Resenhas IME-USP 2004, Vol. 6, No. 2/3, 231 - 239.

Generalized representations of Jordan algebras

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Abstract: In the talk we will introduce the notion of a generalized representation of a Jordan algebra with unit which has the following properties: 1) Usual representations and Jacobson representations correspond to special cases of generalized representations. 2) Every simple Jordan algebra has infinitely many nonequivalent generalized representations. 3) There is a one-to-one correspondence between irreducible generalized representations of a Jordan algebra A and irreducible representations of a graded Lie algebra $L(A) = U_{-1} \oplus U_0 \oplus U_1$ corresponding to A (the Lie algebra $L(A)$ coincides with TKK-construction when A has a unit). The last correspondence allows to use the theory of representations of Lie algebras to study generalized representations of Jordan algebras. In particular, one can classify irreducible generalized representations of semisimple Jordan algebras and also obtain classical results about usual representations and Jacobson representations in a simple way.

Introduction

Jordan algebras were introduced by P. Jordan, J. von Neumann and E. Wigner (see [1]) in the connection with some problems of quantum mechanics. Already there it was found that simple Jordan algebras have only a finite number of nonequivalent irreducible representations (homomorphisms into a space of linear operators with operation $X * Y = XY + YX$). Moreover (A. Albert [2]), the exceptional Jordan algebra E_3 has no such representations at all.

This situation demonstrates a big difference between Lie and Jordan algebras. (As is known a Lie algebra has infinitely many nonequivalent irreducible representations.)

To improve the situation, N.Jacobson had introduced [3] another notion of a representation of a Jordan algebra (see below the definition of Jacobson representation) and had shown that every simple Jordan algebra has at least one nontrivial Jacobson representation. But still the number of Jacobson representations is also finite.

In the talk we will introduce a notion of a generalized representation of a Jordan algebra with unit and describe the irreducible generalized representations. The irreducible generalized representations are in a one-to-one correspondence with the irreducible representations of the 3-graded Lie algebra $L(A)$ corresponding to A.

[1] The first author was partially supported by FINEP.

Particularly, every simple Jordan algebra has infinitely many nonequivalent irreducible generalized representations.

Usual and Jacobson representations correspond to special cases of generalized representations. Moreover, this correspondence preserves the irreducibility and the equivalence of representations. In particular, it allows to classify irreducible usual and Jacobson representations.

Authors are very grateful to Bruce Allison, Kevin McCrimmon and Ivan Shestakov for useful discussions.

§0 A graded Lie algebra $L(A)$, defined by a Jordan algebra A

We need a construction of a 3-graded Lie algebra $L(A)$, defined by a Jordan algebra A.

The construction of $L(A)$ is presented as it was originally given in [4],[5] (see also [6]).

This construction coincides with what is called TKK-construction, when Jordan algebras A has a unit, but does not coincide with it in general (for example $dimU_{-1}$ is not equal in general to $dimU_1$). The Lie algebra $L(A)$ has two following important properties:

1) There is an element $\bar{A} \in U_1$ such that $[[\bar{A}, x], y] = x * y \ \ \forall x, y \in U_{-1}$, where $*$ is the multiplication in the given algebra A. (The space U_{-1} is identified with the space of algebra A.)

2) The Lie algebra $L(A)$ is generated by the space U_{-1} and the element $\bar{A} \in U_1$.

To construct the Lie algebra $L(A)$ let us denote by $\mathcal{U}$ the space of algebra A and by $\bar{A}(x, y) = x * y$ the multiplication in A . We denote also

$$L_a(x) = a * x, \quad A_a(x, y) = (x * a) * y + (y * a) * x - a * (x * y), \ (0.1)$$

$$\mathcal{S} = \{L_a, [L_a, L_b] \mid \forall a, b \in \mathcal{U}\}, \quad \bar{\mathcal{U}} = \{\bar{A}, A_a \mid \forall a \in \mathcal{U}\}, \ (0.2)$$

where $\{...\}$ is the linear span of elements in the braces.

