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Groups generated by 3-state automata over a 2-letter alphabet, I

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1. Introduction

Groups generated by finite automata (groups of automata or automaton groups) were formally introduced at the beginning of 1960's [Hoř63], but more substantial work on this remarkable class of groups started only in 1970's after Aleshin [Ale72] confirmed a conjecture by Glushkov [Glu61] that these groups could be used to study problems of Burnside type (note that groups of automata should not be confused with automatic groups as described in [ECH+92]). It was observed in 1960's and 1970's that groups of automata are closely related to iterated wreath products (pioneering work in this direction is due to Kaloujnin [Kal45]) and that the theory of such groups could be studied by using the language of tables developed by Kaloujnin [Kal48] and Sushchanskii [Suš79].

Even more intensive study of groups of finite automata started in the beginning of 1980's after the development of some new ideas such as self-similarity, contracting properties, and a geometric realization as groups

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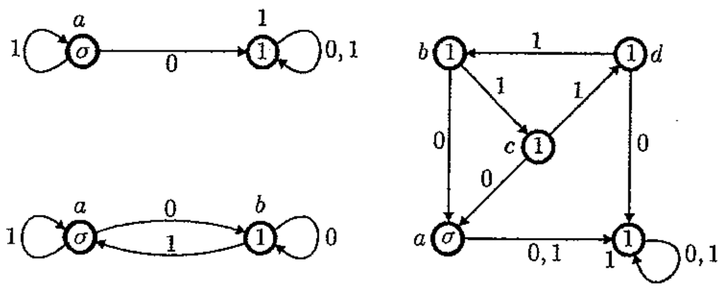


FIGURE 1. Adding machine, lamplighter automaton and automaton generating $\mathcal{G}$

acting on rooted trees. These developments allowed for elegant constructions of Burnside groups [Gri80, GS83b, GS83a] and pushed the study of groups of automata in many directions: analysis [Gri84, Ers04], geometry [BGN03], probability [BV05, Ers04, AV05], dynamics [BG00a, GŻ01], formal languages [HR04], etc.

Two well known and important problems were solved using groups of automata in the early 1980's, namely Milnor Problem [Mil68] on intermediate growth and Day Problem [Day57] on amenability. A 5-state automaton constructed in [Gri80] (on the right in Figure 1) generates a 2-group, denoted $\mathcal{G}$. It was shown in [Gri84] that $\mathcal{G}$ has intermediate growth (between polynomial and exponential). This led to construction of other examples of this type [FG91, BŠ01] and also made important contribution to and impact on the theory of invariant means on groups [Gre69, Wag85, Pat88] initiated by von Neumann [vN29] by providing an example of amenable, but not elementary amenable group (in the sense of Day [Day57]).

Among the most interesting newer developments is the spectral theory of groups generated by finite automata and graphs associated to such groups [BG00a, GŻ01]. For instance, automaton groups provided first examples of regular graphs realized as Schreier graphs of groups for which the spectrum of the combinatorial Laplacian is a Cantor set [BG00a]. Further, the realization of the lamplighter group $\mathbb{Z} \wr C_2$ as automaton group (bottom left in Figure 1) was crucial in the proof that this group has a pure point spectrum (with respect to a system of generators related to the states of the automaton) and thus has discrete spectral measure, which was completely described [GŻ01]. This, in turn, led to a construction [GLSŻ00] of a 7-dimensional closed manifold with non-integer third L^2-Betti number providing a counterexample to the Strong Atiyah Conjecture [Lüc02].

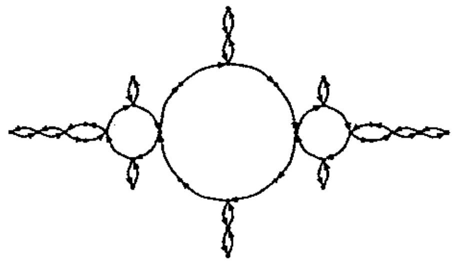


FIGURE 2. Schreier graph of the action of Basilica group $\mathcal{B}$ on the 5-th level of the tree

Another fundamental recent discovery is the relation of groups of automata to holomorphic dynamics [BGN03, Nek05]. Namely, it is shown that to every rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann Sphere with finite postcritical set one can associate a finite automaton generating a group, denoted $IMG(f)$ and called iterated monodromy group of f. The geometry and the topology of the Schreier graphs of $IMG(f)$ is closely related to the geometry of the Julia set of f. Figure 2 depicts a Schreier graph associated to an automaton group, denoted $\mathcal{B}$ and called Basilica group. Its reminiscence to the Julia set of the map $z \mapsto z^2 - 1$ is related to the fact that $\mathcal{B}$ is the iterated monodromy group $IMG(z^2 - 1)$ of the holomorphic map $z \mapsto z^2 - 1$ [BGN03, Nek05]). Groups of automata represent the basis of the theory of self-similar groups and actions [Nek05] and are related to the study of Belyi polynomials and dessins d'enfants of Grothendieck [Pil00]. The use of iterated monodromy groups was crucial in the recent solution of Hubbard's Twisted Rabbit Problem in [BN06].

One of the most important developments in the theory of automaton groups is the introduction of groups with branch structure, providing a link to just-infinite groups [Gri00, Wil00, BGŠ03] and groups of finite width [BG00b]. In particular, a problem suggested by Zelmanov was solved by using the profinite completion of $\mathcal{G}$ [BG00b]. The problem of Gromov on uniformly exponential growth was solved recently by using branch automaton groups [Wil04]. An unexpected link of groups of automata and their profinite completions to Galois theory was found by R. Pink (private communication) and Aitken, Hajir and Maire [AHM04], while N. Boston [Bos06, Bos05] related branch groups of automata to Fontaine-Mazur Conjecture and other problems in number theory. The class of branch groups is also a new source for infinitely presented groups, for which the presentation can be written in a recursive form (see [Lys85, Sid87, GSŠ]).

A recent observation [GŠ06] is that automaton groups and their Schreier graphs stand behind the famous Hanoi Towers Problem (see [Hin89]) and some of its generalizations [Sto94].

There are indications that spectral properties of groups generated by finite automata could be used in the study of Kaplansky Conjecture on Idempotents (and thus also Baum-Connes Conjecture [Con94] and Novikov Conjecture [FRR95]), Dixmier Unitarizability Problem [Dix50, Pis05], and for construction of new families of expanders, and perhaps even Ramanujan, graphs [Lub94].

In this article we are going to describe some progress which was achieved during the last few years in the problem of classification of automaton groups.

Two important characteristics of an automaton are the cardinalities m and n of the set of states and the alphabet, respectively, and the pair (m, n) is a natural measure of complexity of an automaton and of the group it generates.

The groups of complexity $(2,2)$ are classified [GNS00] and there are only 6 such groups (see Theorem 8 in Section 7 here). The problem of classification of $(3,2)$ groups or $(2,3)$ groups is much harder.

The current text represents the progress being made by the research group at Texas A&M University over the last few years toward classification of $(3,2)$ groups.

