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Generalised Euler Characteristics of Varieties of Tori in Lie Groups

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Contents

1 The variety of maximal tori. Rational points, twisting.

Let $\bar{K} = K \supseteq k$ be fields, $\Gamma := \mathrm{Gal}(K/k)$, G be a connected reductive algebraic group over K, defined over k. Γ acts on G, $G(k) := G^\Gamma$, the group of k-points of G.

Our main examples: $k = \mathbb{R}$, $K = \mathbb{C}$, and $k = \mathbb{F}_q$, $K \subset \overline{\mathbb{F}}_q$. In these examples $\Gamma = \langle\gamma\rangle$ is cyclic, and we shall assume this henceforth.

The variety of maximal tori of G is denoted $\mathcal{T}$. It has an action of Γ, and we refer to Γ-fixed tori as "rational". In our two examples, $\mathcal{T}^\Gamma \neq \emptyset$.

1.1 Twisting of tori

Let S_0 be a maximal $k-$*split* torus of G, and let T_0 be any rational maximal torus which contains S_0. It is known in our two cases that T_0 is unique up to conjugacy by $G(k)$.

Write $W := N_G(T_0)/T_0$, the Weyl group. It has a Γ-action. Say $v \sim_\Gamma w$ if $\exists u \in W$ such that $w = uv\gamma(u)^{-1}$. The set of Γ-classes of W is $H^1(\Gamma, W)$ (Galois cohomology).

Suppose $T = gT_0g^{-1} \in \mathcal{T}(k) = \mathcal{T}^\Gamma$. Then $\gamma(gT_0g^{-1}) = gT_0g^{-1} \implies g^{-1}\gamma(g) \in N_G(T_0)$, so $= \dot{v}$ for $v \in W$. The Γ-class of $v \in W$ is independent of g, and is called the *type* of $T \in \mathcal{T}(k)$. We say T is 'twisted by v'.

Let ϵ be the alternating character of W. Then ϵ is constant on Γ-classes, so that it makes sense to speak of the *sign* $\epsilon(T) := \epsilon(v)$ of $T \in \mathcal{T}(k)$. This gives $\epsilon : \mathcal{T}(k) \longrightarrow \{\pm 1\}$.

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2 The main theorems for real Lie groups. (Joint work with J. van Hamel)

2.1 The local system $\mathcal{S}_\epsilon$

For any Γ-space X, a local system $\mathcal{L}$ (of $\mathbb{C}$-vector spaces) on X is Γ-*equivariant* if there is an isomorphism $\gamma : \gamma^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$. A similar definition applies to any sheaf, or complex of sheaves on X.

$\mathcal{P}$:= variety of "Killing couples" $(T \subset B)$, where T, B are respectively a maximal torus and Borel subgroup of G.

$p : \mathcal{P} \longrightarrow \mathcal{T}$, the first projection $(T, B) \mapsto T$, is an unramified covering with group W. If $\mathbb{C}$ denotes the constant sheaf on $\mathcal{P}$, then

$$p_*\mathbb{C} = \oplus_{E\in\hat{W}} E \otimes \mathcal{S}_E,$$

where $\mathcal{S}_E$ is the irreducible local system on $\mathcal{T}$ which corresponds to E.

The local system $\mathcal{S}_\epsilon$ is Γ-equivariant.

2.2 Weighted Euler Characteristics

Let X quasi-projective, defined over $\mathbb{R}$ variety; $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ acts via σ, 'complex conjugation'.

The fixed point variety X^σ has a finite number of connected components: $X^\sigma = \amalg C$.

If $\mathcal{S}$ is a Γ-equivariant local system on X define the *Lefschetz number of σ on $(X, \mathcal{S})$*:

$$\Lambda_c(\sigma, X, \mathcal{S}) := \sum_i (-1)^i \operatorname{Trace}(\sigma, H^i_c(X, \mathcal{S})).$$

For any $x \in X^\sigma$, $\operatorname{Trace}(\sigma, \mathcal{S}_x)$ depends only on the connected component C in which x lies. Write $\operatorname{Trace}(\sigma, \mathcal{S}|_C)$ for this common value.