Consider a direct sum

$$\mathcal{U} \oplus \mathcal{S} \oplus \bar{\mathcal{U}}. \ (0.3)$$

Let

$$a, b \in \mathcal{U}, \qquad S, S_1, S_2 \in \mathcal{S}. \qquad (0.4)$$

Define

$$[a, b] = 0, \ [\bar{A}, A_b] = 0, \ [A_a, A_b] = 0, \ [S, a] = S(a), \ [S_1, S_2] = S_1 S_2 - S_2 S_1, \ \ (0.5)$$

$$[\bar{A}, L_a] = A_a, \ [A_a, L_b] = A_{a*b}. \ \ (3.6)$$

Theorem 1. *The space (0.3) with the commutation relations (0.5) and (0.6) is a graded Lie algebra*

$$L(A) = U_{-1} \oplus U_0 \oplus U_1, \quad (0.7)$$

where $U_{-1} = \mathcal{U}$, $U_0 = \mathcal{S}$, $U_1 = \bar{\mathcal{U}}$.

Remark 1. The multiplication in the Jordan algebra A can be restored as a double commutator $[[\bar{A}, x], y] \quad \forall x, y \in U_{-1}$.

Example 1. Let A_n be a Jordan algebra of matrices of order n with operation $B * C = BC + CB$.

Then the Lie algebra $L(A_n) = A_{2n-1}$, i.e. linear Lie algebra of matrices of order $2n$

$$A_{2n-1} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

(where A, B, C, D are square matrices of order n) with the following grading

$$U_{-1} = \left\{\left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)\right\}, \; U_0 = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)\right\}, \; U_1 = \left\{\left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right)\right\}.$$

The role of element $\bar{A}$ plays the matrix $\left(\begin{array}{c|c} 0 & 0 \\ \hline -E & 0 \end{array}\right)$ and the multiplication in the Jordan algebra A_n can be restored as a double commutator

$$\left[\left[\left(\begin{array}{c|c} 0 & 0 \\ \hline -E & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & x \\ \hline 0 & 0 \end{array}\right)\right], \left(\begin{array}{c|c} 0 & y \\ \hline 0 & 0 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & xy + yx \\ \hline 0 & 0 \end{array}\right).$$

§1 Jacobson definition of a representation of a Jordan algebra

Let A be a Jordan algebra with multiplication $(x, y) \to xy$ and $\phi : A \to End\, V$ be a linear map.

Consider a new algebra $\hat{A}$ on the space $A \oplus V$ with multiplication:

$$x * y = xy, \; u * v = 0, \; x * v = v * x = \phi(x)v \quad \forall (x, y \in A, \; u, v \in V). \quad (1.1)$$

In other words, $\hat{A}$ is a semi-direct sum of the given Jordan algebra A and an ideal V with zero product.

Definition 1. A linear map ϕ of a Jordan algebra A into $End\, V$ is called *a Jacobson representation* in V if $\hat{A}$ is a Jordan algebra.

Similar definition in the Lie algebra case is equivalent to the ordinary one. In the Jordan algebra case the notions of an usual representation and of a Jacobson representation are different.

The notions of irreducibility and of equivalence of Jacobson representations can be given in a natural way.

Let A be a Jordan algebra with multiplication $x * y$. Let $L_a(x) = a * x$. It is easy to show that mapping $a \mapsto L_a$ is a Jacobson representation in the space of algebra A. It is natural to call this representation *the adjoint representation.* Thus any Jordan algebra has at least one Jacobson representation.

Theorem 2. (N.Jacobson [2]) *There is only a finite number of classes of irreducible Jacobson representations up to equivalence.*

§2 Generalized representations of a Jordan algebra

Let V be a linear space and $a \in End\, V$. We denote by $*_a$ a multiplication on $End\, V$ given by the formula

$$x *_a y = [[x, a], y] = xay + yax - axy - yxa. \qquad (2.1)$$

Let us call the space $End\, V$ with the multiplication $*_a$ *the algebra* $End_a\, V$.

