

Symmetries in Constrained Systems

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Abstract: As it is known, singular theories are theories with constraints in the Hamiltonian formulation. In particular, theories with first-class constraints are gauge theories. Our aim is to describe symmetry structure of a general singular theory, and, in particular, to relate the structure of gauge transformations with the constraint structure.

1 Introduction

Our aim is to study the symmetry structure of a general singular theory, and, in particular, to relate this structure to the constraint structure in the Hamiltonian formulation. For simplicity, we consider finite-dimensional models whose actions are of the form

$$S[q] = \int L(q, \dot{q}) dt, \quad q = (q^a, a = 1, \dots, n),$$

$$\frac{\delta S}{\delta q^a} = \frac{\partial L}{\partial q^a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} = 0 - \text{Euler} - \text{Lagrange equations}, \quad (1)$$

where $L(q, \dot{q})$ is a Lagrange function. All such theories can be divided into two classes according to the Hessian's value,

$$\text{Hessian} = \det \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} = \begin{cases} \neq 0 & \text{nonsingular theory} \\ = 0 & \text{singular theory} \end{cases}. \quad (2)$$

Singular Lagrangian theories are theories with constraints in the Hamiltonian formulation [1]. In particular, theories with first-class constraints (FCC) are gauge theories.

A finite transformation $q(t) \rightarrow q'(t)$ is a symmetry of S if

$$L(q, \dot{q}) \rightarrow L'(q, \dot{q}) = L(q, \dot{q}) + \frac{dF}{dt}, \quad (3)$$

where F is a local function (such transformations are called Noether symmetries). The finite symmetry transformations can be discrete, continuous global, gauge, and trivial. Continuous global symmetry transformations are parametrized by a set of time-independent parameters ν^α , $\alpha = 1, \dots, r$. The infinitesimal form of a continuous global symmetry transformation reads $\delta q^a(t) = \rho_\alpha^a(t) \nu^\alpha$, where $\rho_\alpha^a(t)$ are generators of the global symmetry transformations. Continuous symmetry

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transformations are gauge transformations (or local symmetry transformations) if they are parametrized by arbitrary functions of time, the gauge parameters (in the case of field theory, the gauge parameters depend on all space-time variables). The infinitesimal form of a gauge transformation reads

$$\delta q^a(t) = \hat{\mathcal{R}}_\alpha^a(t) \nu^\alpha(t), \quad (4)$$

where $\nu^\alpha(t)$, $\alpha = 1, \dots, r$ are time-dependent gauge parameters. The quantities $\hat{\mathcal{R}}_\alpha^a(t)$ are generators of gauge transformations. Under some natural suppositions about the structure of the Lagrange function, **one can prove that $\hat{\mathcal{R}}_\alpha^a(t)$ are local operators** [5]. The existence of infinitesimal gauge transformations with generators implies the existence of the corresponding gauge identities, which present identities between the Euler-Lagrange equations.

For any action there exist trivial symmetry transformations,

$$\delta_{\text{tr}} q^a = \hat{U}^{ab} \frac{\delta S}{\delta q^b}, \quad (5)$$

where $\hat{U}$ is an antisymmetric local operator, that is $(\hat{U}^T)^{ab} = -\hat{U}^{ab}$. The trivial symmetry transformations do not affect genuine trajectories. Two symmetry transformations $\delta_1 q$ and $\delta_2 q$ are called equivalent ($\delta_1 q \sim \delta_2 q$) whenever they differ by a trivial symmetry transformation,

$$\delta_1 q \sim \delta_2 q \iff \delta_1 q - \delta_2 q = \delta_{\text{tr}} q. \quad (6)$$

Thus, all the symmetry transformations of an action S can be divided into equivalence classes.

Any symmetry transformation implies a conservation law (Nöether theorem):

$$\begin{aligned} \frac{dG}{dt} &= -\delta q^a \frac{\delta S}{\delta q^a} = O\left(\frac{\delta S}{\delta q}\right) \implies G = \text{const. on extremals}, \\ G &= P - F, \quad P = \frac{\partial L}{\partial \dot{q}^a} \delta q^a, \quad \delta L = \frac{dF}{dt}. \end{aligned} \quad (7)$$

The local function G is referred to as the conserved charge related to the symmetry δq of the action S . The quantities δq , S , and G are related by the equation (7). In what follows, we call this equation the symmetry equation.

