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Symmetric Composition Algebras and Freudenthal's Magic Square

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Abstract: The well-known Tits construction provides models of the exceptional simple Lie algebras using, as ingredients, a unital composition algebra and a degree three central simple Jordan algebra. By varying both ingredients, Freudenthal's Magic Square is obtained. On the other hand, since degree three simple Jordan algebras can be obtained as 3×3 matrices over unital composition algebras, Tits construction can be interpreted as a construction based on two unital composition algebras. But, even though the construction is not symmetric on the two algebras, so is the outcome (the Magic Square). Several more symmetric constructions have been proposed, based on the triality phenomenon. Here a new construction of the Magic Square will be surveyed, based on a pair of the so called "symmetric composition algebras", which provide very simple formulas for triality. Finally, some of these symmetric composition algebras can be constructed in terms of copies of the two-dimensional natural module for $\mathfrak{sl}_2$. This gives a unified construction of the simple exceptional Lie algebras in terms of these tiny ingredients: copies of $\mathfrak{sl}_2$ and of its natural module.

Key words: Freudenthal Magic Square, symmetric composition algebra, triality, exceptional Lie algebra

1 Tits construction

In 1966 Tits gave a unified construction of the exceptional simple Lie algebras which uses a couple of ingredients: a unital composition algebra C and a simple Jordan algebra J of degree 3 [Tit66]. At least in the split cases, this is a construction which depends on two unital composition algebras, since the Jordan algebra involved consists of the 3×3-hermitian matrices over a unital composition algebra. Even though the construction is not symmetric in the two composition algebras that are being used, the outcome (Freudenthal's Magic Square [Sch95, Fre64]) is indeed symmetric. Let us first review his construction.

Assume throughout the paper a ground field F of characteristic $\neq 2,3$.

Let A be a unital composition algebra over F. Thus A is endowed with a nondegenerate quadratic form $n : A :\rightarrow F$ admitting composition:

$$n(ab) = n(a)n(b)$$

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for any $a, b \in A$. It is well-known (see [Sch95]) that in this case, for any $a, b \in A$, the linear map

$$D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \tag{1.1}$$

is a derivation of A, where L_a and R_a denote the left and right multiplications by the element $a \in A$. The subspace orthogonal to the unity will be denoted by A^0, and $D_{A,A}$ will denote the linear span of the derivations in (1.1), which fill the whole Lie algebra $\operatorname{Der} A$ of derivations of A. Any unital composition algebra is quadratic:

$$x^2 - n(x,1)x + n(x)1 = 0$$

for any x, and it is endowed with a standard involution $x \mapsto \bar{x} = n(x,1)1 - x$. Moreover, it is either the ground field F, a quadratic étale extension K of F, a generalized quaternion algebra Q (dimension 4) or a Cayley algebra C (dimension 8).

On the other hand, let J be a unital Jordan algebra over F with a normalized trace $t : J \to F$ (that is, t is linear and satisfies $t(1) = 0$ and $t\big((ab)c\big) = t\big(a(bc)\big)$ for any $a, b, c \in J$). Then, for any $x, y \in J$,

$$xy = t(xy)1 + x * y,$$

where $x * y \in J^0 = \ker t$. Here, given any two elements $x, y \in J$, the linear map

$$d_{x,y} = [l_x, l_y] \tag{1.2}$$

is a derivation of J, where l_x denotes the left multiplication by $x \in J$. Denote by $d_{J,J}$ the linear span of the derivations in (1.2), which is a subalgebra of $\operatorname{Der} J$.

Tits considered the vector space

$$\mathcal{T}(A, J) = D_{A,A} \oplus \Big(A^0 \otimes J^0\Big) \oplus d_{J,J}$$

with the anticommutative product [,] specified by:

- $D_{A,A}$ and $d_{J,J}$ are Lie subalgebras,
- $[D_{C,C}, d_{J,J}] = 0$,
- $[D, a \otimes x] = D(a) \otimes x$, $[d, a \otimes x] = a \otimes d(x)$, for any $D \in D_{A,A}$, $d \in d_{J,J}$, $a \in A$ and $x \in J$,
- $[a \otimes x, b \otimes y] = t(xy)D_{a,b} + [a,b] \otimes x * y - 2n(a,b)d_{x,y}$, for any $a, b \in A$ and $x, y \in J$. (Here $n(a,b) = n(a+b) - n(a) - n(b)$.)