Definition 2. A homomorphism π of an algebra A with unit e into some algebra $End\,_a V$ is called *a generalized representation* of A if

$$[a, [\pi(e), a]] = a. \quad (2.2)$$

Example 2. Let A be a Jordan algebra and ϕ be its usual representation in an n-dimensional linear space V, i.e. ϕ is a linear map of A in $End\, V$ such that

$$\phi(x \cdot y) = \phi(x)\phi(y) + \phi(y)\phi(x). \quad (2.3)$$

Consider a 2n-dimensional space $\hat{V} = V_- \oplus V_+$, where V_- and V_+ are copies of V. Consider in this space the following linear operators with matrices:

$$a = \left(\begin{array}{c|c} 0 & 0 \\ \hline E & 0 \end{array}\right), \quad \pi(x) = \left(\begin{array}{c|c} 0 & \phi(x) \\ \hline 0 & 0 \end{array}\right).$$

Then it is easy to check that π is a generalized representation of the algebra A into $End_a \hat{V}$. Indeed,

$$\pi(x \cdot y) = \left(\begin{array}{c|c} 0 & \phi(x \cdot y) \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & \phi(x)\phi(y) + \phi(y)\phi(x) \\ \hline 0 & 0 \end{array}\right) =$$

$$= [[\pi(x), a], \pi(y)] = \pi(x) *_a \pi(y),$$

and it is easy to see that (2.2) is also fulfilled.

Thus with any usual representation ϕ of a Jordan algebra A in a linear space V can be associated a generalized representation of this algebra A in a double linear space $\hat{V}$, which we call *a generalized representation of A associated with a usual representation ϕ* .

Example 3. Let A be a Jordan algebra with a multiplication $(x, y) \mapsto xy$ and

ϕ be its Jacobson representation in the space V. In other words, the algebra $\hat{A}$ defined on the space $A \oplus V$ with the multiplication (1.1) is a Jordan algebra.

Let us consider the Lie algebra $L(\hat{A}) = W_{-1} \oplus W_0 \oplus W_1$. It is easy to see that $L(\hat{A}) = L(A) \oplus \hat{V}$, where $L(A) = U_{-1} \oplus U_0 \oplus U_1$ and $\hat{V} = V_{-1} \oplus V_0 \oplus V_1$ is a graded commutative ideal in $L(\hat{A})$ with $V_{-1} = V$.

Denote by a an element $x \mapsto [\bar{A}, x] \in End(\hat{V})$ where $\bar{A}$ is the element of $U_1 \subset W_1$ such that $[[\bar{A}, x], y] = x * y \;\; \forall x, y \in U_{-1}$, where * is the multiplication in algebra A. Then the mapping π of $A = U_{-1}$ into $End(\hat{V})$ defined by $\pi : b \mapsto [b, x]$, $x \in \hat{V}$ is a generalized representation of A into algebra $End_a(\hat{V})$.

Thus with any Jacobson representation ϕ of a Jordan algebra A in a linear space V can be associated a generalized representation of this algebra A in a linear space $\hat{V}$, which we call *a generalized representation of A associated with a Jacobson representation ϕ* .

Let $\pi : A \to End_a\, V$ be a generalized representation of a Jordan algebra A and $W \subset V$ be an invariant subspace with respect to a and $\pi(A)$. Denote by $\phi : \pi(A) \cup \{a\} \to End\, W$ a homomorphism defined by the restriction of elements of $\pi(A) \cup \{a\}$ on the space W.

Then a mapping $\pi_W : A \to End_{\phi(a)} W$ defined by $\pi_W(x) = \phi(\pi(x))$ is a generalized representation of A which is called *a subrepresentation* of π.

Definition 3. A generalized representation π is called *irreducible* if π has no subrepresentations.

Definition 4. Two generalized representations $\pi : A \to End_a\, V$ and $\nu : A \to End_b\, W$ are called *equivalent* if there exists an isomorphism $f : V \to W$ such that $f\pi(x) = \nu f(x)$ if $x \in V$ and $fa = bf$.