The total number of invertible automata of complexity $(3,2)$ is $2^3 \cdot 3^6 = 5832$. However, the number of non-isomorphic groups generated by these automata is much smaller.

Theorem 1. *There are no more than* 124 *pairwise non-isomorphic groups of complexity* $(3,2)$.

The proof of this theorem is too long to be presented here (even the list of all groups takes a lot of space).

Instead, we have chosen for this article a set of 24 automata generating 20 groups (among the most interesting in this class, in our opinion), which we list in the form of a table. The table provided here is a part of the table listing the whole set of 124 groups. We keep the numeration system from the whole table (the rule for numeration is explained in Section 5).

Major results obtained for the whole family are the following theorems. The numbers in the brackets indicate the numbers of corresponding automata in the class.

Theorem 2. *There are* 6 *finite groups in the class:* $\{1\}$ *[1],* C_2 *[1090],* $C_2 \times C_2$ *[730],* D_4 *[847],* $C_2 \times C_2 \times C_2$ *[802] and* $D_4 \times C_2$ *[748].*

Theorem 3. *There are 6 abelian groups in the class:* $\{1\}$ *[1],* C_2 *[1090],* $C_2 \times C_2$ *[730],* $C_2 \times C_2 \times C_2$ *[802],* $\mathbb{Z}$ *[731] and* $\mathbb{Z}^2$ *[771].*

Note that there are also virtually abelian groups in this class (having $\mathbb{Z}$, $\mathbb{Z}^2$ [2212], $\mathbb{Z}^3$ [752] or $\mathbb{Z}^5$ [968] as subgroups of finite index).

Theorem 4. *The only free non-abelian group in the class is the free group of rank 3 generated by the Aleshin-Vorobets automaton [2240]. Moreover, the isomorphism class of this automaton group coincides with its equivalence class under symmetry.*

The definition of symmetric automata is given in Section 5.

Theorem 5. *There are no infinite torsion groups in the class.*

We do not provide the complete proofs of these theorems (by the reason explained above). Instead, we give here some information about each of the chosen groups and include proofs of most facts.

Properties that are in our focus are the contracting property, self-replication, torsion, relations (we list the relators up to length 10), rank of quotients of the stabilizers series, shape of the related Schreier graphs. The article is organized as follows. We start with a general information about rooted trees and their automorphisms. Then we provide quick introduction to the theory of automaton groups. We continue with the definition of Schreier graphs and explain how they naturally appear for the actions on rooted trees. Then we list 24 automata generating 20 groups together with some of their properties. In the last section we give proofs of many facts related to the groups in the list.

The last part also contains some more general results (such as an algorithm detecting transitivity of an element and a criterion for group transitivity on the binary tree).

We recommend the articles [GNS00] and the book [Nek05] to the reader who is interested in becoming more familiar with automaton groups.

2. Regular rooted tree automorphisms and self-similarity

Let $d \geq 2$ be fixed and let X be the alphabet $X = \{0, 1, \ldots, d-1\}$. The set of words X^* over X (the free monoid over X) can be given the structure of a *regular rooted labeled d-ary tree* $\mathcal{T}$ in which the empty word $\emptyset$ is the *root*, the *level* n in $\mathcal{T}$ consists of the words of length n over X and every vertex v has d children, labeled by vx, for $x \in X$. Denote by $\mathrm{Aut}(\mathcal{T})$ the group of automorphisms of $\mathcal{T}$. Let f be an automorphism in $\mathrm{Aut}(\mathcal{T})$. Any such automorphism can be decomposed as

$$f = \alpha_f(f_0, \ldots, f_{d-1}) \tag{1}$$

where f_x, for $x \in X$, are automorphisms of $\mathcal{T}$ and α_f is a permutation of X. The automorphisms f_x (also denoted by $f|_x$), $x \in X$, are called (the first level) *sections* of f and each one acts as an automorphism on the subtree $\mathcal{T}_x$ hanging below the vertex x in $\mathcal{T}$ consisting of the words in X^* that start with x (any such subtree is canonically isomorphic to the whole tree). The action of f is decomposed in two steps. First the d-tuple $(f_0, \dots, f_{d-1})$ acts on the d subtrees hanging below the root, and then the permutation α_f, called the *root permutation* of f, permutes these d subtrees. Thus the action of f from (1) on X^* is given by

$$f(xw) = \alpha_f(x) f_x(w), \tag{2}$$

for x a letter in X and w a word over X. Further iterations of the decomposition (1) yield the second level sections $f_{xy} = (f_x)_y$, $x, y \in X$, and so on. Algebraically, we have

$$\begin{aligned} \mathrm{Aut}(\mathcal{T}) = \mathsf{Sym}(X) \ltimes (\mathrm{Aut}(\mathcal{T}), \dots, \mathrm{Aut}(\mathcal{T})) = \\ \mathsf{Sym}(X) \ltimes \mathrm{Aut}(\mathcal{T})^X = \mathsf{Sym}(X) \wr \mathrm{Aut}(\mathcal{T}), \end{aligned} \tag{3}$$

where $\wr$ is the *permutational wreath product* in which the coordinates of $\mathrm{Aut}(\mathcal{T})^X$ are permuted by $\mathsf{Sym}(X)$.

Iterations of the decomposition (3) show that $\mathrm{Aut}(\mathcal{T})$ has the structure of an iterated wreath product $\mathrm{Aut}(\mathcal{T}) = \mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr \dots))$. Thus $\mathrm{Aut}(\mathcal{T})$ is a pro-finite group and in particular, all of its subgroups are residually finite. An obvious and natural sequence of normal subgroups of finite index intersecting trivially is the sequence of level stabilizers. The n-th *level stabilizer* $\mathrm{St}_G(n)$ of a group $G \leq \mathrm{Aut}(\mathcal{T})$ consists of those tree automorphisms in G that fix the vertices in $\mathcal{T}$ up to level n. The group $\mathrm{Aut}(\mathcal{T})$ is obviously an uncountable object. We are interested in finitely generated subgroups of $\mathrm{Aut}(\mathcal{T})$ that exhibit some important features of $\mathrm{Aut}(\mathcal{T})$. One such feature is self-similarity.

Definition 1. A group G of tree automorphisms is *self-similar* if, for every g in G and a letter x in X there exists a letter y in X and an element h in G such that

$$g(xw) = yh(w),$$

for all words w over X.