By imposing the Jacobi identity to elements in $A^0 \otimes J^0$, it is easily checked that $\mathcal{T}(A, J)$ is a Lie algebra if and only if:

(i) $0 = \sum_{\text{cyclic}} n\big([a_1, a_2], a_3\big) d_{(x_1 * x_2), x_3}$,

(ii) $0 = \sum_{\text{cyclic}} t\big((x_1 * x_2)x_3\big) D_{[a_1,a_2],a_3}$

(iii) $0 = \sum_{\text{cyclic}} \big(D_{a_1,a_2}(a_3) \otimes t(x_1x_2)x_3 + [[a_1, a_2], a_3] \otimes (x_1 * x_2) * x_3 - 2n(a_1, a_2)a_3 \otimes d_{x_1,x_2}(x_3)\big)$,

for any $a_i \in A^0$, $x_i \in J^0$ $(i = 1, 2, 3)$.

These conditions hold if J satisfies the Cayley-Hamilton equation $ch_3(x) = 0$, where

$$ch_3(x) = x^3 - 3t(x)x^2 + \big(\tfrac{9}{2}t(x)^2 - \tfrac{3}{2}t(x^2)\big)x - \Big(t(x^3) - \tfrac{9}{2}t(x^2)t(x) + \tfrac{9}{2}t(x)^3\Big)1 .$$

In particular, this happens if J is a central simple Jordan algebra of degree 3. In the split cases (for instance, over algebraically closed fields), any such algebra consists of the set of hermitian 3×3 matrices $H_3(A)$ over a unital composition algebra A. The Lie algebras thus obtained form the famous Freudenthal's Magic Square:

$A \setminus J$	$H_3(F)$	$H_3(K)$	$H_3(Q)$	$H_3(C)$
F	A_1	A_2	C_3	F_4
K	A_2	$A_2 \oplus A_2$	A_5	E_6
Q	C_3	A_5	D_6	E_7
C	F_4	E_6	E_7	E_8

This wonderful construction gives models of the exceptional simple Lie algebras of type F_4, E_6, E_7 and E_8. The remaining exceptional simple Lie algebra, G_2 appears as $\operatorname{Der} C = D_{C,C}$ for a Cayley algebra C. The Magic Square depends on two unital composition algebras in a very nonsymmetric way, however the outcome is symmetric.

Remark 1.3. Jordan algebras may be substituted by Jordan superalgebras above, by imposing that the odd part be contained in the kernel of the normalized trace. However, only the following finite dimensional simple Jordan superalgebras satisfy that the superanticommutative superalgebra $\mathcal{T}(A, J)$ thus constructed is a Lie superalgebra (see [BE03]):

(i) $J(V) = F1 \oplus V$, the Jordan superalgebra of a supersymmetric bilinear form such that $V = V_{\bar{1}}$ and $\dim V = 2$, and

(ii) $D_\mu = (Fe \oplus Ff) \oplus (Fx \oplus Fy)$ $(\mu \neq 0, -1)$, with multiplication given by

$$\begin{gathered} e^2 = e, \quad f^2 = f, \quad ef = 0 \\ ex = \tfrac{1}{2}x = fx, \quad ey = \tfrac{1}{2}y = fy, \\ xy = e + \mu f = -yx. \end{gathered}$$

The following extension of Freudenthal's Magic Square, where all the exceptional simple classical Lie superalgebras appear, is obtained:

$A \setminus J$	$J(V)$	D_μ $(\mu\neq 0,-1)$
F	A_1	$B(0,1)$
K	$B(0,1)$	$A(1,0)$
Q	$B(1,1)$	$D(2,1;\mu)$
C	$G(3)$	$F(4)$ $(\mu=2,1/2)$

Over the years, more symmetric constructions have been given of Freudenthal's Magic Square, starting with a construction by Vinberg in 1966 [OV94]. He considered two unital composition algebras C and C' and defined a (somehow involved) Lie bracket on the direct sum

$$\mathcal{V}(C,C') = \operatorname{Der} C \oplus \operatorname{Der} C' \oplus \operatorname{Skew}\Big(\operatorname{Mat}_3(C \otimes C'), *\Big)$$

where $*$ is the natural involution on $\operatorname{Mat}_3(C \otimes C')$.

