$$

We write A^- for the vector space A under this new operation. Then A^- is a **Lie algebra**: it satisfies anticommutativity and the Jacobi identity.

The (Lie) subalgebra L of A^- generated by the original alphabet $X = \{a_1, \ldots, a_r\}$ is (isomorphic to) the **free Lie algebra** on the alphabet X. Elements of L are called **Lie polynomials** in the alphabet X. If every (associative) term in a Lie polynomial has the same degree we call the polynomial **homogeneous**. Here are the first few homogeneous Lie polynomials on two generators:

$$a, \quad b, \quad [a,b], \quad [[a,b],a], \quad [[a,b],b]$$

We write L_n for the subspace of A_n consisting of all Lie polynomials of degree n. We have

$$L = \bigoplus_{n \geq 1} L_n, \qquad [L_m, L_n] \subseteq L_{m+n}.$$

The dimension of L_n is given by the **Witt dimension formula**:

$$\dim L_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d}.$$

Here the sum is over all (positive) divisors d of the degree n. The **Möbius function** $\mu(d)$ is defined by the rule

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^k & \text{if } d = p_1 \cdots p_k \text{ (distinct } p_i) \\ 0 & \text{if } d \text{ has a square factor} \end{cases}$$

Here is a short table for $r = 2$:

n	1	2	3	4	5	6	7	8	9	10
$\dim L_n$	2	1	2	3	6	9	18	30	56	99

(When r is a prime power the same formula counts the number of monic irreducible polynomials of degree n over the field with r elements.)

How do we prove the Witt dimension formula? Write $\ell_n = \dim L_n$. In each homogeneous subspace L_n of the free Lie algebra choose an ordered basis $\{f_{n1}, \ldots, f_{n\ell_n}\}$. Put these finite bases together into an infinite ordered basis of L:

$$f_{11}, \ldots, f_{1\ell_1}, f_{21}, \ldots, f_{2\ell_2}, f_{31}, \ldots, f_{3\ell_3}, \ldots$$

For simplicity write this last basis as

$$g_1, g_2, g_3, \ldots, g_i, \ldots$$

The **PBW Theorem** (after Poincaré, Birkoff and Witt) states that a basis for A consists of all products of the form

$$g_{i_1} g_{i_2} \cdots g_{i_k} \quad (i_1 \leq i_2 \leq \cdots \leq i_k). \tag{1}$$

Now we need to do some combinatorics. There are ℓ_d Lie polynomials g_i of degree d. Each g_i contributes degree d to the total degree of the associative word in (1). But each g_i may occur any number k of times (consecutively) in (1). The generating function for the contribution of these k elements of degree d to the total degree of (1) is

$$1 + x^d + x^{2d} + \cdots + x^{kd} + \cdots = \frac{1}{1 - x^d}.$$

Therefore the generating function for the contribution of the ℓ_d basis elements of degree d is

$$\left(\frac{1}{1 - x^d} \right)^{\ell_d}.$$

Combining these factors for all d gives the generating function

$$\prod_{d \geq 1} \left(\frac{1}{1 - x^d} \right)^{\ell_d}.$$

The PBW Theorem says that this must equal the generating function for the dimensions of the homogeneous subspaces of the free associative algebra:

$$\prod_{d \geq 1} \left(\frac{1}{1 - x^d} \right)^{\ell_d} = \sum_{n \geq 0} r^n x^n.$$

Therefore

$$\frac{1}{1 - rx} = \prod_{d \geq 1} \left(\frac{1}{1 - x^d} \right)^{\ell_d}.$$

Taking reciprocals and then logarithms gives:

$$\log(1 - rx) = \sum_{d \geq 1} \ell_d \log(1 - x^d).$$

Using the power series for the logarithm we get

$$\sum_{n \geq 1} \frac{r^n x^n}{n} = \sum_{d \geq 1} \ell_d \sum_{k \geq 1} \frac{x^{dk}}{k}.$$

Equating coefficients of x^n on both sides gives

$$\frac{r^n}{n} = \sum_{d|n} \frac{\ell_d}{n/d} = \sum_{d|n} \frac{d\ell_d}{n}.$$

Therefore

$$r^n = \sum_{d|n} d\ell_d.$$

Now an application of the Möbius Inversion Formula gives

$$n\ell_n = \sum_{d|n} \mu(d) r^{n/d}.$$

Dividing by n gives the Witt dimension formula.

4 Free nonassociative algebras

Now we generalize all this to the case of a free nonassociative algebra.