Proposition 1. *We have*

$$\Lambda_c(\sigma, X, \mathcal{S}) = \sum_C \chi_c(C) \operatorname{Trace}(\sigma, \mathcal{S}|_C).$$

We refer to either side as the "weighted Euler characteristic" of X^σ.

2.3 Statement of Main Results - The Real Case

Let G connected, reductive over $\mathbb{C}$, defined over $\mathbb{R}$ algebraic group, so $\Gamma = \langle\sigma\rangle$ acts. T_0 a maximally split rational maximal torus; $B_0 \supseteq T_0$ a Borel subgroup. Then $\sigma(B_0) = v_0 B_0 v_0^{-1}$, $v_0 \in W$.

The *real index* $\epsilon_0(G) := \epsilon(v_0) = \pm 1$ is well defined.

$\mathfrak{G}$, the Lie algebra of G (over $\mathbb{R}$) inherits G's Γ-action.

$\mathfrak{G}$ has an $\mathrm{Ad}(G)$ invariant non-degenerate form $\langle\ ,\ \rangle : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathbb{C}$, such that $\langle \mathfrak{G}(\mathbb{R}), \mathfrak{G}(\mathbb{R}) \rangle \subset \mathbb{R}$.

For $\xi \in \mathfrak{G}(\mathbb{R})$, define $\mathbb{R}$-varieties $\mathcal{T}^\xi$ and $\mathcal{T}_\xi$ as follows.

$$\mathcal{T}^\xi = \{T \in \mathcal{T} \mid \mathrm{Lie}\, T \ni \xi\} \tag{2}$$

$$\mathcal{T}_\xi = \{T \in \mathcal{T} \mid \langle \mathrm{Lie}\, T, \xi \rangle = 0\}. \tag{3}$$

Theorem 4. *If $\xi \in \mathfrak{G}(\mathbb{R})$,*

$$\epsilon_0(G)\Lambda_c(\sigma, \mathcal{T}^\xi, \mathcal{S}_\epsilon) = \epsilon_0(Z_G(\xi)^0)\epsilon(v_\xi)(-1)^{N(\xi)},$$

where:
$\epsilon_0(H)$ is the real index of H,
$v_\xi \in W$ is the type of a maximally split torus of $Z_G(\xi)^0$, and
$N(\xi)$ is the number of positive roots of $Z_G(\xi)^0$.

Theorem 5. *In the same notation, we have*

$$\Lambda_c(\sigma, \mathcal{T}_\xi, \mathcal{S}_\epsilon) = \begin{cases} (-1)^N \text{ if } \xi \text{ is nilpotent} \\ 0 \text{ otherwise}, \end{cases}$$

where N is the number of positive roots of G.

To see the connection with weighted Euler characteristics, observe that if $T \in \mathcal{T}^\sigma$ has type $w \in Z^1(\Gamma, W) \subseteq W$, then

$$\mathrm{Trace}(\sigma, \mathcal{S}_{\epsilon,T}) = \epsilon_0(G)\epsilon(w).$$

So both sides in Theorems 4 and 5 may be expressed in the form

$$\sum_{\substack{C \subset X(\mathbb{R}) \\ \text{connected} \\ \text{component}}} \chi_c(C) \cdot \epsilon(C),$$

where $\epsilon(C) = \epsilon(T)$ for any $T \in C$.

Theorem 4 is an analogue of the Steinberg character formula for groups over $\mathbb{F}_q$.

Theorem 5 is an analogue of the fact that over $\mathbb{F}_q$, the Fourier transform of the Steinberg character is the characteristic function of the nilpotent set (on $\mathfrak{G}^F$, $F = $ Frobenius).