Later on, a quite general construction was given by Allison and Faulkner [AF93] of Lie algebras out of structurable ones. In the particular case of the tensor product of two unital composition algebras, this construction provides another symmetric construction of Freudenthal's Magic Square. Quite recently, Barton and Sudbery [BS00, BS03] (see also Landsberg and Manivel [LM02, LM02']) gave a simple recipe to obtain the Magic Square in terms of two unital composition algebras and their triality Lie algebras which, in perspective, is subsumed in Allison-Faulkner's construction. This construction is based on a Lie bracket on the direct sum:

$$\mathcal{BS}(C,C') = \mathfrak{Tri}(C) \oplus \mathfrak{Tri}(C') \oplus 3 \text{ copies of } C \otimes C',$$

where the *triality Lie algebra* is defined as follows:

$$\mathfrak{Tri}(C) = \{(f, g, h) \in \mathfrak{o}(C)^3 : f(xy) = h(x)y + xg(y)\ \forall x, y \in C\}.$$

For a nice survey of the octonions, the Magic Square, the exceptional Lie algebras and geometries associated to them, see [Bae02].

2 Symmetric composition algebras

As shown in [KMRT98], the triality phenomenon is better dealt with by means of the so called *symmetric composition algebras*, instead of the classical unital composition algebras.

Let us recall the definitions. A *symmetric composition algebra* is a triple $(S, *, q)$, where $(S, *)$ is a (nonassociative) algebra over F with multiplication denoted by $x * y$ for $x, y \in S$, and where $q : S \to F$ is a regular quadratic form satisfying:

$$\begin{aligned} q(x * y) &= q(x)q(y) \\ q(x * y, z) &= q(x, y * z) \end{aligned}$$

for any $x, y, z \in S$.

The main examples of symmetric composition algebras are the following:

Para-Hurwitz algebras [OkM80] Given a Hurwitz algebra C with norm n, consider the triple $(C, *, q)$ with

$$\begin{cases} x * y = \bar{x}\bar{y} \\ q(x) = n(x) \end{cases}$$

for any $x, y \in C$, where $x \mapsto \bar{x}$ denotes the standard involution in C. $(C, *, q)$ is a symmetric composition algebra, termed the *para-Hurwitz* algebra associated to C.

Okubo algebras [Ok78] Assume that the ground field is algebraically closed and (since we are assuming that the characteristic is $\neq 2, 3$) take $\omega \neq 1$ a cubic root of 1. Let $S = \mathfrak{sl}_3(F)$ be the Lie algebra of 3×3 matrices of trace 0. Then the triple $(S, *, q)$, where

$$\begin{cases} x * y = \omega xy - \omega^2 yx - \dfrac{\omega - \omega^2}{3} \operatorname{tr}(xy)1 \\ q(x) = -\frac{1}{2} \operatorname{tr}(x^2) \end{cases}$$

for any $x, y \in S$, is a symmetric composition algebra. Its forms are called *Okubo algebras*.

The classification of the symmetric composition algebras was obtained in [EM93]. With a few exceptions in dimension 2, they are either para-Hurwitz or Okubo algebras, and the latter ones are obtained in a precise way from simple degree 3 associative algebras.

If $(S,*,q)$ is any symmetric composition algebra, consider the corresponding orthogonal Lie algebra

$$\mathfrak{o}(S,q)=\{d\in \mathrm{End}_F(S): q\big(d(x),y\big)+q\big(x,d(y)\big)=0\ \forall x,y\in S\},$$

and the subalgebra of $\mathfrak{o}(S,q)^3$ defined by

$$\begin{aligned}\mathfrak{tri}(S,*,q)=\{(d_0,d_1,d_2)\in \mathfrak{o}(S,q)^3:\\ d_0(x*y)=d_1(x)*y+x*d_2(y)\ \forall x,y\in S\}.\end{aligned}$$

If $\dim S=8$, then the projections $(d_0,d_1,d_2)\mapsto d_i$ $(i=0,1,2)$ give three different isomorphisms $\mathfrak{tri}(S,*,q)\cong \mathfrak{o}(S,q)$, thus obtaining the natural and the two half-spin representations of the Lie algebra $\mathfrak{o}(S,q)$ (of type D_4).

