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Non-linear Lie conformal algebras

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In this talk we shall introduce the notion of a non-linear Lie conformal algebra, provide some examples, and discuss their categorical relation with the freely generated vertex algebras, their universal envelopes. This relationship provides a method for constructing non-linear Lie conformal algebras via a quantization of the classical Drinfeld-Sokolov reduction.

Let Γ be a discrete additive semigroup in $\mathbb{R}_{\geq 0}$ containing 0, and let

$$U = \bigoplus_{\alpha \in \Gamma \setminus 0} U_\alpha$$

be a graded vector space. We write $\Delta(a) = \alpha$ if $a \in U_\alpha$. Since Γ is a semigroup, the $\Gamma \setminus 0$-grading on U can be extended to a Γ-grading on the tensor algebra $T(U)$ by defining

$$\Delta(a_1 \otimes .. \otimes a_k) = \Delta(a_1) + .. + \Delta(a_k),$$

for all $k \geq 0$ and homogenous $a_1, .., a_k \in U$. A $\Gamma \setminus 0$-graded vector space $\mathfrak{g}$ is called a *non-linear Lie algebra* if it is endowed with a bilinear bracket

$$\mathfrak{g} \times \mathfrak{g} \to T(\mathfrak{g}), \quad (a, b) \mapsto [a, b] \in T(\mathfrak{g})$$

such that for all a, b, $c \in \mathfrak{g}$

$$\begin{array}{rl} \text{(grading condition)} & \Delta([a,b]) < \Delta(a) + \Delta(b) \\ \text{(Jacobi identity)} & [a,[b,c]] - [[a,b],c] - [b,[a,c]] \in M_\alpha(\mathfrak{g}) \\ & \quad \text{for some} \quad \alpha \in \Gamma, \alpha < \Delta(a) + \Delta(b) + \Delta(c) \end{array}$$

where $M_\alpha(\mathfrak{g})$ is the linear span of

$$\left\{ A \otimes (b \otimes c - c \otimes b - [b, c]) \otimes D \;\middle|\; \begin{array}{l} b, c \in \mathfrak{g}, \quad A, D \in T(\mathfrak{g}), \\ \Delta(A \otimes b \otimes c \otimes D) < \alpha \end{array} \right\}.$$

The triple products in the Jacobi identity are understood via the Leibniz rule with respect to the tensor algebra product.

As in the case of the usual "linear" Lie algebras, we have a Poincaré-Birkhoff-Witt theorem that provides a basis for the enveloping algebra of a non-linear Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ of a non-linear Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided $T(\mathfrak{g})$-ideal $M(\mathfrak{g})$, generated by

$$\{ b \otimes c - c \otimes b - [b, c] \mid b, c \in \mathfrak{g} \}.$$

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The PBW theorem states that the ordered monomials in any Γ-homogenous basis of $\mathfrak{g}$ form a basis for the enveloping algebra $U(\mathfrak{g})$, and can be proved in an analogous manner to the "linear" case.

As an example we shall construct the Zamolodchikov W_3-algebra (see [13]). Firstly, let V denote the Virasoro algebra, with the symbols c, L_m, $m \in \mathbb{Z}$, as a basis and the bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{m^3-m}{12}\delta_{m,-n}c \quad \text{and} \quad [L_m, c] = 0,$$

for all m, $n \in \mathbb{Z}$. The Virasoro algebra is a $\mathbb{Z}$-graded linear Lie algebra. By identifying the symbol c with a scalar, the Virasoro algebra can be seen living inside the Zamolodchikov W_3-algebra, which has as a basis the symbols c, L_m, W_m, $m \in \mathbb{Z}$, and further bracket relations:

$$\begin{aligned} [L_m, W_n] &= (2m-n)W_{m+n}, \\ [W_m, W_n] &= \tfrac{16(m-n)}{22+5c}\Lambda_{m+n} + (m-n)\left(\tfrac{(m+n+2)(m+n+3)}{15} - \tfrac{(m+2)(n+2)}{6}\right)L_{m+n} \\ &\quad + (m^2-4)(m^2-1)\delta_{m,-n}\tfrac{c}{360}, \end{aligned}$$

where $\Lambda_n = d_n L_n + \sum_{m\in\mathbb{Z}} : L_m\ L_{n-m} :$, and

$$d_{2m} = \tfrac{1-m^2}{5}, \quad d_{2m-1} = \tfrac{(1+m)(2-m)}{5}, \qquad : L_m\ L_n := \begin{cases} L_m L_n & \text{if } m \le n, \\ L_n L_m & \text{otherwise.} \end{cases}$$