Any gauge symmetry generates a conserved charge G which depends locally on gauge parameters and on their time-derivatives, and vanishes on the extremals² (the latter fact was already familiar to Nöether)

$$G = O\left(\frac{\delta S}{\delta q}\right). \quad (8)$$

An important inverse statement holds true. Namely: **If a global symmetry transformation generates a conserved charge that vanishes on the**

²For us, extremals are local functions $\delta S/\delta q$ and any linear combinations of these functions and their time derivatives.

extremals then the corresponding action obeys a gauge symmetry. At the same time the initial global symmetry is a reduction of the corresponding gauge symmetry to constant values of the gauge parameters [6].

At present almost all modern physical models are formulated as gauge theories. Thus, the study of the general gauge theory is an important mathematical and physical problem. In particular, the following questions are of especial interest:

How many gauge transformations (with independent gauge parameters) are there for a given action?

What is the structure of the gauge generators (how many time derivatives they contain) for a given action? What is the structure of an arbitrary symmetry of the action of a singular theory?

Is there a constructive procedure to find all the gauge transformations for a given action?

How can one relate the constraint structure in the Hamiltonian formulation with the symmetry structure of the Lagrangian action?

These problems were partially considered in the works [2, 3, 4]. In this talk, we represent the recent progress in attempts to answer the above questions.

2 Symmetry equation and orthogonal constraint basis

For the study of the symmetry structure, we start with the consideration of the Hamiltonian action S_H (there exists an isomorphism between symmetry classes of the Lagrangian S and the Hamiltonian S_H actions).

$$S[q] = \int L(q, \dot{q}) dt \iff S_H[\eta] = \int [p\dot{q} - H^{(1)}(\eta)] dt, \quad \eta = (\eta, \lambda), \quad \eta = (q, p),$$

$$H^{(1)}(\eta) = H(\eta) + \lambda \Phi^{(1)}(\eta);$$

$$\frac{\delta S_H}{\delta \eta} = 0 \implies \begin{cases} \dot{\eta} = \{\eta, H^{(1)}\}, \\ \Phi^{(1)}(\eta) = 0, \end{cases}$$

where $\Phi^{(1)}(\eta)$ are primary constraints, $\eta = (q, p)$ are phase-space variables, and λ are Lagrange multipliers to primary constraints.

One can see that if $\delta\eta = (\delta q, \delta p, \delta\lambda)$ is a symmetry of the Hamiltonian action S_H , then $\delta_L q$ is a symmetry of the Lagrangian action S ,

$$\delta_L q = \delta q|_{p(q, \dot{q}), \lambda(q, \dot{q})}.$$

The symmetry equation for the action S_H reads

$$\delta\eta \frac{\delta S_H}{\delta \eta} + \frac{dG}{dt} = 0, \quad (9)$$

where G is the conserved charge. The charge G and all the variations depend on all the variables and their time derivatives locally. One can study the symmetry of an action by solving the symmetry equation.

It turns out that the symmetry equation can be easily analyzed (solved) by algebraic methods if one chooses the so called orthogonal constraint basis. In the work [7], we have demonstrated that there exists a constraint reorganization of the first-class constraints (FCC) and of the second-class constraints (SCC) consistent with the Dirac procedure, i.e., the reorganization does not violate the decomposition of the constraints according to their stages in the Dirac procedure. Namely:

It is possible to reorganize the independent constraints Φ obtained in the Dirac procedure such that: the complete set of constraints is divided into SCC φ and FCC χ . At the same time, it is decomposed into groups according to the stages of the Dirac procedure,

$$\begin{aligned}\Phi &= (\varphi, \chi) = \left(\Phi^{(i)} \right), \quad i = 1, \dots, N, \\ \Phi^{(i)} &= (\varphi^{(i)}; \chi^{(i)}), \quad \varphi = (\varphi^{(i)}), \quad \chi = (\chi^{(i)}).\end{aligned}$$