3 Example: The Case SL_2

Take $G = \mathrm{SL}_2$ with standard complex conjugation; T_0 = diagonal subgroup; $W = \{1, r\}$.

Here $\mathcal{P} \cong G/T_0$ with the induced real structure, and this is isomorphic to the G-orbit in $\mathfrak{G}$ of $\xi_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. So

$$\mathcal{P} \cong \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} :\ a^2 + bc = 1 \right\} \subset \mathfrak{G} \cong \mathbb{A}^3.$$

The W-action on $\mathcal{P}$ is given by $(a,b,c) \overset{r}{\mapsto} (-a,-b,-c)$, and so $\mathcal{T} \cong \{[a,b,c] \in \mathbb{P}^2 \mid a^2 + bc \neq 0\}$ and $p : \mathcal{P} \longrightarrow \mathcal{T}$ is given by $(a,b,c) \mapsto [a,b,c]$.

The Killing form is given by $\langle \xi, \eta \rangle = \mathrm{Trace}(\xi\eta)$, so if $\xi = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathfrak{G}$ the subvariety $\mathcal{P}_\xi \subset \mathcal{P}$ is given by

$$\mathcal{P}_\xi := \{(a,b,c) \in \mathcal{P} :\ 2xa + yc + zb = 0\}.$$

Note that we have a 2-fold covering $p : \mathcal{P}_\xi^\sigma \amalg \mathcal{P}_\xi^{r\sigma} \longrightarrow \mathcal{T}_\xi^\sigma$, so that $\Lambda_c = \frac{1}{2}\left(\chi_c(\mathcal{P}_\xi^\sigma) - \chi_c(\mathcal{P}_\xi^{r\sigma})\right)$.

One now easily constructs the following table, which lists the various cases. In the table, $\Lambda_c = \Lambda_c(\sigma, \mathcal{T}_\xi, \mathcal{S}_\epsilon)$.

ξ	Λ_c	$\mathcal{P}_\xi^\sigma$	$\chi_c(\mathcal{P}_\xi^\sigma)$	$\mathcal{P}_\xi^{r\sigma}$	$\chi_c(\mathcal{P}_\xi^{r\sigma})$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	-1	$a^2 + bc = 1$	0	$a^2 + bc = -1$	2
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	-1	$\begin{cases} a^2 = 1 \\ c = 0 \end{cases}$	-2	$\emptyset$: $\begin{cases} a^2 = -1 \\ c = 0 \end{cases}$	0
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	0	$\begin{cases} a = 0 \\ bc = 1 \end{cases}$	-2	$\begin{cases} a = 0 \\ bc = -1 \end{cases}$	-2
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	0	$\begin{cases} b = c \\ a^2 + b^2 = 1 \end{cases}$	0	$\emptyset$: $\begin{cases} b = c \\ a^2 + b^2 = -1 \end{cases}$	0

4 (Kashiwara-Sato) Fourier transforms of conical sheaves

Let $\tau : E \longrightarrow X$ be a Γ-equivariant real vector bundle; a Γ-(equivariant) sheaf $\mathcal{S}$ on E is *conical* if $\mathcal{S}$ is constant on each $\mathbb{R}^{>0}$-orbit.

In general Γ may be any discrete group acting compatibly on X and E; for us Γ defines real structures on the complex analytic varieties X, E.

$\mathcal{D}^b_{cc}(E,\Gamma)$ is the category of bounded complexes of sheaves on E which (a) are Γ-equivariant, and (b) have cohomology sheaves which are conical and constructible on some semi-algebraic stratification of E.

Let $\check{\tau} : \check{E} \longrightarrow X$ be the dual bundle of E, $\mu : E \times_X \check{E} \longrightarrow \mathbb{R}$ the canonical pairing and write $P := \mu^{-1}(\mathbb{R}^{\geq 0})$; p_1, p_2 are the first and second projections, $E \times_X \check{E} \longrightarrow E, \check{E}$.