The map

$$\theta:\mathfrak{tri}(S,*,q)\to \mathfrak{tri}(S,*,q),\quad (d_0,d_1,d_2)\mapsto (d_2,d_0,d_1),$$

is an automorphism of $(S,*,q)$ of order 3, the *triality automorphism.* Its fixed subalgebra is (isomorphic to) the derivation algebra of $(S,*)$ which, if the dimension is 8 and the characteristic of the ground field is $\neq 2,3$, is a simple Lie algebra of type G_2 in the para-Hurwitz case and a simple Lie algebra of type A_2 (a form of $\mathfrak{sl}_3$) in the Okubo case.

The simplicity of these formulae lead the author [Eld04] to reinterpret the Barton-Sudbery's construction in terms of two symmetric composition algebras. The construction in [Eld04] using para-Hurwitz algebras reduces naturally to the construction by Barton and Sudbery (although with slightly simpler formulas). Okubo algebras provide new constructions that highlights different order 3 automorphisms and different subalgebras of the exceptional simple Lie algebras. For the relationship between the constructions with either para-Hurwitz or Okubo algebras, you may consult [Eld04'].

For any $x,y\in S$, the triple

$$t_{x,y}=\left(\sigma_{x,y},\frac{1}{2}q(x,y)id-r_xl_y,\frac{1}{2}q(x,y)id-l_xr_y\right)\tag{2.1}$$

is in $\mathfrak{tri}(S,*,q)$, where $\sigma_{x,y}(z)=q(x,z)y-q(y,z)x$, $r_x(z)=z*x$, and $l_x(z)=x*z$ for any $x,y,z\in S$.

The construction given in [Eld04] starts with two symmetric composition algebras $(S,*,q)$ and $(S',*,q')$. Then define $\mathfrak{g}=\mathfrak{g}(S,S')$ to be the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded anticommutative algebra such that $\mathfrak{g}_{(\bar 0,\bar 0)}=\mathfrak{tri}(S,*,q)\oplus\mathfrak{tri}(S',*,q')$, $\mathfrak{g}_{(\bar 1,\bar 0)}=\mathfrak{g}_{(\bar 0,\bar 1)}=\mathfrak{g}_{(\bar 1,\bar 1)}=S\otimes S'$.

For any $a \in S$ and $x \in S'$, denote by $\iota_i(a \otimes x)$ the element $a \otimes x$ in $\mathfrak{g}_{(\bar{1},\bar{0})}$ (respectively $\mathfrak{g}_{(\bar{0},\bar{1})}$, $\mathfrak{g}_{(\bar{1},\bar{1})}$) if $i = 0$ (respectively, $i = 1, 2$). Thus

$$\mathfrak{g} = \mathfrak{g}(S, S') = \left(\mathfrak{tri}(S, *, q) \oplus \mathfrak{tri}(S', *, q')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(S \otimes S')\right).$$

The anticommutative multiplication on $\mathfrak{g}$ is defined by means of:

- $\mathfrak{g}_{(\bar{0},\bar{0})}$ is a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(a \otimes x)] = \iota_i\big(d_i(a) \otimes x\big)$, $[(d'_0, d'_1, d'_2), \iota_i(a \otimes x)] = \iota_i\big(a \otimes d'_i(x)\big)$, for any $(d_0, d_1, d_2) \in \mathfrak{tri}(S, *, q)$, $(d'_0, d'_1, d'_2) \in \mathfrak{tri}(S', *, q')$, $a \in S$ and $x \in S'$.
- $[\iota_i(a \otimes x), \iota_{i+1}(b \otimes y)] = \iota_{i+2}\big((a * b) \otimes (x * y)\big)$ (indices modulo 3), for any $a, b \in S$, $x, y \in S'$.
- $[\iota_i(a \otimes x), \iota_i(b \otimes y)] = q'(x, y)\theta^i(t_{a,b}) + q(a, b)\theta'^i(t'_{x,y})$, for any $i = 0, 1, 2$, $a, b \in S$ and $x, y \in S'$, where $t_{a,b} \in \mathfrak{tri}(S, *, q)$ (respectively $t'_{x,y} \in \mathfrak{tri}(S', *, q')$) is the element in (2.1) for $a, b \in S$ (resp. $x, y \in S'$) and θ (resp. θ') is the triality automorphism of $\mathfrak{tri}(S, *, q)$ (resp. $\mathfrak{tri}(S', *, q')$).