Here $\Phi^{(i)}$ are constraints of the i -th stage, $\varphi^{(i)}$ are SCC of the i -th stage, $\chi^{(i)}$ are FCC of the i -th stage, and N is the number of stages of the Dirac procedure. It may turn out that after a certain stage new independent FCC (SCC) do not appear anymore. We are going to denote this stages by N_χ (N_φ). Obviously, $N = \max(N_\chi, N_\varphi)$. In addition, the constraints in each stage are divided into groups,

$$\begin{aligned}\varphi^{(i)} &= \left(\varphi^{(i|s)} \right), \quad s = i, \dots, N_\varphi; \\ \chi^{(i)} &= \left(\chi^{(i|a)} \right), \quad a = i, \dots, N_\chi.\end{aligned}\tag{10}$$

Such a division creates chains of constraints. Thus, there exist N_φ chains of SCC

$$\varphi^{(\dots|s)} = \left(\varphi^{(i|s)}, \quad i = 1, \dots, s \right), \quad s = 1, \dots, N_\varphi,$$

labeled by the index s , and N_χ chains of FCC

$$\chi^{(\dots|a)} = \left(\chi^{(i|a)}, \quad i = 1, \dots, a \right), \quad a = 1, \dots, N_\chi$$

labeled by the index a . Within the Dirac procedure, the group $\varphi^{(1|s)}$ of primary SCC produces SCC of the second stage, third stage, and so on, which belong to the same chain, $\varphi^{(1|s)} \rightarrow \varphi^{(2|s)} \rightarrow \varphi^{(3|s)} \rightarrow \dots \rightarrow \varphi^{(s|s)}$. The chain of SCC labeled by the number s ends with the group of the s -th-stage constraints. The consistency conditions for the latter group determine the Lagrange multipliers λ_φ to be $\bar{\lambda}$. At the same time, the group $\chi^{(1|a)}$ of primary FCC produces FCC of the second stage, third

stage, and so on, which belong to the same chain, $\chi^{(1|a)} \rightarrow \chi^{(2|a)} \rightarrow \chi^{(3|a)} \rightarrow \dots \rightarrow \chi^{(a|a)}$. We call such organized set of constraints the orthogonal constraint basis. The described hierarchy of constraints in the orthogonal basis (in the Dirac procedure) looks schematically as follows:

$$\begin{array}{ccccccc}
 \varphi^{(1|1)} & \rightarrow & \bar{\lambda}_1 & & & & \\
 \varphi^{(1|2)} & \rightarrow & \varphi^{(2|2)} & \rightarrow & \bar{\lambda}_2 & & \\
 \vdots & \rightarrow & \vdots & \rightarrow & \vdots & & \\
 \varphi^{(1|N-1)} & \rightarrow & \varphi^{(2|N-1)} & \rightarrow & \varphi^{(3|N-1)} & \dots & \varphi^{(N-1|N-1)} \rightarrow \bar{\lambda}_{N-1} \\
 \varphi^{(1|N)} & \rightarrow & \varphi^{(2|N)} & \rightarrow & \varphi^{(3|N)} & \dots & \varphi^{(N-1|N)} \rightarrow \varphi^{(N|N)} \rightarrow \bar{\lambda}_N \\
 \chi^{(1|N)} & \rightarrow & \chi^{(2|N)} & \rightarrow & \chi^{(3|N)} & \dots & \chi^{(N-1|N)} \rightarrow \chi^{(N|N)} \rightarrow O(\Phi) \\
 \chi^{(1|N-1)} & \rightarrow & \chi^{(2|N-1)} & \rightarrow & \chi^{(3|N-1)} & \dots & \chi^{(N-1|N-1)} \rightarrow O(\Phi^{(\dots N-1)}) \\
 \vdots & \rightarrow & \vdots & \rightarrow & \vdots & & \vdots \\
 \chi^{(1|2)} & \rightarrow & \chi^{(2|2)} & \rightarrow & O(\Phi^{(\dots 2)}) & & \\
 \chi^{(1|1)} & \rightarrow & O(\Phi^{(1)}) & & & &
 \end{array}$$

The chain of FCC labeled by the number a ends with the group of the a -th-stage constraints. Their consistency conditions do not determine any multipliers and any new constraints. The Lagrange multipliers λ_χ are not determined by the Dirac procedure (and by the complete set of equations of motion). Thus, all the constraints in a chain are of the same class. One ought to say that the numbers of constraints in each stage in the same chain are the same. At the same time, each chain may be either empty or contain several functions. Thus, whenever FCC (SCC) exist, the corresponding primary FCC (SCC) do exist.