Let $x_1 x_2 \cdots x_n$ be an associative word of degree n. The number of distinct ways to put parentheses into this word is given by the **Catalan number**:

$$K_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Here is a short table:

n	1	2	3	4	5	6	7	8	9
K_n	1	1	2	5	14	42	132	429	1430

Let X be a nonempty countable set. We form all nonassociative words in these letters. A word of length n is an ordered n-tuple of elements of X together with one of the K_n association types of degree n. For example, if $X = \{a\}$, then here are the words of length $1 \leq n \leq 4$:

$$a, \quad aa, \quad (aa)a, \quad a(aa)$$
$$((aa)a)a, \quad (a(aa))a, \quad (aa)(aa), \quad a((aa)a), \quad a(a(aa))$$

If $|X| = r$ then there are $K_n r^n$ words of length n. We can multiply two nonassociative words v, w by juxtaposition but we must include parentheses around each factor. For example, the product of $(aa)a$ and $a(aa)$ is just $((aa)a)(a(aa))$. Let N be the vector space with basis consisting of all nonassociative words in the alphabet X. We extend the product of nonassociative words to N by distributivity. This makes N into a nonassociative algebra (without a unit element). If $|X| = r$, say $X = \{a_1, \ldots, a_r\}$, then we call N the **free nonassociative algebra** over $\mathbb{F}$ on r (free) generators. Write N_n for the subspace of N spanned by the words of length n. Then we have the direct sum decomposition

$$N = \bigoplus_{n \geq 1} N_n,$$

and the algebra N is graded in the sense that

$$N_m N_n \subseteq N_{m+n}.$$

5 Free Akivis algebras

On the free nonassociative algebra N we can define two new operations, the **commutator** and the **associator**:

$$[f,g] = fg - gf, \text{ for any } f,g \in A,$$
$$\{f,g,h\} = (fg)h - f(gh), \text{ for any } f,g,h \in A.$$

These operations satisfy the identities

$$\begin{aligned}
&[f,f] = 0,\\
&[[f,g],h] + [[g,h],f] + [[h,f],g]\\
&\quad = \{f,g,h\} + \{g,h,f\} + \{h,f,g\} - \{f,h,g\} - \{g,f,h\} - \{h,g,f\}.
\end{aligned}$$

The second identity is called the **Akivis identity**. It relates the commutator and the associator in any nonassociative algebra. We write N^- for the vector space N under these two new operations (one binary, one ternary). Then N^- is an **Akivis algebra** [1, 3]: it satisfies anticommutativity and the Akivis identity. If the underlying algebra is associative, the associator is identically zero. So the Akivis identity reduces to the Jacobi identity, and an Akivis algebra is just a Lie algebra.

Shestakov [5] has shown that the subalgebra Ak of N^- generated (using commutators and associators) by the original alphabet $X = \{a_1, \ldots, a_r\}$ is (isomorphic to) the **free Akivis algebra** on the alphabet X. Elements of Ak are called **Akivis elements** in the alphabet X. Here are the first few homogeneous Akivis elements on one generator:

$$\begin{aligned}
&a, \quad \{a,a,a\}, \quad [\{a,a,a\},a]\\
&[[\{a,a,a\},a],a], \quad \{\{a,a,a\},a,a\}, \quad \{a,\{a,a,a\},a\}, \quad \{a,a,\{a,a,a\}\}
\end{aligned}$$

We write Ak_n for the subspace of N_n consisting of all Akivis polynomials of degree n. We have

$$Ak = \bigoplus_{n \geq 1} Ak_n.$$

In joint work with I. Hentzel and L. Peresi, we have found a recursive formula for $\dim Ak_n$. However, to get the natural generalization of the Witt dimension formula, we need to consider another subalgebra of N^-.

6 Primitive elements

We make the tensor product $N \otimes N$ into an algebra by defining

$$(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 f_2 \otimes g_1 g_2,$$

and extending to sums of tensors using distributivity. We define an algebra homomorphism $\Delta\colon N \to N \otimes N$ by setting

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \text{ for all } a \in X,$$

and extending the domain to N by linearity and the homomorphism property $\Delta(fg) = \Delta(f)\Delta(g)$. (In the associative case, this is the usual definition of the comultiplication for the Hopf algebra structure on the free associative algebra.) We say that an element $f \in N$ is **primitive** if it satisfies the condition

$$\Delta(f) = f \otimes 1 + 1 \otimes f.$$

(In the associative case, the primitive elements coincide with the Lie polynomials, by a theorem of Friedrichs.)

For example, we calculate $\Delta(a^2)$ as follows:

$$\begin{aligned}\Delta(a^2) &= \Delta(a)^2 = (a \otimes 1 + 1 \otimes a)(a \otimes 1 + 1 \otimes a) \\ &= a^2 \otimes 1 + 2a \otimes a + 1 \otimes a^2 \neq a^2 \otimes 1 + 1 \otimes a^2.\end{aligned}$$

It follows that a^2 is not primitive, at least when the characteristic of $\mathbb{F}$ is not 2.

Let Pr denote the subspace of all primitive elements in the free nonassociative algebra N. We write Pr_n for the subspace of N_n consisting of all primitive elements of degree n. We have

$$Pr = \bigoplus_{n \geq 1} Pr_n.$$

It is not difficult to show that the commutator of primitive elements is again primitive, and that the associator of primitive elements is again primitive. It follows that every Akivis element is primitive, and that the subspace of primitive elements is closed under commutators and associators; that is, the subspace of primitive elements is an Akivis subalgebra of N^-.