The main result in [Eld04] asserts that, with this multiplication, $\mathfrak{g}(S, S')$ is a Lie algebra (which turns out to be very easy to check, due to the symmetry of the construction) and Freudenthal's Magic Square is recovered:

		$\dim S$			
		1	2	4	8
$\dim S'$	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

3 Exceptional simple Lie algebras

The split para-Hurwitz algebras of dimension 4 and 8 can be easily described in terms of the smallest three dimensional simple split Lie algebra $\mathfrak{sl}_2(F)$ and its natural two-dimensional module. As a consequence, all the exceptional split simple Lie algebras in the Magic Square can be constructed in very simple terms using copies of $\mathfrak{sl}_2(F)$ and of its natural module. Here we will indicate how this is done for the Lie algebra of type E_7. For complete details see [Eld04"].

Let V be a two dimensional vector space over F, endowed with a nonzero skew-symmetric bilinear form $\langle .|. \rangle$. Consider the symplectic Lie algebra

$$\mathfrak{sp}(V) = \operatorname{span}\left\{\gamma_{a,b} = \langle a|.\rangle b + \langle b|.\rangle a : a, b \in V\right\},$$

which coincides with $\mathfrak{sl}(V) \cong \mathfrak{sl}_2(F)$. The bilinear form allows $Q = V \otimes V$ to be identified with $\mathrm{End}_F(V)$ (the split quaternion algebra over F) by means of:

$$\begin{aligned} V \otimes V &\longrightarrow \mathrm{End}_F(V) \\ a \otimes b &\mapsto \langle a|.\rangle b\,. \end{aligned}$$

Then the split para-Hurwitz algebra of dimension 8 is, up to isomorphism, $(S_8, *, q_8)$ with

$$\begin{cases} S_8 = V \otimes V \oplus V \otimes V, \\ (a \otimes b, c \otimes d) * (a' \otimes b', c' \otimes d') \\ \quad = \Big(\langle b|a'\rangle b' \otimes a - \langle d|d'\rangle c \otimes c', -\langle a|c'\rangle b \otimes d' - \langle c|b'\rangle a' \otimes d\Big), \\ q_8\Big((a \otimes b, c \otimes d), (a' \otimes b', c' \otimes d')\Big) = \langle a|a'\rangle\langle b|b'\rangle + \langle c|c'\rangle\langle d|d'\rangle\,. \end{cases}$$

The Lie algebra $\mathfrak{sp}(V)^4$ acts naturally on S_8, where the i^{th} component of $\mathfrak{sp}(V)^4$ acts on the i^{th} copy of V in $S_8 = V \otimes V \oplus V \otimes V$. This gives an embedding into the orthogonal Lie algebra of S_8 relative to q:

$$\rho : \mathfrak{sp}(V)^4 \longrightarrow \mathfrak{o}(S_8, q).$$

Actually, this is an isomorphism of $\mathfrak{sp}(V)^4$ onto the subalgebra $\mathfrak{o}(V \otimes V) \oplus \mathfrak{o}(V \otimes V)$ of $\mathfrak{o}(S_8, q)$, which is the even part of $\mathfrak{o}(S_8, q)$ relative to the $\mathbb{Z}_2$-grading given by the orthogonal decomposition $S_8 = V \otimes V \oplus V \otimes V$.

Consider also the linear map (denoted by ρ too):

$$\begin{aligned} \rho :\ V^{\otimes 4} &\longrightarrow \mathfrak{o}(S_8, q) \\ v_1 \otimes v_2 \otimes v_3 \otimes v_4 &\mapsto \rho(v_1 \otimes v_2 \otimes v_3 \otimes v_4)\,, \end{aligned}$$

such that

$$\begin{aligned} &\rho(v_1 \otimes v_2 \otimes v_3 \otimes v_4)\big((w_1 \otimes w_2, w_3 \otimes w_4)\big) \\ &\qquad = \Big(\langle v_3|w_3\rangle\langle v_4|w_4\rangle v_1 \otimes v_2, -\langle v_1|w_1\rangle\langle v_2|w_2\rangle v_3 \otimes v_4\Big)\,, \end{aligned}$$

for any $v_i, w_i \in V$, $i = 1, 2, 3, 4$.

It can be proved that, by means of ρ, $\mathfrak{o}(S_8, q_8)$ is isomorphic to the direct sum $\mathfrak{d}_4 = \mathfrak{sp}(V)^4 \oplus V^{\otimes 4}$, endowed with a 'natural' Lie bracket.