Let A be a Jordan algebra with unit and π be a representation of Lie algebra $L(A)$. Denote by $\Phi(\pi)$ the restriction of π on the subspace U_{-1}.

Lemma. *The mapping Φ associates to any representation π of $L(A)$ in the space V a generalized representation of Jordan algebra A in $End_{\pi(\bar{A})}\, V$.*

The proof is very simple:

$$\Phi(\pi)(A(x, y)) = \pi([[[\bar{A}, x], y]) = [[\pi([\bar{A}), \pi(x)], \pi(y)] = \Phi(\pi(x)) *_a \Phi(\pi(y)), \quad (2.4)$$

where $a = \pi(\bar{A})$.

The second condition (2.2) is also fulfilled:

$$[\pi(\bar{A}), [\pi(e), \pi(\bar{A})]] = \pi([\bar{A}, [e, \bar{A}]] = \pi(\bar{A}), \quad (2.5)$$

because $[\bar{A}, [e, \bar{A}]] = \bar{A}$ in the algebra Lie $L(A)$.

The following statement allows to describe all irreducible representations of Jordan algebras with unit.

Theorem 3. *The mapping Φ defines a one-to-one correspondence between* *irreducible linear representations of Lie algebra $L(A)$ and irreducible generalized representations of Jordan algebra A.*

§3 The graded representations of graded Lie algebras and the order of generalized representations

Let G be a graded Lie algebra

$$G = U_{-k} \oplus \cdots \oplus U_{-1} \oplus U_0 \oplus U_1 \oplus \cdots \oplus U_m, \quad (3.1)$$

Let π be a representation of G in the linear space V. *An l-grading of π* is a presentation of V as a direct sum:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_l, \quad (3.2)$$

such that

$$\pi(U_i)V_j \subset V_{i+j}. \quad (3.3)$$

A representation π of G in the linear space V equipped with l-grading is called *l-graded representation.* The number l is called *the length* of the graded representation π.

Example 4. Let A_n be a Jordan algebra of matrices of order n with the operation $B * C = BC + CB$. Consider the graded Lie algebra $A_{2n-1} = U_{-1} \oplus U_0 \oplus U_1$ from the example 1. The representation of this algebra given there is 2-graded:

$$V = V_1 \oplus V_2,$$

where V_1 is the linear subspace spanned by the n first coordinate vectors and V_2 is the linear subspace spanned by the n last coordinate vectors.

Definition 5. A generalized representation of a Jordan algebra A is called *of order l* if it has form $\Phi(\pi)$, where π is an l-graded representation of the Lie algebra $L(A)$.

Remark 2. As we have obtained in Examples 2 and 3, generalized representation of Jordan algebra A associated with usual (Jacobson) representation is of order 2 (order 3).

Theorem 4. *To find all usual (Jacobson) representations of Jordan algebra A it is enough to find all 2-graded (3-graded) representations π of Lie algebra $L(A) = U_{-1} \oplus U_0 \oplus U_1$ and consider $\Phi(\pi)$.*

Particularly, a Jordan algebra A is special iff the Lie algebra $L(A)$ has a 2-graded representation.

Corollary (See [1],[2],[3]). *The exceptional Jordan algebra E_3 has no usual representations.*

This classical result follows also from the following simple fact: the graded Lie algebra $E_7 = L(E_3) = U_{-1} \oplus U_0 \oplus U_1$ has no 2-graded representations.

§4 The classification of irreducible generalized representations of simple Jordan algebras.

To classify irreducible generalized representations of simple Jordan algebras we first will describe all gradings of irreducible representations of graded simple Lie algebras.

We remind that a grading of a semisimple Lie algebra G is defined by linear function f(x) on the dual space H^* of Cartan subalgebra $H \subset G$ with nonnegative integer values on simple roots of algebra G (see [6],[7]).