The Zamolodchikov W_3-algebra finds more elegant expression in the language of (non-linear) Lie conformal algebras. A (linear) *Lie conformal algebra* R is a $\mathbb{C}[\partial]$-module with a $\mathbb{C}$-bilinear product

$$R \times R \to \mathbb{C}[\lambda] \otimes R, \quad (a, b) \mapsto [a_\lambda b],$$

such that for all a, b, $c \in R$,

$$\begin{array}{rl} \text{(sesquilinearity)} & [\partial a_\lambda b] = -\lambda[a_\lambda b] \ \text{ and } \ [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b] \\ \text{(skew-commutativity)} & [a_\lambda b] = -[b_{-\partial-\lambda} a] \\ \text{(Jacobi identity)} & [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c] \end{array}$$

where ∂ is a distinguished symbol and λ, μ are indeterminants. This product is called a *λ-bracket*. There is a close relationship between the category of Lie conformal algebras and the category of Lie algebras. In particular, given a Lie algebra $\mathfrak{g}$, a corresponding Lie conformal algebra $\operatorname{Cur}\mathfrak{g}$, called the *current algebra* of $\mathfrak{g}$, can be constructed using formal distributions with values in the Kac-Moody affinization $\hat{\mathfrak{g}}$. We set:

$$a(z) = \sum_{n\in\mathbb{Z}} (a \otimes t^n) z^{-n-1} \ \text{ for all } a \in \mathfrak{g}, \quad \operatorname{Cur}\mathfrak{g} = \mathbb{C}[\partial]\,\{\, a(z) \mid a \in \mathfrak{g} \,\},$$

where $\partial = \frac{d}{dz}$.

The λ-bracket of a, $b \in \mathrm{Cur}\,\mathfrak{g}$ is defined as an encoding of the singular terms of the "operator product expansion" of $[a(z), b(w)]$. Simply:

$$[a_\lambda b] = [a, b] \in \mathrm{Cur}\,\mathfrak{g}.$$

The Virasoro conformal algebra Vir is constructed from the Virasoro Lie algebra in a similar manner, by setting

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad \mathrm{Vir} = \mathbb{C}[\partial]L + \mathbb{C}c.$$

The λ-bracket relations are given by

$$[L_\lambda L] = (\partial + 2\lambda)L + \lambda^3 \tfrac{c}{12}, \quad [L_\lambda c] = 0.$$

The general rule of translation from the language of Lie conformal algebras to that of Lie algebras is given by the formula

$$[a_{(m)}, b_{(n)}] = \sum_{j\in\mathbb{Z}_+} \tbinom{m}{j} c^j_{(m+n-j)},$$

for all integers m, $n \in \mathbb{Z}$. The formal distributions c^j associated with a and b are given by

$$[a_\lambda b] = \sum_{j\in\mathbb{Z}_+} \tfrac{\lambda^j}{j!} c^j,$$

and we write $f_{(n)}$ for the coefficient of z^{-n-1} in any formal distribution $f(z)$ (for example, $L_{(n)} = L_{n-1}$). Equivalently,

$$[a(z), b(w)] = \sum_{j\in\mathbb{Z}_+} c^j(w) (\tfrac{d}{dw})^j \tfrac{\delta(z-w)}{j!} \quad \text{where} \quad \delta(z-w) = z^{-1} \sum_{n\in\mathbb{Z}} (\tfrac{w}{z})^n.$$

The classification of the finitely generated simple Lie conformal algebras was acheived in 1998 (see [3]), and states that the only such algebras are the (centreless) Virasoro conformal algebra and the current algebras associa

ted to the simple finite dimensional Lie algebras. The classification in the super algebra case has richer variety, and can be found in [6].

Both the Virasoro and current conformal algebras are linear, whilst the Zamolodchikov W_3-conformal algebra is a primary example in the non-linear case. We use the formal distributions

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-3}.$$

The ordinary bracket relations translate to the following λ-bracket relations

$$\begin{aligned}
[L_\lambda L] &= (\partial + 2\lambda)L + \lambda^3 \tfrac{1}{12} c, \quad [L_\lambda W] = (\partial + 3\lambda)W, \\
[W_\lambda W] &= (\partial + 2\lambda)(\tfrac{16}{22+5c} :L\,L: + \tfrac{c-10}{3(22+5c)} \partial^2 L \\
&\quad + \tfrac{1}{6}\lambda(\partial + \lambda)L) + \lambda^5 \tfrac{c}{360},
\end{aligned}$$

where the term $: L\, L :$, a normally ordered product, is not expressible as a linear combination of L and W, and thus constitutes a non-linearity. With the grading $\Delta(L) = 2$, $\Delta(W) = 3$, we therefore have that

$$W_3 = \mathbb{C}[\partial]L + \mathbb{C}[\partial]W + \mathbb{C}c$$

is a non-linear Lie conformal algebra. In fact, W_3 is essentially the unique non-linear Lie conformal extension of the Virasoro conformal algebra by a field W of conformal weight 3 (see [4]).