The *(Kashiwara-Sato-)Fourier transform* $\mathcal{F}_E : \mathcal{D}^b_{cc}(E) \longrightarrow \mathcal{D}^b_{cc}(\check{E})$ is defined by

$$\mathcal{F}_E(K^\bullet) := Rp_{2*} \circ \mathrm{Res}_P \circ p_1^*(K^\bullet),$$

where $\mathrm{Res}_P = Ri_* i^!$, i being the inclusion $P \hookrightarrow E \times_X \check{E}$.

The Fourier transform has familiar properties, including: it is involutory modulo Tate twists, shifts and inversion; it commutes with Verdier duality; it behaves well with respect to base change, and morphisms of varieties.

5 Characteristic functions and Fourier transforms

For $K^\bullet$ in $\mathcal{D}^b_{cc}(E(\mathbb{C}),\sigma)$ define the *orbit characteristic function* $\chi_{K^\bullet}$ of $K^\bullet$ to be the function whose value at the orbit $\overline{x} = \{x, \sigma(x)\}$ of a point $x \in E(\mathbb{C})/\sigma$ is the element of $R(\sigma)$ (the representation ring of $\langle\sigma\rangle$) given by

$$\chi_{K^\bullet}(\overline{x}) = \oplus_{y\in\overline{x}} \sum_i (-1)^i [\mathcal{H}^i(K^\bullet)_y]. \tag{6}$$

This clearly determines the *characteristic function* $\Lambda_{K^\bullet} : X^\sigma = X(\mathbb{R}) \longrightarrow \mathbb{C}$, defined by (for $x \in X^\sigma$)

$$\Lambda_{K^\bullet}(x) = \sum_i (-1)^i \,\mathrm{Trace}(\sigma, \mathcal{H}^i(K^\bullet)_x). \tag{7}$$

For our purpose, a key result is:

Proposition 8. *Suppose $M^\bullet, N^\bullet$ in $\mathcal{D}^b_{cc}(E(\mathbb{C}),\sigma)$ satisfy $\chi_{M^\bullet} = \chi_{N^\bullet}$. Then $\chi_{\mathcal{F}_E M^\bullet} = \chi_{\mathcal{F}_E N^\bullet}$.*

In particular, the characteristic functions $\Lambda_{\mathcal{F}_E M^\bullet}$ and $\Lambda_{\mathcal{F}_E N^\bullet}$ are equal.

6 Connection with Springer representations

Define varieties $\tilde{V}, V$:

$$\begin{aligned} \tilde{V} &= \{(\xi, (T \subseteq B)) \in \mathfrak{G} \times \mathcal{P} \mid \xi \in \mathrm{Lie}\, T\} \\ V &= \{(\xi, T) \in \mathfrak{G} \times \mathcal{T} \mid \xi \in \mathrm{Lie}\, T\} \end{aligned} \tag{9}$$

Then consider $\tilde{V} \xrightarrow{\omega} V \xrightarrow{\rho} \mathfrak{G}$, where ω is a Galois W-covering and ρ is the first projection.

$\mathcal{S}_\epsilon^V$ is the local system on V corresponding to $\epsilon \in \hat{W}$, and we define $K^\bullet := \rho_! \mathcal{S}_\epsilon^V$.

Theorem 10. *$K^\bullet \in \mathcal{D}_{cc}^b(\mathfrak{G}, \sigma)$, and for $\xi \in \mathfrak{G}(\mathbb{R})$,*

$$\Lambda_{\mathcal{F}_{\mathfrak{G}} K^\bullet}(\xi) = (-1)^{rank\ \mathfrak{G}} \Lambda_c(\sigma, \mathcal{T}_\xi, \mathcal{S}_\epsilon).$$

Our strategy: *Find a perverse complex $M^\bullet$ with the same orbit characteristic function as $K^\bullet$, whose Fourier transform can be computed by other means.*