Theorem 5. *Let $G = \sum_{i=-k}^{i=k} U_i$ be a semisimple Lie algebra graded by a function f. Then a linear space V of a finite dimensional irreducible representation π of G with highest weight Λ can be equipped with a grading such that π becomes an l-graded representation with*

$$l = f(\Lambda) + f(i(\Lambda)) + 1,$$

where i is a Tits involution on simple roots of G. The length l is uniquely defined.

Remark 3. We remind that the Tits involution is defined by a nontrivial involutive automorphism of the Dynkin diagram of G in the cases $G = A_n, D_{2n}, E_6$ and the identity in other cases.

We will apply the theorem 5 to the special cases when $G = L(A) = U_{-1} \oplus U_0 \oplus U_1$. In all these cases the function f is equal to 1 on some simple root α_j and is equal to zero on other simple roots, i.e. $f(\Lambda) = \lambda_j$, where $\Lambda = \sum_1^n \lambda_i \alpha_i$ is the presentation of Λ as a linear combination of simple roots α_i.

Let A be a simple Jordan algebra and π_Λ be an irreducible representation of the Lie algebra $L(A)$ with a highest weight Λ.

The following theorems follow from theorems 4 and 5.

Theorem 6. *The finite dimensional irreducible generalized representations of a simple Jordan algebra A over field C have form $\Phi(\pi_\Lambda)$.*

The orders l of these representations are given in the following list.

1) $A = A_n$, $L(A) = A_{2n-1}$, $f(\Lambda) = \lambda_n$, $l = 2\lambda_n + 1$.

2) $A = B_n$, $L(A) = C_n$, $f(\Lambda) = \lambda_n$, $l = 2\lambda_n + 1$.

3) $A = C_n$, $L(A) = D_{2n}$, $f(\Lambda) = \lambda_{2n}$, $l = \lambda_{2n} + \lambda_{2n-1} + 1$.

4) $A = D_{2n}$, $L(A) = D_n$, $f(\Lambda) = \lambda_1$, $l = 2\lambda_1 + 1$.

5) $A = D_{2n+1}$, $L(A) = B_n$, $f(\Lambda) = \lambda_1$, $l = 2\lambda_1 + 1$.

6) $A = E_3$, $L(A) = E_7$, $f(\Lambda) = \lambda_7$, $l = 2\lambda_7 + 1$.

Theorem 7. *Let A be a simple Jordan algebra. There is a finite number of* *irreducible generalized representations of the given order l.*

Particularly, there is a finite number of usual representations ($l = 2$) and Jacobson representations ($l = 3$).

§5 One interpretation of the generalized representations

First we will define a series of algebras $A_{k_1 k_2 \cdots k_l}$. Let U_{st} be the space of $s \times t$ matrices.

The space of algebra $A_{k_1 k_2 \cdots k_l}$ is the direct sum

$$A_{k_1 k_2} \oplus A_{k_2 k_3} \oplus \cdots \oplus A_{k_{l-1} k_l},$$

i.e. the elements of the algebra are tuples

$$x = (x_1, x_2, \cdots, x_{l-1}), \qquad x_i \in U_{k_i k_{i+1}}.$$

Let us fix $a = (a_1, a_2, \cdots, a_{l-1})$. The multiplication $*_a$ in the algebra $A_{k_1 k_2 \cdots k_l}$ is given by the formula

$$(x *_a y)_i = x_i a_i^t y_i + y_i a_i^t x_i - a_{i-1}^t x_{i-1} y_i - y_i x_{i+1} a_{i+1}^t,$$

where a^t is the transpose of matrix a.

Remark 4. The algebra A_{kk} is the Jordan algebra of matrices of order k. Indeed

$$x *_a y = x a^t y + y a^t x.$$

It is easy to see that this multiplication is isomorphic to the multiplication $xy + yx$ if a is not a degenerate matrix.

Theorem 8. *There is a one-to-one correspondence between generalized representations π of order l of Jordan algebra A and homomorphic embeddings π^* of A in algebras $A_{k_1 k_2 \cdots k_l}$.*

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