The Poisson brackets of SCC from different chains of the orthogonal basis vanish on the constraint surface

$$\{\varphi^{(i|s)}, \varphi^{(j|v)}\} = O(\Phi), \quad s \neq v.$$

In addition,

$$\begin{aligned}
 \{\varphi^{(i|s)}, H^{(1)}\} &= \varphi^{(i+1|s)} + O(\Phi^{(1)}, \dots, \Phi^{(i)}), \quad i = 1, \dots, N_\varphi - 1, \quad s = i + 1, \dots, N_\varphi, \\
 \{\varphi^{(1|s)}, \varphi^{(s|s)}\} &= \theta, \quad \det \theta^s \neq 0; \\
 \{\chi^{(i|a)}, H^{(1)}\} &= \chi^{(i+1|a)} + O(\Phi^{(1)}, \dots, \Phi^{(i)}), \quad i = 1, \dots, N_\chi - 1, \quad a = i + 1, \dots, N_\chi, \\
 \{\chi^{(a|a)}, H^{(1)}\} &= O(\Phi^{(1)}, \dots, \Phi^{(a)}).
 \end{aligned}$$

The consistency conditions for SCC $\varphi^{(i|i)}$ of the i -th stage

$$\{\varphi^{(s|s)}, H^{(1)}\} = 0$$

allows one to determine λ_φ^s multipliers.

We stress, that the consistency conditions for SCC $\varphi^{(i|s)}$, $s > i$ of the i -th stage produce SCC $\varphi^{(i+1|s)}$ of the $i+1$ -th stage. The consistency conditions for FCC $\chi^{(i|s)}$, $s > i$ of the i -th stage produce FCC $\chi^{(i+1|s)}$ of the $i+1$ -th stage. The consistency conditions for FCC $\chi^{(i|i)}$ of the i -th stage do not produce any new constraints and do not determine any Lagrange multipliers.

Such properties of the constraint basis are extremely helpful for analyzing the symmetry equation. In particular, they allow one to guess (and then to strictly prove) the form of the conserved charges as decompositions in the orthogonal constraint basis. For example, these properties imply that SCC $\varphi^{(i|i)}$ cannot enter linearly into the conserved charges. At the same time, one can see that only FCC $\chi^{(i|i)}$ enter the gauge charges multiplied by independent gauge parameters, other FCC $\chi^{(i|a)}$, $a > i$ are multiplied by factors that must contain derivatives of the same gauge parameters.

3 What can be proved solving the symmetry equation in orthogonal constraint basis?

I. For any theory (singular or non-singular) any symmetry transformations that vanish on the equations of motion are trivial.

II. In theories with FCC there exist nontrivial symmetries $\delta_\nu \eta$, G_ν of the Hamiltonian action S_H that are gauge transformations. These symmetries are parametrized by the gauge parameters ν_i . The latter parameters are arbitrary functions of time t .

The number of the gauge parameters $[\nu]$ is equal to the number of the primary FCC $[\chi^{(1)}]$,

$$[\nu] = [\chi^{(1)}].$$

The corresponding conserved charge (the gauge charge) is a local function $G_\nu = G_\nu(\eta, \lambda^{[l]}, \nu^{[l]})$, which vanishes on the extremals. The gauge charge has the following decomposition with respect to the orthogonal constraint basis:

$$G_\nu = \sum_{i=1}^{N_X} \nu_i \chi^{(i|i)} + \sum_{i=1}^{N_X-1} \sum_{a=i+1}^{N_X} C_{i|a}^X \chi^{(i|a)} + \sum_{i=1}^{N_X-1} \sum_{s=i+1}^N C_{i|s}^\varphi \varphi^{(i|s)}. \quad (11)$$