Another way to express self-similarity of a group G of tree automorphisms is to say that every section g_x of every element g in G is again an element of G. The full tree automorphism group $\mathrm{Aut}(\mathcal{T})$ is clearly self-similar (see (2)). A self-similar group G embeds in the permutational wreath product $\mathsf{Sym}(X) \wr G = \mathsf{Sym}(X) \ltimes G^X$ by

$$g \mapsto \alpha_g(g_0, g_1, \dots, g_{d-1}). \tag{4}$$

3. Definition of automaton groups

Consider a finite system of recursive relations

$$\begin{cases} f^{(1)} & = & \alpha_1\left(f_0^{(1)}, f_1^{(1)}, \dots, f_{d-1}^{(1)}\right), \\ \dots & & \\ f^{(k)} & = & \alpha_k\left(f_0^{(k)}, f_1^{(k)}, \dots, f_{d-1}^{(k)}\right), \end{cases} \tag{5}$$

where each symbol $f_j^{(i)}$, $i = 1, \dots, k$, $j = 0, \dots, d-1$, is equal to one of the symbols $f^{(1)}, \dots, f^{(k)}$ and $\alpha_1, \dots, \alpha_k \in \mathsf{Sym}(X)$. The system (5) has a unique solution in $\mathrm{Aut}(\mathcal{T})$. The action of $f^{(i)}$ on $\mathcal{T}$ is given recursively by $f^{(i)}(xw) = \alpha_i(x)f_x^{(i)}(w)$. The group generated by the automorphisms $f^{(1)}, \dots, f^{(k)}$ is finitely generated self-similar group of automorphisms of $\mathcal{T}$. This group can be described by a finite invertible automaton (just called automaton in the rest of the article).

Definition 2. A *finite invertible automaton* A is a 4-tuple $A = (Q, X, \rho, \tau)$ where Q is a finite set of *states*, X is a finite *alphabet* of cardinality $d \geq 2$, $\rho : Q \times X \to X$ is a map, called *output map*, $\tau : Q \times X \to Q$ is a map, called *transition map*, and for each state q in Q, the restriction $\rho_q : X \to X$ given by $\rho_q(x) = \rho(q, x)$ is a permutation, i.e. $\rho_q \in \mathsf{Sym}(X)$.

The automaton $A = (Q, X, \rho, \tau)$ reads words from X^* and provides output words that are also in X^*. The behavior is encoded in the output and transition maps. An *initial automaton* A_q is just an automaton A with a distinguished state $q \in Q$ selected as an initial state. We first informally describe the action of the initial automaton A_q on X^*. The automaton starts at the state q, reads the first input letter x_1, outputs the letter $\rho_q(x_1)$ and changes its state to $q_1 = \tau(q, x_1)$. The rest of the input word is handled by the new state q_1 in the same fashion (in fact it is handled by the initial automaton A_{q_1}). Formally, the action of the states of the automaton A on X^* can be described by extending the output function ρ to a function $\rho : Q \times X^* \to X^*$ recursively by

$$\rho(q, xw) = \rho(q, x)\rho(\tau(q, x), w) \tag{6}$$

for all states q in Q, letters $x \in X$ and words w over X. Then the action of the initial automaton A_q is defined by $A_q(u) = \rho(q, u)$, for words u over X. In fact, (6) shows that each initial automaton A_q, $q \in Q$, defines a tree automorphism, denoted by q, defined by

$$q(xw) = \alpha_q(x)q_x(w), \tag{7}$$

where the section q_x is the state $\tau(q, x)$ and the root permutation α_q is the permutation ρ_q.

Definition 3. Given an automaton $A = (Q, X, \rho, \tau)$, the group of tree automorphisms generated by the states of A is denoted by $G(A)$ and called the *automaton group* defined by A. The generating set Q is called the *standard* generating set of $G(A)$.

Boundary of the tree $\mathcal{T}$, denoted $\partial\mathcal{T}$, is the set X^ω of words over X that are infinite to the right (infinite geodesic rays in $\mathcal{T}$ starting at the root). It has a natural metric (infinite words are close if they agree on long finite prefixes) and the group of isometries $\mathsf{Isom}(\partial\mathcal{T})$ is canonically isomorphic to $\mathrm{Aut}(\mathcal{T})$. Thus the action of the automaton group $G(A)$ on $\mathcal{T}$ can be extended to an isometric action on $\partial\mathcal{T}$. In fact, (6) and (7) are valid for infinite words w as well.

An automaton A can be represented by a labeled directed graph, called Moore diagram, in which the vertices are the states of the automaton, each state q is labeled by its own root permutation α_q and, for each pair $(q,x) \in Q \times X$, there is an edge from q to $q_x = \tau(q,x)$ labeled by x. For example, the 5-state automaton in the right half of Figure 1 generates the group $\mathcal{G}$ mentioned in the introduction (σ denotes the permutation exchanging 0 and 1). The two 2-state automata given on the left of Figure 1 are the so called *adding machine* (top), which generates the infinite cyclic group $\mathbb{Z}$ and the *lamplighter automaton* (bottom) generating $L_2 = \mathbb{Z} \wr C_2$. Recursion relations of type (5) for the adding machine and the lamplighter automaton are given by

$$\begin{aligned} a &= \sigma(1,a) & \qquad\qquad a &= \sigma(b,a) \\ 1 &= \ (1,1) & b &= \ (b,a), \end{aligned} \tag{8}$$

respectively.

Various classes of automaton groups deserve special attention. An automaton group $G = G(A)$ is *contracting* if there exist constants κ, C, and N, with $0 \le \kappa < 1$, such that $|g_v| \le \kappa|g| + C$, for all vertices v of length at least N and $g \in G$ (the length is measured with respect to the standard generating set Q). For sufficiently long elements g this means that the length of its sections at vertices on levels deeper than N is strictly shorter than the length of g. This length shortening leads to an equivalent definition of a contracting group. Namely, a group G of tree automorphisms is contracting if there exists a finite set $\mathcal{N} \subset G$, such that for every $g \in G$, there exists $N > 0$, such that $g_v \in \mathcal{N}$ for all vertices $v \in X^*$ of length not shorter than N. The minimal set $\mathcal{N}$ with this property is called the *nucleus* of G. The contraction property is a key feature of various inductive arguments and algorithms involving the decomposition $g = \alpha_g(g_0, \dots, g_{d-1})$.

Another important class is the class of automaton groups of *branch type*. Branch groups arise as one of the three [Gri00] possible types of just-infinite groups (infinite groups for which all proper homomorphic images are finite). Every infinite, finitely generated group has a just-infinite image. Thus if a class of groups $\mathcal{C}$ is closed under homomorphic images and if it contains infinite, finitely generated examples then it contains just-infinite examples. Such examples are, in a sense, minimal infinite examples in $\mathcal{C}$. For example, $\mathcal{G}$ is a branch automaton group that is a just-infinite 2-group. i.e., it is an infinite, finitely generated, torsion group that has no proper infinite quotients. Also, the Hanoi Towers group [GŠ06] and the iterated monodromy group $IMG(z^2+i)$ [GSŠ] are branch groups, while $\mathcal{B} = IMG(z^2-1)$ is not a branch group, but only weakly branch (for definitions see [Gri00, BGŠ03]).

The class of *polynomially growing automata* was introduced by Sidki in [Sid04], where it is proved that no group $G(A)$ defined by such an automaton contains free subgroups of rank 2. Moreover, for a subclass of so called *bounded automata* it is known that the corresponding groups are amenable [BKNV05] (this class of automata, for instance, includes the automata generating $\mathcal{G}$, $\mathcal{B}$ and Hanoi Towers group on 3 pegs, but not for more pegs).

