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An example of simple Lie superalgebra with several invariant bilinear forms

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Abstract: Simple associative superalgebra with 2 independent supertraces is presented. Its commutant is a simple Lie superalgebra which has at least 2 independent invariant bilinear forms.

1 Introduction

Shchepochkina proved [1] that the dimension of the space of invariant supersymmetric nondegenerate bilinear forms on any finite dimensional simple Lie superalgebra is ≤ 1. Here a simple infinite dimensional associative superalgebra with 2 independent supertraces is described. The Lie superalgebra, associated with it is simple and has at least 2 independent invariant bilinear forms.

The construction is based on the associative superalgebra $H_{W(\mathfrak{R})}(\nu)$, the superalgebra of observables of the rational Calogero model based on the root system $\mathfrak{R}$ [2]; here $W(\mathfrak{R})$ is the Coxeter group generated by $\mathfrak{R}$ and ν is a coupling constant(s) of this Calogero model.

It is shown in [3] that $H_{W(\mathfrak{R})}(\nu)$ possesses $Q(\mathfrak{R})$-dimensional space of the supertraces, where $Q(\mathfrak{R})$ is the number of conjugacy classes of the elements of $W(\mathfrak{R})$ which have no eigenvalue -1. In particularly $Q(A_{N-1})$ is equal to the number of decompositions N into the sum of odd integers [4] and so $H_{W(A_2)}(\nu)=H_{S_3}(\nu)$ has 2-dimensional space of the supertraces.

Each supertrace $str(\cdot)$ generates invariant supersymmetric bilinear form $B_{str}(f,g) \stackrel{def}{=} str(fg)$. It is shown in [5] that all the supertraces on $H_{S_3}(\nu)$ generate nondegenerate bilinear forms if and only if $\nu \neq n+\frac{1}{2}$ and $\nu \neq n \pm \frac{1}{3}$ for every integer n. The conjecture arises that the associative superalgebras $H_{S_3}(\nu)$ are simple if $\nu \neq n+\frac{1}{2}$ and $\nu \neq n \pm \frac{1}{3}$ which is proved in [6] for the case $\nu = 0$.

In this report based on [6] the associative superalgebra $H_{S_3}(0)$ with two-dimensional space of supertraces is presented. It is shown that (i) it is simple, (ii) its commutant $\mathfrak{A}_1 = [H_{S_3}(0), H_{S_3}(0)\}$ is a simple Lie superalgebra and (iii) $\mathfrak{A}_1$ has at least 2-dimensional space of nondegenerate bilinear invariant forms.

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2 Algebra declared in the title

Consider the associative algebra which is semidirect product of Weyl algebra W^2 generated by elements x^α, $y^\alpha, 1$ $(\alpha = 0, 1)$ and $\mathbb{C}[S_3]$ generated by elements L_i, Q_i $(i \in \mathbb{Z}_3)$ where the generating elements satisfy the following relations:

$$[x^\alpha, x^\beta] = [y^\alpha, y^\beta] = 0, \qquad [y^\alpha, x^\beta] = \varepsilon^{\alpha\beta}, \tag{1}$$

$$L_i L_j = \delta_{i+j}\, Q_j, \qquad L_i Q_j = \delta_{i-j}\, L_j, \tag{2}$$

$$Q_i L_j = \delta_{i+j}\, L_j, \qquad Q_i Q_j = \delta_{i-j}\, Q_j, \tag{3}$$

$$Q_0 + Q_1 + Q_{-1} = 1, \tag{4}$$

(here $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{01} = 1$ and $\delta_0 = 1$, $\delta_i = 0$ if $i \neq 0$)

$$L_i x^\alpha = y^\alpha L_{i+1}, \qquad L_i y^\alpha = x^\alpha L_{i-1}, \tag{5}$$

$$Q_i x^\alpha = x^\alpha Q_{i+1}, \qquad Q_i y^\alpha = y^\alpha Q_{i-1}. \tag{6}$$

Definition 1. *$H_{S_3}(0)$ is the associative superalgebra of all polynomials in the operators x^α, y^α, Q_i and L_i satisfying the relations (1)-(6) with $\mathbb{Z}_2$-grading π defined by the formula: $\pi(x^\alpha) = \pi(y^\alpha) = 1$, $\pi(L_i) = \pi(Q_i) = 0$.*

This algebra can be realized in the terms of the operators acting on the space of complex valued smooth functions on $\mathbb{R}^3$.