Let us consider now the order three automorphism $\theta : \mathfrak{d}_4 \to \mathfrak{d}_4$ such that:

$$\begin{cases} \theta\big((s_1, s_2, s_3, s_4)\big) = (s_3, s_2, s_1, s_4), \\ \theta(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_3 \otimes v_1 \otimes v_2 \otimes v_4, \end{cases}$$

for any $s_i \in \mathfrak{sp}(V)$ and $v_i \in v$, $i = 1, 2, 3, 4$. Then, for any $f \in \mathfrak{d}_4$ and any $x, y \in S_8$:

$$\rho(f)(x * y) = \Big(\rho(\theta^{-1}(f))(x)\Big) * y + x * \Big(\rho(\theta^{-2}(f))(y)\Big)\,.$$

so, with $\rho_i = \rho \circ \theta^{-i}$, $i = 0, 1, 2$,

$$\mathfrak{tri}(S_8, *, q_8) = \left\{ \big((\rho_0(f), \rho_1(f), \rho_2(f) \big) : f \in \mathfrak{d}_4 \right\}.$$

Denote by V_i the $\mathfrak{sp}(V)^4$-module V on which only the i^{th} component acts: $(s_1, s_2, s_3, s_4).v = s_i(v)$ for any $s_i \in \mathfrak{sp}(V)$, $i = 1, 2, 3, 4$, and $v \in V$. Then, as $\mathfrak{sp}(V)^4$-module, S_8 is isomorphic to:

$$\begin{cases} V_1 \otimes V_2 \,\oplus\, V_3 \otimes V_4 & \text{through } \rho_0, \\ V_2 \otimes V_3 \,\oplus\, V_1 \otimes V_4 & \text{through } \rho_1, \\ V_1 \otimes V_3 \,\oplus\, V_2 \otimes V_4 & \text{through } \rho_2. \end{cases}$$

Notice that for the split 'para-quaternion' algebra $S_4 = V \otimes V$, by restriction we obtain:

$$\mathfrak{tri}(S_4, *, q_4) = \left\{ \big(\tilde\rho_0(f), \tilde\rho_1(f), \tilde\rho_2(f) \big) : f \in \mathfrak{sp}(V)^3 \right\},$$

where $\tilde\rho_i : \mathfrak{sp}(V)^3 \to \mathfrak{o}(S_4, q_4)$ $(i = 0, 1, 2)$ is obtained by restriction of ρ_i.

At this point, some extra notation is needed. For any $n \in \mathbb{N}$ and any subset $\sigma \subseteq \{1, 2, \dots, n\}$ consider the $\mathfrak{sp}(V)^n$-modules given by

$$V(\sigma) = \begin{cases} \mathfrak{sp}(V)^n & \text{if } \sigma = \emptyset, \\ V_{i_1} \otimes \cdots \otimes V_{i_r} & \text{if } \sigma = \{i_1, \dots, i_r\},\ 1 \le i_1 < \cdots < i_r \le n. \end{cases}$$

As before, V_i denotes the module V for the i^{th} component of $\mathfrak{sp}(V)^n$, annihilated by the other $n - 1$ components.

Identify any subset $\sigma \subseteq \{1, \dots, n\}$ with the element $(\sigma_1, \dots, \sigma_n) \in \mathbb{Z}_2^n$ such that $\sigma_i = 1$ if and only if $i \in \sigma$. Then for any $\sigma, \tau \in \mathbb{Z}_2^n$, consider the natural $\mathfrak{sp}(V)^n$-invariant maps

$$\varphi_{\sigma,\tau} : V(\sigma) \times V(\tau) \longrightarrow V(\sigma + \tau) \tag{3.1}$$

defined as follows:

- If $\sigma \ne \tau$ and $\sigma \ne \emptyset \ne \tau$, then $\varphi_{\sigma,\tau}$ is obtained by contraction, by means of $\langle .|. \rangle$ in the indices i with $\sigma_i = 1 = \tau_i$. Thus, for instance,

  $$\varphi_{\{1,2,3\},\{1,3,4\}}(v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_3 \otimes w_4) = \langle v_1 | w_1 \rangle \langle v_3 | w_3 \rangle v_2 \otimes w_4$$

  for any $v_1, w_1 \in V_1$, $v_2 \in V_2$, $v_3, w_3 \in V_3$ and $w_4 \in V$.

- $\varphi_{\emptyset,\emptyset}$ is the Lie bracket in $\mathfrak{sp}(V)^n$.