Finally, self-replicating groups play an important role. A self-similar group G is called *self-replicating* if, for every vertex u, the homomorphism $\varphi_u : \mathsf{St}_G(u) \to G$ from the stabilizer u in G to G, given by $\varphi(g) = g_u$, is surjective. This condition is usually easy to check and, together with transitivity of the action on level 1, it implies transitivity of the action on all levels. Another way to show that a group of automorphisms of the binary tree is level transitive is to use Proposition 2.

4. Limit spaces, Schreier graphs and iterated monodromy groups

Let us fix some self-similar contracting group acting on X^* by automorphisms. Denote by $X^{-\omega}$ the space of left infinite sequences over X.

Definition 4. Two elements $\ldots x_3x_2x_1, \ldots y_3y_2y_1 \in X^{-\omega}$ are said to be *asymptotically equivalent* with respect to the action of the group G, if there exist a finite set $K \subset G$ and a sequence $\{g_k\}_{k=1}^{\infty}$ of elements in K such that

$$g_k(x_kx_{k-1}\ldots x_2x_1) = y_ky_{k-1}\cdots y_2y_1$$

for every $k \geq 1$.

The asymptotic equivalence is an equivalence relation. Moreover, sequences $\ldots x_2x_1, \ldots y_2y_1 \in X^{-\omega}$ are asymptotically equivalent if and only if there exists a sequence $\{h_k\}$ of the elements in the nucleus of G such that $h_k(x_k) = y_k$ and $h_k|_{x_k} = h_{k-1}$, for all $k \geq 1$.

Definition 5. The quotient space $\mathcal{J}_G$ of the topological space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the self-similar action of G.

The limit space $\mathcal{J}_G$ is metrizable and finite-dimensional. If the group G is finitely-generated and level-transitive, then the limit space $\mathcal{J}_G$ is connected.

The last decade witnessed a shift in the attention payed to the study of Schreier graphs. Let G be a group generated by a finite set S and let G act on a set Y. The *Schreier graph* of the action (G, Y) is the graph $\Gamma(G, S, Y)$ with set of vertices Y and set of edges $S \times Y$, where the arrow (s, y) starts in y and ends in $s(y)$. If $y \in Y$ then the Schreier graph $\Gamma(G, S, y)$ of the action of G on the G-orbit of y is called *orbital Schreier graph.*

Let G be a subgroup of $\mathrm{Aut}(\mathcal{T})$ generated by a finite set S (not necessary self-similar). The levels X^n, $n \geq 0$, are invariant under the action of G and we can consider the Schreier graphs $\Gamma_n(G, S) = \Gamma(G, S, X^n)$. Let $\omega = x_1x_2x_3\ldots \in X^\omega$. Then the pointed Schreier graphs $(\Gamma_n(G, S), x_1x_2\ldots x_n)$ converge in the local topology (topology defined in [Gri84]) to the pointed orbital Schreier graph $(\Gamma(G, S, \omega), \omega)$.

The limit space of a finitely generated contracting self-similar group G can be viewed as a hyperbolic boundary in the following way. For any given finite generating system S of G define the self-similarity graph $\Sigma(G, S)$ as the graph with set of vertices X^* in which two vertices $v_1, v_2 \in X^*$ are connected by an edge if and only if either $v_i = xv_j$, for some $x \in X$ (vertical edges), or $s(v_i) = v_j$ for some $s \in S$ (horizontal edges). If the group is contracting then the self-similarity graph $\Sigma(G, S)$ is Gromov-hyperbolic and its hyperbolic boundary is homeomorphic to the limit space $\mathcal{J}_G$. The set of horizontal edges of $\Sigma(G, S)$ spans the disjoint union of all Schreier graphs $\Gamma_n(G, S)$.Thus, the Schreier graphs $\Gamma_n(G, S)$ in some sense approximate the limit space $\mathcal{J}_G$ of the group G. Moreover, for many examples of self-similar contracting groups there exists a sequence of numbers λ_n such that the metric spaces $(\Gamma_n, d(\cdot, \cdot)/\lambda_n)$, where d is the combinatorial metric on the graph, converge in the Gromov-Hausdorff metric to the limit space of the group.

We recall the definition and basic properties of iterated monodromy groups (IMG). Let $\mathcal{M}$ be a path connected and locally path connected topological space and let $\mathcal{M}_1$ be its open path connected subset. Let $f : \mathcal{M}_1 \to \mathcal{M}$ be a d-fold covering. By f^n we denote the n-th iteration of the map f. The map $f^n : \mathcal{M}_n \to \mathcal{M}$, where $\mathcal{M}_n = f^{-n}(\mathcal{M})$, is a d^n-fold covering.

Choose an arbitrary base point $t \in \mathcal{M}$. Let $\mathcal{T}_t$ be the disjoint union of the sets $f^{-n}(t), n \geq 0$ (these sets are note necessarily disjoint by themselves). The set of pre-images $\mathcal{T}_t$ has a natural structure of a rooted d-ary tree with root $t \in f^{-0}(t)$ in which every vertex $z \in f^{-n}(t)$ is connected to the vertex $f(z) \in f^{-n+1}(t)$, $n \geq 1$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally on every set $f^{-n}(t)$ and, in fact, acts by automorphisms on $\mathcal{T}_t$.

Definition 6. *Iterated monodromy group $IMG(f)$* of the covering f is the quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of its action on $\mathcal{T}_t$.

It is proved in [Nek05] that all iterated monodromy groups are self-similar. This fact provides a connection between holomorphic dynamics and groups generated by automata.

Theorem 6. *The iterated monodromy group of a sub-hyperbolic rational function is contracting and its limit space is homeomorphic to the Julia set of the rational function.*

In particular, the sequence of Schreier graphs Γ_n of the iterated monodromy group of a sub-hyperbolic rational function can be drawn on the Riemann sphere in such a way that they converge in the Hausdorff metric to the Julia set of the function.

Schreier graphs also play a role in computing the spectrum of the Markov operator M on the group. Namely, given a group G generated by a finite set $S = \{s_1, s_2, \ldots, s_k\}$, acting on a tree X^* there is a natural unitary representation of G in the space of bounded linear operators $\mathcal{H} = B(L_2(X^\omega))$ given by $\pi_g(f)(x) = f(g^{-1}x)$.

The Markov operator $M = \frac{1}{2k}(\pi_{s_1} + \cdots + \pi_{s_k} + \pi_{s_1^{-1}} + \cdots + \pi_{s_k^{-1}})$ corresponding to this unitary representation plays an important role. The spectrum of M for a self-similar group G is approximated by the spectra of finite dimensional operators arising from the action of G on the levels of the tree X^*. For more on this see [BG00a].