Indeed, let $a_k^\alpha = \frac{1}{\sqrt{2}}(z_k + (-1)^\alpha \frac{\partial}{\partial z_k})$ and $\sigma f(z_1, z_2, z_3) = f(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ for all $\sigma \in S_3$. Let $K_{ij} = K_{ji} \in S_3$ be the operators of elementary transpositions and $\lambda = exp(2\pi i/3)$.

Then the operators

$$\begin{aligned}
x^\alpha &= \frac{1}{\sqrt{3}}(a_1^\alpha + \lambda a_2^\alpha + \lambda^2 a_3^\alpha), \\
y^\alpha &= \frac{1}{\sqrt{3}}(a_1^\alpha + \lambda^2 a_2^\alpha + \lambda a_3^\alpha), \\
L_k &= \frac{1}{3}(\lambda^k K_{12} + K_{23} + \lambda^{-k} K_{31}), \\
Q_k &= \frac{1}{3}(1 + \lambda^k K_{12} K_{13} + \lambda^{-k} K_{12} K_{23}), \\
& \quad L_{i+3} = L_i, \quad Q_{i+3} = Q_i
\end{aligned}$$

satisfy commutation relations (1)-(6).

3 Supertraces on $H_{S_3}(0)$

Definition 2. *The supertrace on an arbitrary associative superalgebra $\mathcal{A}$ is complex-valued linear function $str(\cdot)$ on $\mathcal{A}$ which satisfies the condition*

$$str(fg) = (-1)^{\pi(f)\pi(g)} str(gf)$$

for every $f, g \in \mathcal{A}$ with definite parity.

To describe all supertraces on $H_{S_3}(0)$ let us note that $H_{S_3}(0)$ has the Lie algebra $\mathfrak{sl}_2$ of inner automorphisms. This algebra is generated by the generators $T^{\alpha\beta}$

$$T^{\alpha\beta} = (x^\alpha y^\beta + x^\beta y^\alpha) \tag{7}$$

which satisfy the usual commutation relations

$$[T^{\alpha\beta}, T^{\gamma\delta}] = \epsilon^{\alpha\gamma} T^{\beta\delta} + \epsilon^{\alpha\delta} T^{\beta\gamma} + \epsilon^{\beta\gamma} T^{\alpha\delta} + \epsilon^{\beta\delta} T^{\alpha\gamma},$$

and act on generating elements x^α and y^α as follows:

$$\left[T^{\alpha\beta},\, x^\gamma\right] = \epsilon^{\alpha\gamma} x^\beta + \epsilon^{\beta\gamma} x^\alpha, \qquad \left[T^{\alpha\beta},\, y^\gamma\right] = \epsilon^{\alpha\gamma} y^\beta + \epsilon^{\beta\gamma} y^\alpha$$

leaving the group algebra $\mathbb{C}[S_3]$ invariant: $\left[T^{\alpha\beta},\, K_{ij}\right] = 0$.

Clearly, $H_{S_3}(0)$ is decomposed into the infinite direct sum of finitedimensional irreducible representations of this $\mathfrak{sl}_2$. Let $\mathcal{A}_s$ (s is a spin) be the direct sum of all $2s+1$-dimensional irreducible representations in this decomposition. So

$$H_{S_3}(0) = \mathcal{A}_0 \oplus \mathcal{A}_{1/2} \oplus \mathcal{A}_1 \oplus \ldots, \quad \pi(\mathcal{A}_n) = 0,\ \pi(\mathcal{A}_{n+1/2}) = 1.$$

Evidently, $\mathcal{A}_0$ is a subalgebra of $H_{S_3}(0)$.

As every element of the subspace $\mathcal{A}_s$ for $s > 0$ is a linear combinations of the vectors of the form $[f,\, T^{\alpha\beta}]$ where $T^{\alpha\beta}$ is defined in (7), every supertrace vanishes on this subspace and has non-trivial values only on the associative subalgebra $\mathcal{A}_0 \subset H_{S_3}(0)$ of all $\mathfrak{sl}_2$-singlets.