Consider

$$\begin{array}{ccc} \widetilde{\mathfrak{G}} & \xrightarrow{\pi} & \mathfrak{G} \\ {\scriptstyle \mathrm{incl}}\uparrow & & {\scriptstyle \mathrm{incl}}\uparrow \\ \widetilde{\mathfrak{G}}_{rs} & \xrightarrow{\pi_0} & \mathfrak{G}_{rs}, \end{array} \tag{11}$$

where $\widetilde{\mathfrak{G}} = G \times^B \mathfrak{b} = \{(B, \xi) \in \mathcal{B} = G/B_0 \times \mathfrak{G} \mid \xi \in g.\mathrm{Lie}\, B\}$, $\mathfrak{G}_{rs}$ is the variety of regular and semisimple elements of $\mathfrak{G}$.

π_0 is an unramified W-covering, and Lusztig has shown that $\pi_! \mathbb{C} \cong IC^\bullet(\mathfrak{G}, \mathcal{L}_{\mathrm{Reg}})$, where $\mathcal{L}_{\mathrm{Reg}} = \pi_{0*}\mathbb{C}$ is the local system on $\mathfrak{G}_{rs}$ corresponding to the regular representation. This leads to Springer action of W on $\mathcal{H}^*(\pi_!(\mathbb{C}))$, and implies

$$\pi_! \mathbb{C} = \oplus_{E \in \widehat{W}} E \otimes M_E^\bullet,$$

where $M_E^\bullet \in \mathcal{D}_{cc}^b(\mathfrak{G}, \sigma)$, and $M_E^\bullet[\dim \mathfrak{G}]$ is perverse and irreducible.

Theorem 12. *The complexes $K^\bullet$ (7) and $M_\epsilon^\bullet \in \mathcal{D}_{cc}^b(\mathfrak{G}, \sigma)$ have the same orbit characteristic functions.*

Hence by Proposition 8, their Fourier transforms have equal characteristic functions.

Proposition 13. *(MacPherson)*

$$\mathcal{F}_{\mathfrak{G}}(M_{\epsilon E}^\bullet) = \mathfrak{tt}^{N - \dim \mathfrak{G}}(M_E^\bullet[-\dim \mathfrak{G} - r])|_{\mathfrak{G}_{nil}},$$

So, up to Tate twist and shift,

$$\mathcal{F}_{\mathfrak{G}}(M_\epsilon^\bullet) = M_1^\bullet|_{\mathfrak{G}_{nil}}.$$

Theorem 5 now follows from: $\Lambda_{\mathcal{F}_{\mathfrak{G}} M_\epsilon^\bullet}(\xi) = \Lambda_{\mathcal{F}_{\mathfrak{G}} K^\bullet}(\xi) = \pm \Lambda_c(\sigma, \mathcal{T}_\xi, \mathcal{S}_\epsilon)$.

7 Open problems.

- The formula for $\Lambda_c(\sigma, \mathcal{T}^\xi, \mathcal{S}_\epsilon)$ bears a striking resemblance to the character formula for the Steinberg representation of a reductive group over $\mathbb{F}_q$. Is there a representation of $G(\mathbb{R})$ with a "trace" whose value at $x \in G(\mathbb{R})$ is $\pm\Lambda_c(\sigma, \mathcal{T}^x, \mathcal{S}_\epsilon)$?
- Compute $\Lambda_c(\sigma, \mathcal{T}^\xi, \mathcal{S}_\rho)$ for other representations ρ of W, There are analogies with the case of $\mathbb{F}_q$ which suggest that the values of Green functions at $q = -1$ may be involved.
- Is there a "reasonable" Fourier transform on the space of constructible functions on a vector bundle E satisfying the property that for $K^\bullet \in \mathcal{D}^b_{cc}(E)$, the Fourier transform of $\chi_{K^\bullet}$ is $\chi_{\mathcal{F}_E(K^\bullet)}$?

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