Here $C_{i|s}^\varphi(\eta, \lambda^{[l]}, \nu^{[l]})$ and $C_{i|a}^X(\eta, \lambda^{[l]}, \nu^{[l]})$ are some local functions, which are determined by the symmetry equation in an algebraic way. It turns out that $C_{i|s}^\varphi = O(I)$, where $I = \delta S_H / \delta \eta$ are extremals. The gauge charge depends both on the gauge parameters and on their time derivatives

up to a finite order. Namely,

$$G_\nu = \sum_{i=1}^{N_x} \sum_{m=0}^{i-1} G_m^i(\eta, \lambda^{[l]}) \nu_i^{[m]}. \quad (12)$$

where $G_m^i(\eta, \lambda^{[l]})$ are some local functions. The total number of independent gauge parameters together with their time derivatives, that enter essentially in the gauge charge is equal to the number of all FCC $[\chi]$,

$$\sum_{m=0} [\nu^{[m]}] = [\chi].$$

The gauge charge is the generating function for the variations $\delta\eta$ of the phase-space variables,

$$\delta_\nu \eta = \{\eta, G_\nu\} = \{\eta, \eta^A\} \frac{\partial G_\nu}{\partial \eta^A}. \quad (13)$$

(Note that here the Poisson bracket acts only on the explicit dependence on η of the gauge charge.) The variations $\delta_\nu \lambda$ contain additional time derivatives of the gauge parameters, namely, they have the form

$$\delta_\nu \lambda = \sum_{i=1}^{N_x} \sum_{m=0}^i \Upsilon_m^i(\eta, \lambda^{[l]}) \nu_i^{[m]}, \quad (14)$$

where Υ_m^i are some local functions, which can be determined from the symmetry equation in an algebraic way.

Thus, the gauge charge G_ν have the following structure

$$G_\nu = \sum_{m=1}^{N_x} \sum_{b=m}^{N_x} G^{mb}(\eta, \lambda^{[l]}) \nu_b^{[m-1]}, \quad (15)$$

where the local functions $G^{mb}(\eta, \lambda^{[l]})$ have the form

$$G^{mb} = \sum_{k=1}^{N_x} \sum_{a=k}^{N_x} \chi^{(k|a)} \mathcal{C}_{ka}^{mb}(\eta, \lambda^{[l]}) + O(I^2), \quad (16)$$

and $\mathcal{C}_{ka}^{mb}(\eta, \lambda^{[l]})$ are some local functions. Thus,

$$G_\nu = O(\chi) + O(I^2). \quad (17)$$

The form of the variations $\delta_\nu \eta$ follows from (??),

$$\delta_\nu \eta = \left(\sum_{k=1}^{N_x} \sum_{a=k}^{N_x} \right) \left(\sum_{m=1}^{N_x} \sum_{b=m}^{N_x} \right) \left\{ \eta, \chi^{(k|a)} \right\} \mathcal{C}_{ka}^{mb} \nu_b^{[m-1]} + O(I). \quad (18)$$

After the gauge charge has been determined, the variations $\delta_\nu \lambda$ can be found from the equation (??). Their general structure is given by Eq. (14), where $\Upsilon_m^i(\eta, \lambda^{[l]})$ are some local functions. In particular, one can see that

$$\delta_\nu \lambda_\chi^a = \sum_{i=1}^{N_\chi} \mathcal{D}^{ai}(\eta, \lambda^{[l]}) \nu_i^{[i]} + O(\nu_j^{[l]}, l < j). \quad (19)$$

Note that the local functions G_m^i , $\mathcal{C}_{ka}^{im}$, and Υ_m^i do not depend on the gauge parameters and are, in that sense, universal. The matrices $\mathcal{C}$ and $\mathcal{D}$ are not singular.

III. In theories with FCC, any symmetry $\delta\eta$, G of the Hamiltonian action S_H can be represented as the sum of three types of symmetries

$$\begin{pmatrix} \delta\eta \\ G \end{pmatrix} = \begin{pmatrix} \delta_c\eta \\ G_c \end{pmatrix} + \begin{pmatrix} \delta_{\bar{\nu}}\eta \\ G_{\bar{\nu}} \end{pmatrix} + \begin{pmatrix} \delta_{\text{tr}}\eta \\ G_{\text{tr}} \end{pmatrix}, \quad (20)$$

such that:

The set $\delta_c\eta$, G_c is a global symmetry, canonical for the phase-space variables η . The corresponding conserved charge G_c does not vanish on the extremals.