Let $m = \frac{1}{2}\{x^\alpha,\, y_\alpha\}$. In this formula the greek indices are lowered and raised with the help of the antisymmetric tensor $\varepsilon^{\alpha\beta}$: $a^\alpha = \sum_\beta \varepsilon^{\alpha\beta} a_\beta$. Clearly, m is a singlet under the action of $\mathfrak{sl}_2$ (7).

Evidently, the associative algebra $\mathcal{A}_0$ of $\mathfrak{sl}_2$-singlets consists of the polynomials of m with coefficients in $\mathbb{C}[S_3]$.

To describe the restrictions of all the supertraces on $\mathcal{A}_0$, it is convenient to use the generating functions, which are computed in [5] for any values of ν, and for $\nu = 0$ take the following simple form:

$$\begin{aligned} \mathcal{L}_i &\stackrel{def}{=} str(e^{\xi m} L_i) = 0, \\ \mathcal{Q}_i &\stackrel{def}{=} str(e^{\xi m} Q_i) = \frac{P_i}{\Delta}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} P_0 &= -2S_1 + \frac{S_2}{2}\left(e^{2\xi} + e^{-2\xi} - 2e^{\xi} - 2e^{-\xi}\right), \\ P_+ &= \frac{2}{3}S_1\left(-e^{2\xi} + 2e^{-\xi}\right) + \frac{S_2}{2}\left(e^{-2\xi} - 2e^{\xi} + 3\right), \\ P_- &= \frac{2}{3}S_1\left(-e^{-2\xi} + 2e^{\xi}\right) + \frac{S_2}{2}\left(e^{2\xi} - 2e^{-\xi} + 3\right) \\ \Delta &= exp(-3\xi)(exp(3\xi)+1)^2. \end{aligned} \tag{9}$$

Here S_1, S_2 are arbitrary parameters, specifying the supertrace in the two-dimensional space of supertraces:

$$S_1 = -2str(1) - str(K_{12}K_{23}), \qquad S_2 = \frac{8}{3}str(1) - \frac{2}{3}str(K_{12}K_{23}).$$

4 The simplicity of $H_{S_3}(0)$.

To prove the simplicity of $H_{S_3}(0)$ it is sufficient to prove that elements Q_i belong to any nonzero ideal $\mathcal{I}$.

First of all let us note that the existence of $\mathfrak{sl}_2$ algebra of inner derivations implies that each ideal of associative algebra $H_{S_3}(0)$ is graded.

Let $\mathcal{I} \subset H_{S_3}(0)$ be nonzero ideal.

Consider the ideals $\mathcal{J}_i \subset \mathbb{C}[m]$ defined as

$$\mathcal{J}_i = \{f(m) \in \mathbb{C}[m] : \ f(m)Q_i \in \mathcal{I}\}.$$

One can easily prove that for arbitrary element $u \in \mathcal{I}$ there exist such finite sets of elements $v^i_k, w^i_k \in H_{S_3}(0)$ ($i \in \mathbb{Z}_3$, $k = 1, ..., K$) that $0 \neq \sum_{k=1}^{K} v^i_k u w^i_k \in cal J_i$. So the following proposition is valid:

Proposition 1. $\mathcal{J}_i \neq \{0\}$.

Let $\varphi_i(m) = m^{n_i} + \alpha_{i,1} m^{n_i - 1} + ...$ generates the ideal $\mathcal{J}_i$.

¿From the relation $L_i f(m) Q_i L_{-i} = f(-m) Q_{-i}$ the next proposition follows:

Proposition 2. *The polynomials $\varphi_i(m)$ generating the ideals $\mathcal{J}_i$ satisfy the relations*

$$n_i = n_{-i}$$
$$\varphi_i(m) = (-1)^{n_i} \varphi_{-i}(-m).$$

Now we can prove the following theorem:

Theorem 1. *The associative superalgebra $H_{S_3}(0)$ is simple.*

We will prove that arbitrary nonzero ideal $\mathcal{I} \subset H_{S_3}(0)$ contains the elements Q_i and as a consequence $\mathcal{I} \ni 1 = Q_0 + Q_{-1} + Q_1$