By definition, a *non-linear conformal algebra* is a $\mathbb{C}[\partial]$-module R, $\Gamma \setminus 0$-graded by $\mathbb{C}[\partial]$-submodules

$$R = \bigoplus_{\alpha \in \Gamma \setminus 0} R_\alpha$$

endowed with a $\mathbb{C}$-bilinear product

$$R \times R \to \mathbb{C}[\lambda] \otimes T(R)$$

such that for all $a, b \in R$,

$$\begin{array}{rl} \text{(sesquilinearity)} & [\partial a_\lambda b] = -\lambda [a_\lambda b] \text{ and } [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \\ \text{(grading condition)} & \Delta([a_\lambda b]) < \Delta(a) + \Delta(b). \end{array}$$

A non-linear conformal algebra R is called a *non-linear Lie conformal algebra* if in addition, for all $a, b, c \in R$,

$$\begin{array}{rl} \text{(skew-commutativity)} & [a_\lambda b] = -[b_{-\partial-\lambda} a] \\ \text{(Jacobi identity)} & J(a,b,c) \in \mathbb{C}[\lambda, \mu] \otimes M_\alpha(R), \\ \text{for some} & \alpha \in \Gamma, \alpha < \Delta(a) + \Delta(b) + \Delta(c), \\ \text{where} & J(a,b,c) = [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] - [[a_\lambda b]_{\lambda+\mu} c]. \end{array}$$

Here, $M_\alpha(R)$ is the linear span of

$$\left\{ \begin{array}{l|l} A \otimes (b \otimes c - c \otimes b) \otimes D & b, c \in R, \quad A, D \in T(R), \\ -A \otimes : \int_{-\partial}^0 [b_\lambda c] d\lambda \; D : & \Delta(A) + \Delta(b) + \Delta(c) + \Delta(D) < \alpha \end{array} \right\}.$$

In this non-linear case, the triple products are understood via the "quantized" Leibniz rule, called the non-abelian Wick formula (see below). For the Zamolodchikov W_3-conformal algebra, the only non-trivial Jacobi triples are $J(W, W, L)$ and $J(W, W, W)$. The subspace $M_4(W_3)$ contains

$$\partial L \otimes L - L \otimes \partial L - \int_{-\partial}^0 [\partial L_\lambda L] d\lambda$$

and one can check that these triples are zero only modulo this relation.

The importance of the non-linear Lie conformal algebras stems from their relationship with the freely generated vertex algebras. A *vertex algebra* may be defined as a Lie conformal algebra R, with λ-bracket $[\ {}_\lambda\ ]$, further endowed with a $\mathbb{C}$-bilinear "normally ordered product"

$$R \times R \to R, \quad (a,b) \mapsto :a\ b:$$

which makes it a unitial differential algebra, with unit element $|0\rangle \in R$ and derivation ∂, such that for all a, b, $c \in R$,

$$\begin{aligned}
&\text{(quasicommutativity)} && :a\ b: - :b\ a: = \int_{-\partial}^{0} [a_\lambda b] d\lambda, \\
&\text{(quasiassociativity)} && ::a\ b:\ c: - :a\ :b\ c:: \\
& && =: (\int_0^{\partial} a\ d\lambda)\ [a_\lambda b] : + : (\int_0^{\partial} b\ d\lambda)\ [a_\lambda b] :, \\
&\text{(non-abelian Wick formula)} && [a_\lambda :b\ c:] =: [a_\lambda b]\ c: + :b\ [a_\lambda c]: \\
& && + \int_0^{\lambda} [a_\lambda [b_\mu c]] d\mu.
\end{aligned}$$

This definition characterizes vertex algebras as quantum Poisson algebras, and is equivalent to the definition in terms of local fields (see [1]). To each Lie conformal algebra R we can associate a universal enveloping vertex algebra $U(R)$, freely generated by R. However, it is not true that all freely generated vertex algebras are obtained in this manner. The freely generated vertex algebras are the universal enveloping vertex algebras of the non-linear Lie conformal algebras. In fact, the two categories are equivalent [4].

Given a non-linear Lie conformal algebra R, the universal enveloping vertex algebra $U(R)$ is constucted as a quotient of the tensor algebra $T(R)$. It can be shown (see [4]) that the λ-bracket of R can be uniquely extended to a λ-bracket L_λ on $T(R)$, determining a compatiable normally ordered product N on the same. Let

$$M(R) = \bigcup_{\alpha \in \Gamma} M_\alpha(R).$$

Then

(a) $M(R)$ is a ∂-invariant ideal of $T(R)$ with respect to the λ-bracket L_λ and the normally ordered product N, and the quotient $U(R) = T(R)/M(R)$ is a vertex algebra under the induced operations of L_λ and N.