Let $\mathcal{H}_n$ be a subspace of $\mathcal{H}$ spanned by the $|X|^n$ characteristic functions $f_v, v \in X^n$, of the cylindrical sets corresponding to the $|X|^n$ vertices on level n. Then $\mathcal{H}_n$ is invariant under the action of G and $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Denote by $\pi_g^{(n)}$ the restriction of π_g on $\mathcal{H}_n$. Then

$$M_n = \frac{1}{2k}(\pi_{s_1}^{(n)} + \pi_{s_2}^{(n)} + \cdots + \pi_{s_k}^{(n)} + \pi_{s_1^{-1}}^{(n)} + \pi_{s_2^{-1}}^{(n)} + \cdots + \pi_{s_k^{-1}}^{(n)})$$

are finite dimensional operators, whose spectra converge to the spectrum of M in the sense

$$sp(M) = \overline{\bigcup_{n \geq 0} sp(M_n)}.$$

If P is the stabilizer of an infinite word from X^ω, then one can consider the Markov operator $M_{G/P}$ on the Schreier graph of G with respect to P.

The following fact is observed in [BG00a] and can be applied to compute the spectrum of Markov operator on the Cayley graph of a group in case if P is small.

Theorem 7. *If G is amenable or the Schreier graph G/P (the Schreier graph of the action of G on the cosets of P) is amenable then $sp(M_{G/P}) = sp(M)$.*

5. Approach to a classification of groups generated by 3-state automata over a 2-letter alphabet

The next three sections are devoted to the groups generated by 3-state automata over the 2-letter alphabet $X = \{0, 1\}$. Fix $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ as the set of states. Every $(3, 2)$ automaton is given by

$$\begin{cases} \mathbf{1} = \sigma^{a_{11}}(a_{12}, a_{13}), \\ \mathbf{2} = \sigma^{a_{21}}(a_{22}, a_{23}), \\ \mathbf{3} = \sigma^{a_{31}}(a_{32}, a_{33}), \end{cases}$$

where $a_{ij} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, for $j \neq 1$, $a_{i1} \in \{0, 1\}$, $i = 1, 2, 3$, and $\sigma = (01) \in \mathsf{Sym}(X)$. A number is assigned to the automaton above by the following formula

$$\begin{aligned} \mathrm{Number}(\mathcal{A}) = {} & \\ & (a_{12} - 1) + 3(a_{13} - 1) + 9(a_{22} - 1) + 27(a_{23} - 1) + \\ & 81(a_{32} - 1) + 243(a_{33} - 1) + 729(a_{11} + 2a_{21} + 4a_{31}) + 1. \end{aligned}$$

Thus every $(3, 2)$ automaton obtains a unique number in the range from 1 to 5832. The numbering of the automata is induced by the lexicographic ordering of all automata in the class. The automata numbered 1 through 729 act trivially on the tree and generate the trivial group. The automata numbered 5104 through 5832 generate the group C_2 of order 2, because every element in any of these groups is either trivial, or changes all letters in any word over X. Therefore the "interesting" automata have numbers 730 through 5103.

Denote by $\mathcal{A}_n$ the automaton numbered by n and by G_n the corresponding group of tree automorphisms. Sometimes, when the context is clear, we use just the number to refer to the corresponding automaton or group.

The following operations on automata change neither the group generated by this automaton, nor, essentially, the action of the group on the tree.

(*i*) passing to inverses of all generators
(*ii*) permuting the states of the automaton
(*iii*) permuting the letters of the alphabet

Definition 7. Two automata $\mathcal{A}$ and $\mathcal{B}$ that can be obtained from one another using a composition of the operations (*i*)–(*iii*), are called *symmetric.*

Definition 8. If the minimization of an automaton $\mathcal{A}$ is symmetric to the minimization of an automaton $\mathcal{B}$, we say that the automata $\mathcal{A}$ and $\mathcal{B}$ are *minimally symmetric* and write $\mathcal{A} \sim \mathcal{B}$.

Another equivalence relation we consider is the isomorphism of the groups generated by the automata. The minimal symmetry relation is a refinement of the isomorphism relation, since the same abstract group may have different actions on the binary tree.

There are 194 classes of automata, which are pairwise not minimally symmetric, 10 of which are minimally symmetric to automata with fewer than 3 states. These 10 classes of automata are subject of Theorem 8, which states that they generate 6 different groups.

At present, it is known that there are at most 124 non-isomorphic groups in the considered class.

6. Selected groups from the class

In this section we provide information about selected groups in the class of all groups generated by $(3,2)$ automata. The groups are selected in such a way that the corresponding proofs in Section 7 show most of the main methods and ideas that were used for the whole class.

The following notation is used:

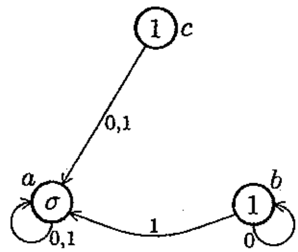
- Rels - this is a list of some relators in the group. All independent relators up to length 20 are included. On some situations additional longer relators are included. For G_{753} and G_{858} there are no relators of length up to 20 and the relators provided in the table are not necessarily the shortest. In many cases, the given relations are not sufficient (for example, some of the groups are not finitely presented).
- SF - these numbers represent the size of the factors $G/\operatorname{St}_G(n)$, for $n \geq 0$.
- Gr - these numbers represent the values of the growth function $\gamma_G(n)$, for $n \geq 0$, and generating system a, b, c.

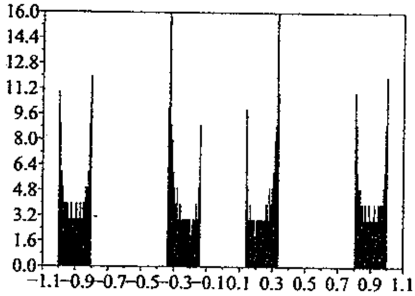
Finally, for each automaton in the list a histogram for the spectral density of the operator M_9 acting on level 9 of the tree is shown.

In some cases, in order to show the main ways to prove the group isomorphism, we provide several different automata generating the same group.

Automaton number 739

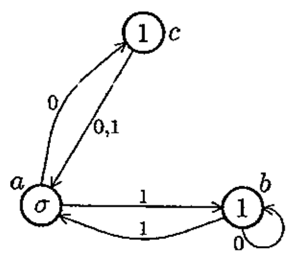
$a = \sigma(a, a)$ Group: $C_2 \ltimes (C_2 \wr \mathbb{Z})$
$b = (b, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, *acac*, *acbabcabab*, *acbabacbab*, *abacbcacbcab*, *acbacbabcabc*, *acbcbabcabcbab*, *acbcbabacbcbab*, *acbacbcacbcabc*, *abcbacbcacbcbcab*, *acbcbacbabcbcabc*, *acbcbcbabcabcbcbab*, *acbcbcbabacbcbcbab*, *abcbacbcbabcbcabcb*, *acbcbacbcacbcbcabc*
SF: 2^0,2^1,2^3,2^6,2^8,2^{10},2^{12},2^{14},2^{16}
Gr: 1,4,9,17,30,47,68,93,122,155,192