Let $\mathcal{I}$ be nonzero ideal in $H_{S_3}(0)$ and $\varphi_i(m)$ generate the ideals $\mathcal{J}_i$ described above. So $\varphi_i(m)Q_i \in \mathcal{I}$ and hence $(y^0)^3\varphi_i(m)Q_i(x^1)^3 \in \mathcal{I}$. Consider the $\mathfrak{sl}_2$-singlet part of this element which also belongs to $\mathcal{I}$ and can be computed easily:

$$((y^0)^3\varphi_i(m)Q_i(x^1)^3)_0 = \frac{1}{8}(m+1)(m+2)(m+3)\varphi_i(m+3)Q_i \in \mathcal{I}$$

So $(m+1)(m+2)(m+3)\varphi_i(m+3) \in \mathcal{J}_i$ and $(m+1)(m+2)(m+3)\varphi_i(m+3) = p_i(m)\varphi_i(m)$ with some polynomials $p_i(m)$. Further

$$(m+1)^2(m+2)^2(m+3)^2\varphi_i(m+3)\varphi_{-i}(m+3) = q_i(m)\varphi_i(m)\varphi_{-i}(m),$$

where $q_i = p_i p_{-i}$. Since $\varphi_i(m)\varphi_{-i}(m)$ is even polynomial, the real part of the rightest root of $\varphi_i(m)\varphi_{-i}(m)$ is non negative, and so this rightest root is not a root of the polynomial $(m+1)^2(m+2)^2(m+3)^2\varphi_i(m+3)\varphi_{-i}(m+3)$. So the polynomial $\varphi_i(m)\varphi_{-i}(m)$ has no complex roots and hence $\varphi_i(m) = 1$.

In such a way, $Q_i \in \mathcal{I}$ and $1 = Q_0 + Q_1 + Q_2 \in \mathcal{I}$.

5 Lie superalgebra $H_{S_3}(0)^L$

Now let us consider Lie superalgebra $\mathfrak{A} = H_{S_3}(0)^L$ which consists of elements of associative algebra $H_{S_3}(0)$ with operation $[f, g\} = fg - (-1)^{\pi(f)\pi(g)}gf$. This Lie superalgebra has an ideal $\mathfrak{A}_1 = [\mathfrak{A}, \mathfrak{A}\}$ with the following evident properties

(i) *codim* $\mathfrak{A}_1 = 2$,

(ii) $\mathcal{A}_{1/2} \oplus \mathcal{A}_1 \oplus ... \subset \mathfrak{A}_1$.

Let Z be the center of $H_{S_3}(0)$. Obviously, $Z \subset \mathcal{A}_0$ and so every element of Z is a polynomial of m with coefficients in $\mathbb{C}[S_3]$.

Proposition 3. $Z = \mathbb{C}$.

Let $f(m) \in Z$. As m is even then f is even element also. As $mL_i = -L_i m$, the relation $[f, m] = 0$ gives that $f(m) = \sum f_i(m)Q_i$. Due to equalities $[(x^0)^3, Q_i] = 0$ and commutation relations one has $[(x^0)^3, f(m)] = (x^0)^3(f(m) - f(m-3))$ and as a consequence $f(m) = f(m-3)$. So $f(m)$ does not depend on m, $f(m) \in \mathbb{C}[S_3]$. To finish the proof it is sufficient to commute f with x^0 or y^0.

Proposition 4. $Z \bigcap \mathfrak{A}_1 = \{0\}$.

As there exists such supertrace that $str(1) \neq 0$ and $Z = \mathbb{C}$, it follows from $str(Z \bigcap \mathfrak{A}_1) = 0$ that $Z \bigcap \mathfrak{A}_1 = \{0\}$.

Proposition 5. *Let* $\mathcal{I} \subset \mathfrak{A}_1$ *be an ideal and* $\mathcal{I}_{even} = \{0\}$. *Then* $\mathcal{I}_{odd} = \{0\}$.