(b) The ordered monomials of any graded basis of R, in the normally ordered product and associated from left to right, form a basis for $U(R)$ over $\mathbb{C}$. That is, $U(R)$ is "freely generated" by R.

(c) Conversely, any graded vertex algebra V, freely generated by a $\mathbb{C}[\partial]$-submodule R, gives rise to a non-linear Lie conformal algebra structure on R, and we have $V = U(R)$.

In the case where R is a linear Lie conformal algebra, the universal enveloping vertex algebra is given by

$$U(R) = Ind_{\mathfrak{Lie}R_-}^{\mathfrak{Lie}R}\mathbb{C},$$

and is freely generated by R. The Virasoro conformal algebra $\mathrm{Vir} = \mathbb{C}[\partial]L + \mathbb{C}C$ and the current algebra $\mathrm{Cur}\,\mathfrak{g} = \mathbb{C}[\partial]\mathfrak{g} + \mathbb{C}K$ (centrally extended) yield the universal Virasoro and affine vertex algebras, respectively, after specifying a scalar value for the centre.

A large class of non-linear Lie conformal (super)algebras is obtained by the quantum Hamiltonian reduction attached to a simple Lie (super)algebra $\mathfrak{g}$, with good grading, and a nilpotent orbit of $\mathfrak{g}$. The good gradings of the simple Lie algebras have been classified in [5]. The construction begins with a simple Lie algebra $\mathfrak{g}$ equipped with a non-degenerate symmetric bilinear form $(.|.)$, and an element $x \in \mathfrak{g}$ such that the adjoint action $ad(x)$ is diagonalizable with values in $\frac{1}{2}\mathbb{Z}$, so that $\mathfrak{g} = \bigoplus_{j\in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$. Let

$$\mathfrak{g}_\pm = \bigoplus_{j>0} \mathfrak{g}_{\pm j} \quad \text{and} \quad \mathfrak{g}_\leq = \mathfrak{g}_0 + \mathfrak{g}_-,$$

and assume that the grading on $\mathfrak{g}$ is *good*, that is, $\mathfrak{g}^f \subset \mathfrak{g}_\leq$ for some nilpotent $f \in \mathfrak{g}_{-1}$, where $\mathfrak{g}^f$ denotes the centralizer of f in $\mathfrak{g}$. The simplest example is the $sl(2)$-triple $< e, x, f >$ with

$$[x,e] = e, \quad [x,f] = -f, \quad [e,f] = x.$$

To each such $(\mathfrak{g}, x)$ and $k \in \mathbb{C}$, we associate a vertex algebra complex $(C(\mathfrak{g},x,k), d)$ where

$$C(\mathfrak{g},x,k) = U^k(\mathrm{Cur}\,\mathfrak{g}) \otimes F$$

is the tensor product of the universal enveloping vertex algebra $U^k(\mathrm{Cur}\,\mathfrak{g})$ of level k of the current algebra $\mathrm{Cur}\,\mathfrak{g}$ and the universal enveloping vertex algebra

$$F = U(Cl(\mathfrak{g}_{\frac{1}{2}} + \sqcap\mathfrak{g}_+ + \sqcap\mathfrak{g}_+^*))$$

of the Clifford conformal superalgebra associated to $\mathfrak{g}_{1/2} + \sqcap\mathfrak{g}_+ + \sqcap\mathfrak{g}_+^*$ and the form

$$< a, b >= (f|[a,b]),$$

where $\sqcap$ denotes the parity reversing operation on a vector superspace. The zeroth homology of this complex, denoted by $W_k(\mathfrak{g},x)$, is a vertex algebra, and is freely and finitely generated when the $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ is good. This construction is a generalization of the quantized Drinfeld-Sokolov reduction, studied in [7], [8], [9] and many other papers. If f is taken to be the principal nilpotent element, we obtain the principal W-algebras. In particular, $W_k(sl(2), \rho^\vee)$ and $W_k(sl(3), \rho^\vee)$ are the universal enveloping vertex algebras of the Virasoro and Zamolodchikov conformal algebras, respectively. Taking f to be a minimal nilpotent element of $\mathfrak{g}$ yields the minimal W-algebras, including many well-known examples. For instance (see [10]),

$$
\begin{array}{lll}
\mathfrak{g} = & sl(3) & \text{Bershadsky-Polyakov algebra (see [2]),} \\
& spo(2|1) & \text{Neveu-Schwartz algebra,} \\
& sl(2|1) & N = 2 \text{ superconformal algebra,} \\
& spo(2|3) & N = 3 \text{ superconformal algebra,} \\
& sl(2|2)/\mathbb{C}I & N = 4 \text{ superconformal algebra,} \\
& D(2,1;a) & \text{big } N = 4 \text{ superconformal algebra.}
\end{array}
$$

This was used in [11] and in [12] to solve a series of open problems in the representation theory of superconformal algebras.

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