The set $\delta_{\bar{\nu}}\eta$, $G_{\bar{\nu}}$ is a particular gauge transformation given by Eqs. (12), (13), and (14) with fixed gauge parameters (i.e. with specific forms for the functions $\nu_i = \bar{\nu}_i(t, \eta^{[l]}, \lambda^{[l]})$) that do not vanish on the extremals. The corresponding conserved charge $G_{\bar{\nu}}$ vanishes on the extremals, whereas the variations $\delta_{\bar{\nu}}\eta$ do not.

The set $\delta_{\text{tr}}\eta$, G_{tr} is a trivial symmetry. All the variations $\delta_{\text{tr}}\eta$ and the corresponding conserved charge G_{tr} vanish on the extremals. The gauge charge G_{tr} depends on the extremals as $G_{\text{tr}} = O(I^2)$.

As an example, we consider a field model which includes a set of Yang-Mills vector fields $\mathcal{A}_\mu^a$, $a = 1, \dots, r$, and a set of spinor fields $\psi^\alpha = (\psi_i^\alpha, i = 1, \dots, 4)$,

$$S = \int \mathcal{L} dx, \quad \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + i \bar{\psi}^\alpha \gamma^\mu \nabla_{\mu\beta}^\alpha \psi^\beta - V(\psi, \bar{\psi}),$$

$$G_{\mu\nu}^a = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a + f_{bc}^a \mathcal{A}_\mu^b \mathcal{A}_\nu^c, \quad \nabla_{\mu\beta}^\alpha = \partial_\mu \delta_\beta^\alpha - iT_{a\beta}^\alpha \mathcal{A}_\mu^a, \quad (21)$$

where V is the local polynomial in the field, which contains no derivatives. The model is based on a certain global Lie group G ,

$$\psi(x) \xrightarrow{g} \exp(i\nu^\alpha T_a) \psi(x), \quad g \in G, \quad \nu^a, \quad a = 1, \dots, r,$$

$$T_a = T_a^+, \quad [T_a, T_b] = if_{ab}^c T_c, \quad f_{ab}^k f_{kc}^n + f_{bc}^k f_{ka}^n + f_{ca}^k f_{kb}^n = 0.$$

For $V = 0$, the action is invariant under gauge transformations ($\nu^a = \nu^a(x)$)

$$\delta \mathcal{A}_\mu^a = D_{\mu b}^a \nu^b, \quad \delta \psi = iT_a \psi \nu^a, \quad D_{\mu b}^a = \partial_\mu \delta_b^a + f_{cb}^a \mathcal{A}_\mu^c. \quad (22)$$

We assume the polynomial V to be such that the whole action (21) is invariant under the transformations (22) as well. Below we relate the symmetry structure of the model with its constraint structure. To this end we first reveal the constraint structure.

Proceeding to the Hamiltonian formulation, we introduce the momenta

$$p_{0a} = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}^{0a}} = 0, \quad p_{ia} = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}^{ia}} = G_{i0}^a, \quad p_\psi = \frac{\partial_r \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0, \quad p_{\bar{\psi}} = \frac{\partial_r \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0.$$

Thus, there exists a set of primary constraints $\Phi^{(1)} = (\chi_a^{(1)}, \varphi_\sigma^{(1)}, \sigma = 1, 2) = 0$, where

$$\chi_a^{(1)} = p_{0a}, \quad \varphi_1^{(1)} = p_\psi - i\bar{\psi}\gamma^0, \quad \varphi_2^{(1)} = p_{\bar{\psi}}.$$

The total Hamiltonian reads $H^{(1)} = \int \mathcal{H}^{(1)} d\mathbf{x}$,

$$\mathcal{H}^{(1)} = \frac{1}{2}p_{ia}^2 + \frac{1}{4}G_{ik}^{a2} - p_\psi \gamma^0 \gamma^k \nabla_k \psi + \mathcal{A}^{0a} (D_{ib}^a p_{ib} - \bar{\psi} \gamma^0 T_a \psi) + V + \lambda_\chi^a \chi_a^{(1)} + \lambda_\varphi^\sigma \varphi_\sigma^{(1)}.$$