Let $r \in \mathcal{I}_{odd}$. Due to $\mathfrak{sl}_2$ invariance one can choose r in the form

$$r = \sum_{i=0,1,2} \sum_{n=0}^{2s} \left((x^0)^n(y^0)^{2s-n} f_{ni}(m)Q_i + (x^0)^n(y^0)^{2s-n} g_{ni}(m)L_i\right)$$

with odd value of $2s$. As $(x^0)^3 \in \mathcal{A}_{3/2} \subset \mathfrak{A}_1$ then $[(x^0)^3, r\} \in \mathcal{I}_{even}$ and $\{(x^0)^3, r\} = 0$. So the following relations take place:

$$\sum_{i=0,1,2} \sum_{n=0}^{2s} (x^0)^{n+3} (y^0)^{2s-n} \left(f_{ni}(m) + f_{ni}(m+3) \right) Q_i = 0$$

and so $f_{ni} = 0$. Further, it follows from

$$0 = \{(x^0)^3, r\} \simeq \sum_{i=0,1,2} \sum_{n=0}^{2s} \left((x^0)^3 + (y^0)^3 \right) (x^0)^n (y^0)^{2s-n} g_{ni}(m) L_i$$

that $g_{ni}(m) = 0$ also. Here the sign $\simeq$ is used to denote the equality up to polynomials of lesser degrees.

Theorem 2 *Lie superalgebra $\mathfrak{A}_1$ is simple.*

This Theorem follows from the next two general theorems proven by S.Montgomery in [7]:

Theorem 3 *Let $\mathcal{A}$ be associative simple superalgebra and $\mathcal{I} \subset [\mathcal{A}^L, \mathcal{A}^L\}$ be a graded ideal of $[\mathcal{A}^L, \mathcal{A}^L\}$ such that $\mathcal{I} \neq [\mathcal{A}^L, \mathcal{A}^L\}$. Then $\mathcal{I}_3 = \{0\}$, where $\mathcal{I}_1 = [\mathcal{I}, \mathcal{I}\}$, $\mathcal{I}_2 = [\mathcal{I}_1, \mathcal{I}_1\}$, $\mathcal{I}_3 = [\mathcal{I}_2, \mathcal{I}_2\}$.*

Theorem 4. *Let $\mathcal{A}$ be associative simple superalgebra and $\mathcal{I} \subset [\mathcal{A}^L, \mathcal{A}^L\}$ be a graded ideal of $[\mathcal{A}^L, \mathcal{A}^L\}$ such that $[\mathcal{I}_{even}, \mathcal{I}\} \subseteq Z(\mathcal{A})$ where $Z(\mathcal{A})$ is the center of $\mathcal{A}$. Then $\mathcal{I}_{even} \subseteq Z(\mathcal{A})$.*

Indeed, let $\mathcal{I} \subset \mathfrak{A}_1$ be an ideal such that $\mathcal{I} \neq \mathfrak{A}_1$. Due to Theorem 3 one can consider that $[\mathcal{I}, \mathcal{I}\} = \{0\}$. Then it follows from Theorem 4 and Proposition 4 that $\mathcal{I}_{even} = \{0\}$. Further, Proposition 5 gives that $\mathcal{I}_{odd} = \{0\}$, and so $\mathcal{I} = \{0\}$.

Theorem 5 *Simple Lie superalgebra $\mathfrak{A}_1$ has at least 2 independent bilinear invariant forms.*

Two-dimensional space of supertraces $str(\cdot)$ on $H_{S_3}(0)$ generates some space of invariant bilinear forms $B(u, v) = str(uv)$ on $\mathfrak{A}_1$. This space is also 2-dimensional. Indeed, let some supertrace $str_0(\cdot)$ on $\mathcal{A}$ leads to bilinear form B_0 on $\mathfrak{A}_1$ which is equal to zero identically. The elements x^0, y^1, $x^0 Q_1$ and $y^1 Q_1$ belong to $\mathcal{A}_{1/2} \subset \mathfrak{A}_1$. So, $0 = B_0(x^0, y^1) - B_0(y^1, x^0) = str_0([x^0, y^1]) = str_0(1)$ and $0 = B_0(x^0, y^1 Q_1) - B_0(y^1, x^0 Q_1) = str_0([x^0, y^1] Q_1) = str_0(Q_1)$. It follows from (8) that $str_0(1) = -\frac{1}{6}(S_1 - \frac{3}{2} S_2)$ and $str_0(Q_1) = \frac{1}{6}(S_1 + \frac{3}{2} S_2)$. So $S_1 = S_2 = 0$ and str_0 is equal to zero identically.

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