The first degree in which there exist primitive elements which are not Akivis is $n = 4$. A primitive non-Akivis element in degree 4 was first discovered by Shestakov and Umirbaev [6]. For one generator, the space of Akivis elements is one-dimensional and is spanned by $[\{a, a, a\}, a]$. The space of primitive elements has dimension 3. A basis for the primitive elements in degree 4 consists of the Akivis element together with the two primitive non-Akivis elements

$$\begin{aligned} f &= \{a^2, a, a\} - \{a, a, a^2\}, \\ g &= \{a, a, a\}a + a\{a, a, a\} - \{a, a^2, a\}. \end{aligned}$$

The element

$$h = \{a, a^2, a\}a - \{a^2, a, a^2\} + a\{a, a^2, a\}$$

is a new primitive element in degree 5, which is not an Akivis element, and cannot be generated by primitive elements of lower degree. Shestakov and Umirbaev [6] give a recursive construction of a complete set of primitive elements in any number of variables using the hyperalgebra operations from Sabinin and Mikheev [4].

The primitive elements rather than the Akivis elements generalize the Lie polynomials. In joint work with I. Hentzel and L. Peresi, we have found an exact formula for $\dim Pr_n$ which generalizes the Witt dimension formula for Lie algebras. It depends on a theorem of Shestakov and Umirbaev [6] which generalizes the PBW Theorem to free nonassociative algebras.

Let $p_n = \dim Pr_n$. In each homogeneous subspace Pr_n of primitive elements in N choose an ordered basis $\{f_{n1}, \ldots, f_{np_n}\}$. Put these finite bases together into an infinite ordered basis of Pr:

$$f_{11}, \ldots, f_{1p_1}, f_{21}, \ldots, f_{2p_2}, f_{31}, \ldots, f_{3p_3}, \ldots$$

For simplicity write this last basis as

$$g_1, g_2, g_3, \ldots, g_i, \ldots$$

Then the Shestakov-Umirbaev theorem states that a basis for N consists of all left-normalized products of the form

$$(\cdots((g_{i_1} g_{i_2}) g_{i_3}) \cdots) g_{i_k} \quad (i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_k).$$

It follows as in the associative case that

$$\prod_{d \geq 1} \left(\frac{1}{1 - x^d} \right)^{p_d} = \sum_{n \geq 0} K_n r^n x^n.$$

The exponents on the left side are now p_d (the dimensions of the homogeneous subspaces of primitive elements), and the coefficient of x^n on the right side now has the additional complicating factor of K_n (the Catalan number).

This extra K_n factor makes it harder to prove the desired dimension formula for p_n, since the right side no longer has the simple form $1/(1 - rx)$ as in the associative case.

To get around this difficulty it is necessary to consider the generating function of the Catalan numbers:

$$K(x) = \sum_{n \geq 0} K_n x^n = \frac{3 - \sqrt{1 - 4x}}{2}.$$

Then we have

$$\prod_{d \geq 1} \left(\frac{1}{1 - x^d} \right)^{p_d} = K(rx).$$

Recall the proof of the Witt dimension formula: we take reciprocals, and then logarithms, and then compare coefficients of x^n on both sides. In order to do this in the nonassociative case, we have to understand the logarithm of the Catalan generating function. More precisely, we need the logarithmic derivative

$$\frac{d}{dx}\log K(x) = \frac{K'(x)}{K(x)} = \sum_{n\geq 0} \lambda_n x^n.$$

The coefficients λ_n of this power series are integers.

Here is a short table:

n	0	1	2	3	4	5	6	7	8
λ_n	1	1	4	13	46	166	610	2269	8518

We searched for this sequence in Neil Sloane's *On-Line Encyclopedia of Integer Sequences* [7]. It occurs as number A026641: λ_n counts the number of nodes of even outdegree (including leaves) in all ordered trees with n edges. There is a closed formula for these numbers:

$$\lambda_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{2n-2j-1}{n-1}. \tag{2}$$

(This is a simplified version of the on-line formula. It was communicated to us by Emeric Deutsch [2].) The proof of this formula depends on two identities for the Catalan generating function [8]:

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0}\binom{2n+k}{n}x^n,$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0}\frac{k}{n+k}\binom{2n+k-1}{n}x^n.$$

Using formula (2) we can derive a closed formula for the dimensions of the subspaces of primitive elements in the free nonassociative algebra:

$$\dim Pr_n = \frac{1}{n}\sum_{d|n}\mu(d)\lambda_{n/d-1}r^{n/d}.$$

Here is a table of the dimensions of the Akivis and primitive subspaces in the free nonassociative algebra on one generator up to degree 9:

n	1	2	3	4	5	6	7	8	9
$\dim Ak_n$	1	0	1	1	4	7	23	53	157
$\dim Pr_n$	1	0	1	3	9	27	87	282	946
K_n	1	1	2	5	14	42	132	429	1430

The ratio $\dim Ak_n / \dim Pr_n$ becomes smaller as n increases.

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