By performing the Dirac procedure, one can verify that there only appear secondary constraints $\chi_a^{(2)} = 0$,

$$\begin{aligned} \{\varphi_\sigma^{(1)}, H^{(1)}\} &= 0 \Rightarrow \lambda_\varphi^\sigma = \bar{\lambda}_\varphi^\sigma (\mathcal{A}, \psi, \bar{\psi}), \\ \{\chi_a^{(1)}, H^{(1)}\} &= 0 \Rightarrow \chi_a^{(2)} = D_{ib}^a p_{ib} + i(p_\psi T_a \psi + p_{\bar{\psi}} \bar{T}_a \bar{\psi}), \quad (\bar{T}_a)_\beta^\alpha = -\gamma^0 (T_a^*)_\beta^\alpha \gamma^0. \end{aligned}$$

All the constraints φ are second-class and all the χ are first-class. It turns out that the complete set of constraints already forms the orthogonal constraint basis, namely:

$$\varphi^{(1|1)} \equiv \varphi^{(1)}, \quad \chi^{(1|2)} \equiv \chi^{(1)}, \quad \chi^{(2|2)} \equiv \chi^{(2)},$$

and there are no constraints $\chi^{(1|1)}$,

$$\begin{aligned} \varphi^{(1|1)} &\rightarrow \bar{\lambda} \\ \chi^{(1|2)} &\rightarrow \chi^{(2|2)} \rightarrow O(\Phi). \end{aligned}$$

According to the general considerations, we chose the gauge charge in the form

$$G = \int [\nu^a \chi_a^{(2|2)} + C^a \chi_a^{(1|2)}] d\mathbf{x}, \quad C^a = (c_b^a \nu^b + d_b^a \dot{\nu}^b).$$

Solving the symmetry equation (9), we obtain $C^a = \dot{\nu}^a - \nu^c \mathcal{A}^{0b} f_{cb}^a = D_{0b}^a \nu^b$. Thus,

$$\begin{aligned} G &= \int [p_{\mu a} D_b^{\mu a} \nu^b + i(p_\psi T_a \psi + p_{\bar{\psi}} \bar{T}_a \bar{\psi}) \nu^a] d\mathbf{x}, \\ \delta \mathcal{A}_\mu^a &= \{\mathcal{A}_\mu^a, G\} = D_{\mu b}^a \nu^b, \quad \delta \psi = \{\psi, G\} = i T_a \psi \nu^a. \end{aligned}$$

4 Main conclusions

Below we summarize the main conclusions.

Any symmetry transformation can be represented as a sum of three kinds of symmetries: global, gauge, and trivial symmetries. The global part of a symmetry does not vanish on the extremals, and the corresponding charge does not vanish on the extremals as well. This separation is not unique. In particular, the determination of the global charge from the corresponding equation, and thus the determination of the global part of the symmetry is then ambiguous. However, the ambiguity in the global part of a symmetry transformation is always a sum of a gauge transformation and a trivial transformation. The gauge part of a symmetry does not vanish on the extremals, but the gauge charge vanishes on them. We stress that the gauge charge necessarily contains a part that vanishes linearly in the FCC, and the remaining part of the gauge charge vanishes quadratically on the extremals. The trivial part of any symmetry vanishes on the extremals, and the corresponding charge vanishes quadratically on the extremals.

The reduction of symmetry variations to extremals are global canonical symmetries of the physical action, whose conserved charge is the reduction of the complete conserved charge to the extremals.

Any global canonical symmetry of the physical action can be extended to a nontrivial global symmetry of the complete Hamiltonian action.

There are no other gauge transformations that cannot be represented in the form (11).

We stress that in the general case the gauge charge cannot be constructed with the help of any complete set of FCC only, for its decomposition contains SCC as well. A model for which the gauge charge must be constructed both with the help of FCC and of SCC is considered in the *Example 1*.

Note that in our procedure, generators (conserved charges) of canonical and gauge symmetries may depend on Lagrange multipliers and their time derivatives. This happens in the case when the number of stages in the Dirac procedure is more than two. In the *Example 2* we represent models that illustrate this fact.