- For any $\sigma \ne \emptyset$, $\varphi_{\emptyset,\sigma} = -\varphi_{\sigma,\emptyset}$ is given by the natural action of $\mathfrak{sp}(V)^n$ on $V(\sigma)$. Thus, for instance,

  $$\varphi_{\emptyset,\{1,3\}}\big((s_1, \dots, s_n), v_1 \otimes v_3 \big) = s_1(v_1) \otimes v_3 + v_1 \otimes s_3(v_3),$$

  for any $s_i \in \mathfrak{sp}(V)$, $i = 1, \dots, n$, and $v_1 \in V_1$, $v_3 \in V$.

- Finally, for any $\sigma \neq \emptyset$, $\varphi_{\sigma,\sigma}$ is defined as follows:

$$\varphi_{\sigma,\sigma}(v_{i_1} \otimes \cdots \otimes v_{i_r}, w_{i_1} \otimes \cdots \otimes w_{i_r}) = \frac{1}{2} \sum_{j=1}^{r} \Big(\prod_{k \neq j} \langle v_{i_k} | w_{i_k} \rangle \nu_{i_j}(\gamma_{v_{i_j}, w_{i_j}}) \Big),$$

where $\nu_i : \mathfrak{sp}(V) \to \mathfrak{sp}(V)^n$ denotes the inclusion into the i^{th} component.

With all this notations, the split simple Lie algebra of type E_7 is obtained as:

$$\begin{aligned}
\mathfrak{e}_7 &= \mathfrak{g}(S_8, S_4) \\
&= \mathfrak{tri}(S_8, *, q_8) \oplus \mathfrak{tri}(S_4, *, q_4) \oplus \Big(\oplus_{i=0}^{2} \iota_i(S_8 \otimes S_4) \Big) \\
&= \Big(\mathfrak{sp}(V)^4 \oplus V_1 \otimes V_2 \otimes V_3 \otimes V_4 \Big) \oplus \mathfrak{sp}(V)^3 \\
&\qquad\qquad \oplus (V_1 \otimes V_2 \,\oplus\, V_3 \otimes V_4) \otimes (V_5 \otimes V_6) \\
&\qquad\qquad \oplus (V_2 \otimes V_3 \,\oplus\, V_1 \otimes V_4) \otimes (V_6 \otimes V_7) \\
&\qquad\qquad \oplus (V_1 \otimes V_3 \,\oplus\, V_2 \otimes V_4) \otimes (V_5 \otimes V_7) \\
&= \bigoplus_{\sigma \in \mathcal{S}_{E_7}} V(\sigma),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{S}_{E_7} = \Big\{ &\emptyset, \{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}, \\
&\qquad \{2,3,6,7\}, \{1,4,6,7\}, \{1,3,5,7\}, \{2,4,5,7\} \Big\}.
\end{aligned}$$

The multiplication in $\mathfrak{g}(S_8, S_4)$ translates into:

$$[x_\sigma, y_\tau] = \epsilon_7(\sigma, \tau) \varphi_{\sigma,\tau}(x_\sigma, y_\tau)$$

for any $x_\sigma \in V(\sigma)$, $y_\tau \in V(\tau)$, where

$$\epsilon_7 : \mathcal{S}_{E_7} \times \mathcal{S}_{E_7} \to \{\pm 1\}$$

is given by the signs that appear in the multiplication table of the classical octonions $\mathbb{O}$ in its usual basis $\{1, i, j, k, l, il, jl, kl\}$, under the assignment:

$$\begin{array}{ll}
\emptyset \;\; \mapsto 1 & \{1,2,3,4\} \mapsto l \\
\{1,2,5,6\} \mapsto i & \{3,4,5,6\} \mapsto il \\
\{2,3,6,7\} \mapsto j & \{1,4,6,7\} \mapsto jl \\
\{1,3,5,7\} \mapsto k & \{2,4,5,7\} \mapsto kl
\end{array}$$

A precise and simple formula can be given explicitly for ϵ_7 by following ideas of Albuquerque and Majid [AM99], who considered the classical octonions $\mathbb{O}$ as a 'twisted group algebra'. We will not go into details.

Similar constructions appear for all the Lie algebras in Freudenthal's Magic Square (see [Eld04"]).

Remark 3.2. The models thus constructed for the split E_7 and E_8 are strongly related to a very interesting combinatorial construction previously obtained by A. Grishkov in [G01], in which he gives the multiplication table of these Lie algebras in very specific bases.

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