The gauge charge contains time derivatives of the gauge parameters whenever there exist secondary FCC. Namely, the power of the highest time derivative that enters the gauge charge is equal to $N_x - 1$, where N_x is the number of the last stage when new FCC still appear. A simple model for which the gauge charge contains a second-order time derivative of the gauge parameter is considered in the *Example 3*.

Since there is an isomorphism between symmetry classes of the Hamiltonian action S_H and the Lagrangian action S the symmetry structure of Lagrangian

action S coincides with the symmetry structure of the Hamiltonian action S_H , and is given by all the assertions represented above. As to the concrete form of a symmetry transformation (symmetry transformation of the coordinates) of the Lagrangian action S , it can be obtained as a reduction of the symmetry transformation of the coordinates of the Hamiltonian action S_H by the substitution of all the Lagrange multipliers and momenta via coordinates and velocities.

Example 1: Consider a Hamiltonian action S_H that depends on the phase-space variables $(q_i, p_i, i = 1, 2,)$ and $(x_a, \pi_a, \alpha = 1, 2)$, and of two Lagrange multipliers λ_π and λ_p ,

$$S_H = \int [p_i \dot{q}_i + \pi_\alpha \dot{x}_\alpha - H^{(1)}] dt, \quad H^{(1)} = H_0^{(1)} + x_1 q_2^2, \\ H_0^{(1)} = \frac{1}{2} \pi_2^2 + x_1 \pi_2 + \frac{1}{2} p_2^2 + \frac{1}{2} q_2^2 + q_1 p_2 + \lambda_\pi \pi_1 + \lambda_p p_1.$$

One can see that the model has two primary constraints π_1 and p_1 . It is easy to verify that a complete set of constraints can be chosen as $\chi = (\pi_1, \pi_2)$ and $\varphi = (q_1, q_2, p_1, p_2)$. Here χ are FCC and φ are SCC. Thus, the model is a gauge one. The peculiarity of the model is that gauge symmetries of the action S_H have gauge charges which must be constructed with the help of both FCC and SCC.

Example 2: Consider a Hamiltonian action S_H that depends on the phase-space variables $(q_a, p_a, a = 1, 2,)$ and $(x_a, \pi_a, \alpha = 1, 2, 3)$, and on a Lagrange multiplier λ ,

$$S_H = \int [p_i \dot{q}_i + \pi_\alpha \dot{x}_\alpha - H^{(1)}] dt, \quad H^{(1)} = H_0^{(1)} + V, \quad V = q_1 x_1 x_3^2, \\ H_0^{(1)} = \frac{1}{2} (q_i^2 + p_i^2) + x_1 \pi_2 + x_2 \pi_3 + \frac{1}{2} x_3^2 + \frac{1}{2} \pi_2^2 + \frac{1}{2} \pi_3^2 + \lambda \pi_1,$$

The model has one primary constraint π_1 . The peculiarity of the model is that symmetries of the action S_H have charges that must depend on Lagrange multipliers.

Example 3: Consider a Lagrangian action that depends on the coordinates x, y, z ,

$$S = \frac{1}{2} \int [(\dot{x} - y)^2 + (\dot{y} - z)^2] dt.$$

One can easily see that the action is gauge invariant under the following transformations that include first and second-order time derivatives of the gauge parameters,

$$\delta x = \nu, \quad \delta y = \dot{\nu}, \quad \delta z = \ddot{\nu}.$$

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References

- [1] P.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York 1964)
- [2] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin 1990)
- [3] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton 1992)
- [4] V.A. Borochoy and I.V. Tyutin, *Physics of Atomic Nuclei*, **61** (1998) 1603; *ibid* **62** (1999) 1070
- [5] B. Geyer, D.M. Gitman, and I.V. Tyutin, *J. Phys.* **A36** (2003) 6587
- [6] G. Fulop, D. Gitman, and I. Tyutin, *Int. J.Theor. Phys.* **38** (1999) 1953
- [7] D. M. Gitman, and I.V. Tyutin, *Gravitation & Cosmology*, **8**, No.1-2 (2002) 138; *Constraint reorganization consistent with Dirac procedure*, Michael Marinov Memorial Volume: *Multiple Facets of Quantization and Supersymmetry*, ed. M. Olshanetsky and A. Vainstein (World Publishing, Singapore 2002) pp.184-204

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