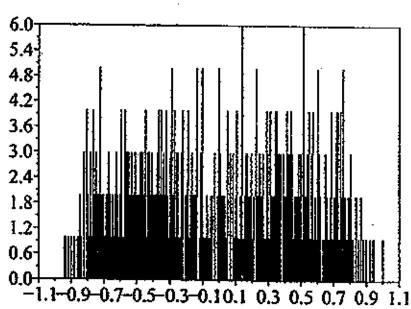




Automaton number 744

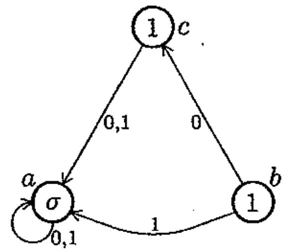
$a = \sigma(c, b)$ Group:
$b = (b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $abcb^{-1}ac^{-1}a^{-2}bcb^{-1}ac^{-1}aca^{-1}bc^{-1}b^{-1}ca^{-1}bc^{-1}b^{-1}$, $abcb^{-1}ac^{-1}a^{-2}bcb^{-1}ab^{-1}aca^{-1}bc^{-1}a^{-1}bc^{-1}b^{-1}$, $abcb^{-1}ab^{-1}a^{-2}bcb^{-1}ac^{-1}aba^{-1}bc^{-1}b^{-1}ca^{-1}bc^{-1}b^{-1}$, $abcb^{-1}ab^{-1}a^{-2}bcb^{-1}ab^{-1}aba^{-1}bc^{-1}a^{-1}bc^{-1}b^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4687

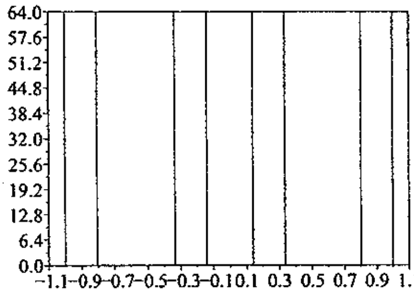




Automaton number 748

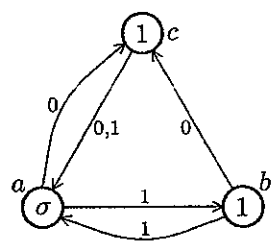
$a = \sigma(a, a)$ Group: $D_4 \times C_2$
$b = (c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $acac$, $bcbc$, $abababab$
SF: $2^0, 2^1, 2^3, 2^4, 2^4, 2^4, 2^4, 2^4, 2^4$
Gr: 1,4,8,12,15,16,16,16,16,16,16

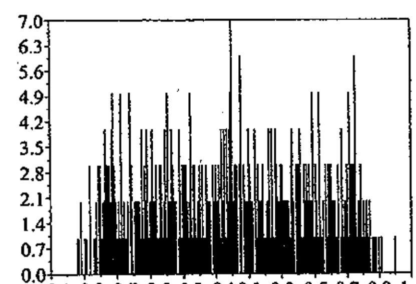




Automaton number 753

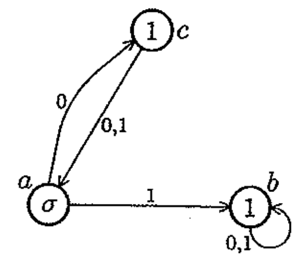
$a = \sigma(c, b)$ Group:
$b = (c, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $aba^{-1}b^{-1}ab^{-1}ca^{-1}ba^{-1}b^{-1}ab^{-1}cac^{-1}ba^{-1}bab^{-1}\cdot a^{-1}c^{-1}ba^{-1}bab^{-1}$, $aba^{-1}b^{-1}ab^{-1}ca^{-1}c^{-1}ba^{-1}c^{-1}b\cdot ab^{-1}cac^{-1}ba^{-1}bab^{-1}a^{-1}c^{-1}ba^{-1}b^{-1}cab^{-1}c$, $ac^{-1}ba^{-1}c^{-1}bab^{-1}ca^{-1}ba^{-1}b^{-1}ab^{-1}cac^{-1}ba^{-1}b^{-1}ca\cdot b^{-1}ca^{-1}c^{-1}ba^{-1}bab^{-1}$
SF: $2^0, 2^1, 2^3, 2^6, 2^{12}, 2^{23}, 2^{45}, 2^{88}, 2^{174}$
Gr: 1,7,37,187,937,4687

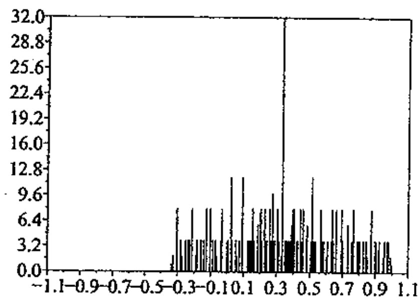




Automaton number 771

$a = \sigma(c, b)$ Group: $\mathbb{Z}^2$
$b = (b, b)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*
Rels: b, $a^{-1}c^{-1}ac$
SF: 2^0,2^1,2^2,2^3,2^4,2^5,2^6,2^7,2^8
Gr: 1,5,13,25,41,61,85,113,145,181,221
Limit space: 2-dimensional torus T_2

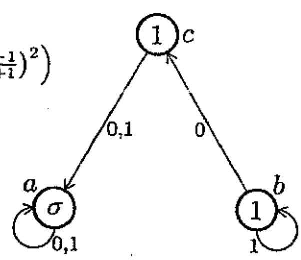


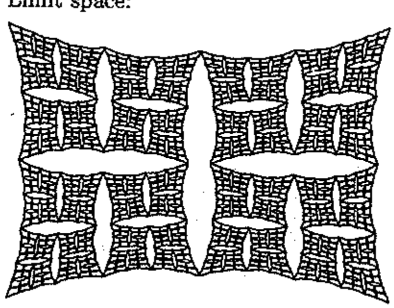


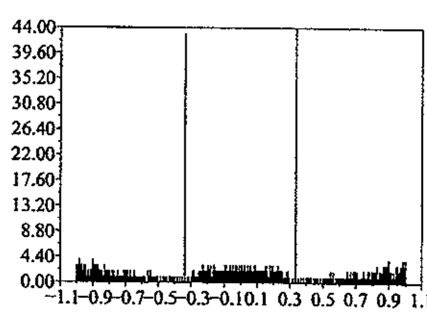
Automata number 775 and 783

$a = \sigma(a, a)$ $a = \sigma(c, c)$ Group: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = (c, b)$ $b = (c, b)$ Contracting: *yes*
$c = (a, a)$ 783: $c = (a, a)$ Self-replicating: *yes*

Rels: a^2, b^2, c^2, *acac*, *acbcbabcbcabcbabcb*, *acbcbabcbacbcbabcb*, *abcbacbcbcacbcbcabcb*, *acbcbacbcbabcbcabcbc*, *acbcbacbcbcacbcbcabcbc*
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,9,17,30,51,85,140,229,367,579
Limit space:

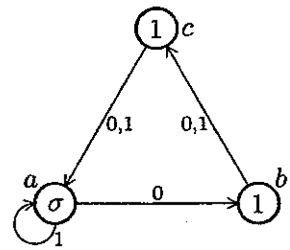


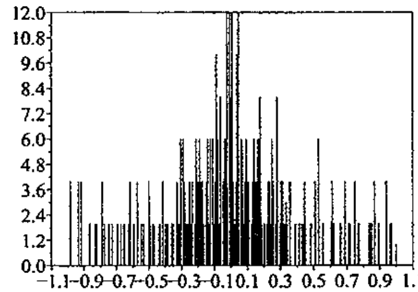




Automaton number 803

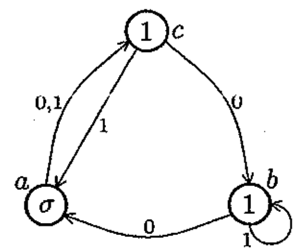
$a = \sigma(b, a)$ Group: $\mathbb{Z}^2$
$b = (c, c)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*
Rels: cba^2, $a^{-1}c^{-1}ac$
SF: $2^0,2^1,2^2,2^3,2^4,2^5,2^6,2^7,2^8$
Gr: 1,7,21,43,73,111,157,211,273,343,421
Limit space: 2-dimensional torus T_2

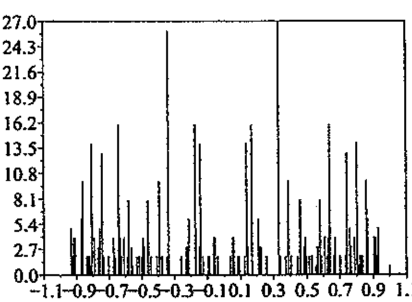




Automaton number 846

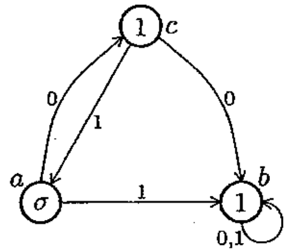
$a = \sigma(c, c)$ Group: $C_2 * C_2 * C_2$
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2
SF: $2^0,2^1,2^3,2^5,2^7,2^{10},2^{13},2^{16},2^{19}$
Gr: 1,4,10,22,46,94,190,382,766,1534



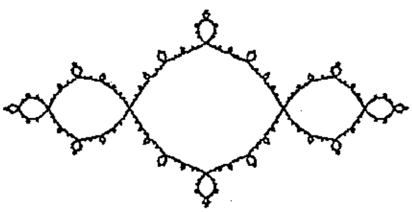


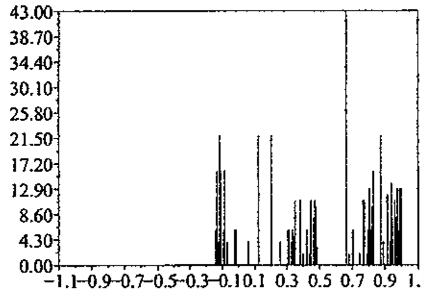
Automaton number 852

$a = \sigma(c, b)$ Group: *Basilica Group* $\mathcal{B} = IMG(z^2 - 1)$
$b = (b, b)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*



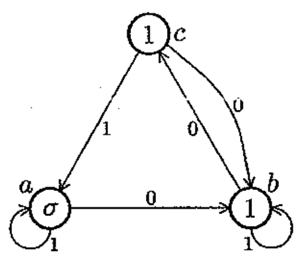
Rels: b, $a^{-1}c^{-1}ac^{-1}a^{-1}cac$, $a^{-1}c^{-2}ac^{-1}a^{-1}c^2ac$, $a^{-1}c^{-1}ac^{-2}a^{-1}cac^2$, $a^{-2}c^{-1}a^{-2}ca^2c^{-1}a^2c$, $a^{-1}c^{-3}ac^{-1}a^{-1}c^3ac$, $a^{-1}c^{-2}ac^{-2}a^{-1}c^2ac^2$, $a^{-1}c^{-1}ac^{-3}a^{-1}cac^3$
SF: $2^0, 2^1, 2^3, 2^6, 2^{12}, 2^{23}, 2^{45}, 2^{88}, 2^{174}$
Gr: 1,5,17,53,153,421,1125,2945,7545
Limit space:



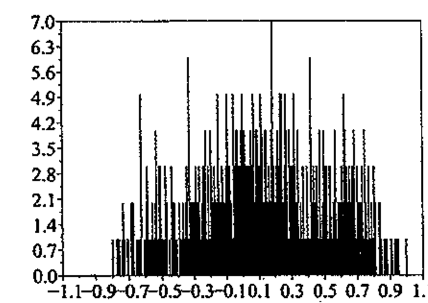


Automaton number 857

$a = \sigma(b, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*



Rels: $a^{-1}ca^{-1}c$, $a^{-1}ba^{-1}ba^{-1}ba^{-1}b$, $a^{-1}b^{-1}aca^{-1}b^{-1}ac$, $a^{-1}b^{-1}a^2c^{-1}b^{-1}ac$, $a^{-2}bca^{-2}bc$, $b^{-1}cb^{-1}cb^{-1}cb^{-1}c$, $a^{-1}ba^{-1}bc^{-1}ac^{-1}ba^{-1}b$, $a^{-1}cac^{-1}b^{-1}aca^{-1}b^{-1}c$, $a^{-1}bac^{-2}ac^{-1}bca^{-1}$
SF: $2^0, 2^1, 2^3, 2^7, 2^{13}, 2^{25}, 2^{47}, 2^{90}, 2^{176}$
Gr: 1,7,35,165,758,3460



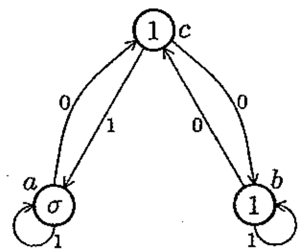
Automaton number 858

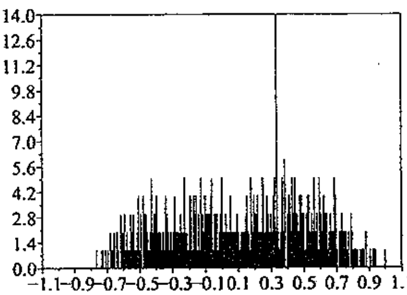
$a = \sigma(c, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

Rels: $abca^{-1}c^{-1}ab^{-1}a^2c^{-1}b^{-1}a^{-1}bca^{-1}c^{-1}ab^{-1}a^2c^{-1}\cdot$
$b^{-1}abca^{-2}ba^{-1}cac^{-1}b^{-1}a^{-1}bca^{-2}ba^{-1}cac^{-1}b^{-1}$,
$abca^{-1}c^{-1}ab^{-1}a^2c^{-1}b^{-1}a^{-1}cba^{-1}b^{-1}ab^{-1}abca^{-2}b\cdot$
$a^{-1}cac^{-1}b^{-1}a^{-1}ba^{-1}bab^{-1}c^{-1}$

SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{90},2^{176}

Gr: 1,7,37,187,937,4687





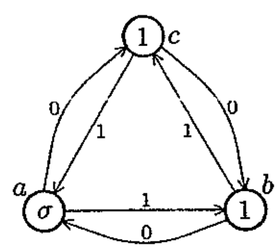
Automaton number 870

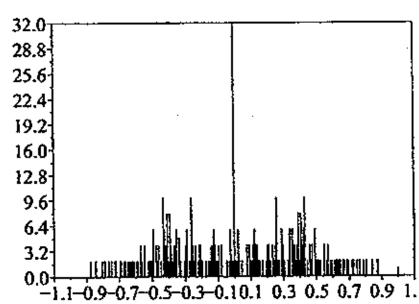
$a = \sigma(c, b)$ Group: *Baumslag-Solitar group* $BS(1, 3)$
$b = (a, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

Rels: $a^{-1}ca^{-1}b$, $(b^{-1}a)^b(b^{-1}a)^{-3}$

SF: 2^0,2^1,2^3,2^4,2^6,2^8,2^{10},2^{12},2^{14}

Gr: 1,7,33,127,433,1415





Automaton number 878

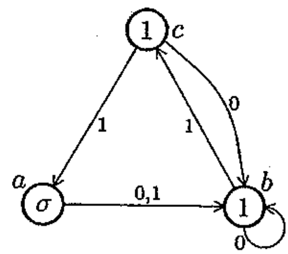
$a = \sigma(b, b)$ Group: $C_2 \ltimes IMG(1 - \frac{1}{z^2})$
$b = (b, c)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*

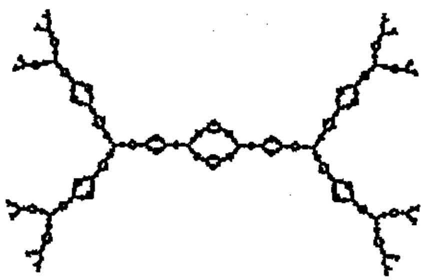
Rels: a^2, b^2, c^2, $abcabcacbacb$, $abcbcabcacbcbacb$

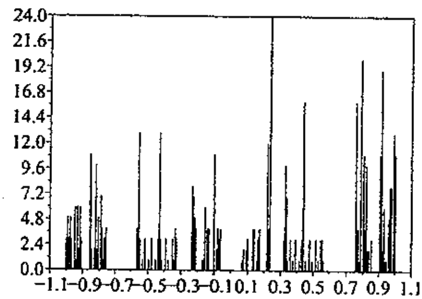
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}

Gr: 1,4,10,22,46,94,184,352,664,1244,2296,4198,7612

Limit space:







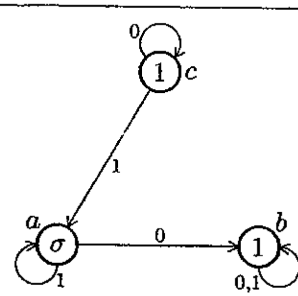
Automaton number 929

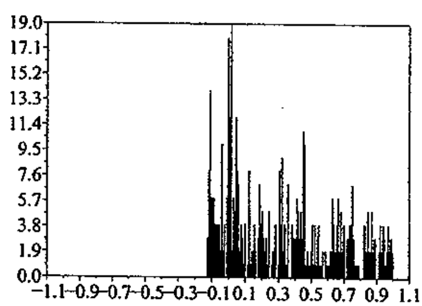
$a = \sigma(b, a)$ Group:
$b = (b, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*

Rels: b, $a^{-3}cac^{-1}ac^{-1}ac$

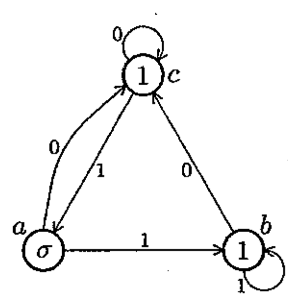
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,5,17,53,161,475,1387

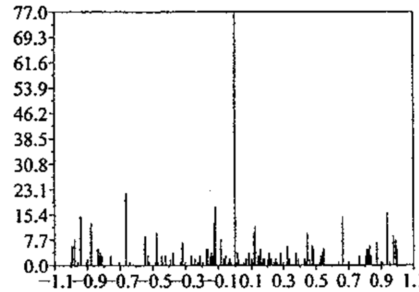




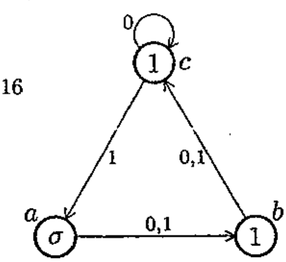
Automaton number 942



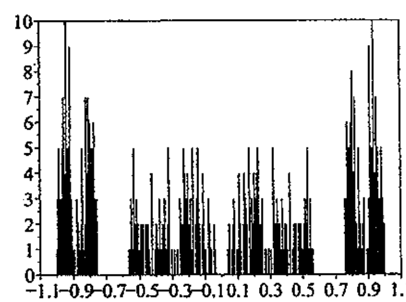
$a = \sigma(c, b)$ Group: Contains the lamplighter group
$b = (c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $a^{-1}ba^{-1}b$, $b^{-1}cb^{-1}c$, $b^{-1}ca^{-1}ba^{-1}c$,
$a^{-2}b^2a^{-1}b^{-1}ab$, $a^{-2}bab^{-2}ab$, $b^{-2}c^2b^{-1}c^{-1}bc$,
$b^{-2}cbc^{-2}bc$, $a^{-1}ca^{-1}ca^{-1}ca^{-1}c$, $b^{-1}ca^{-1}cb^{-1}ca^{-1}c$,
$a^{-1}bab^{-1}a^{-1}b^2a^{-1}$, $b^{-1}cbc^{-1}b^{-1}c^2b^{-1}$,
$a^{-2}bac^{-1}bc^{-1}b^{-1}ab$, $a^{-1}bc^{-1}ac^{-1}ac^{-1}ac^{-1}b$,
$b^{-1}ca^{-2}b^2a^{-1}b^{-1}ac$, $b^{-1}ca^{-2}bab^{-2}ac$,
$b^{-2}c^2a^{-1}ba^{-1}c^{-1}bc$, $b^{-1}cab^{-2}aba^{-2}c$,
$b^{-1}cab^{-1}a^{-1}b^2a^{-2}c$,
$a^{-1}bab^{-1}c^{-1}bc^{-1}aba^{-1}$,
$b^{-1}cbc^{-1}a^{-1}ba^{-1}c^2b^{-1}$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,597,2465



Automaton number 968

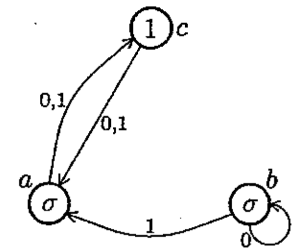


$a = \sigma(b, b)$ Group: contains $\mathbb{Z}^5$ as a subgroup of index 16
$b = (c, c)$ Contracting: *yes*
$c = (c, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $abcabcacbacb$, $acbcbabacbcbab$,
$acacbcbacacbcb$, $abcbcabcacbcbacb$, $acabacbabacabacbab$,
$acbabacacacbacacab$, $acacacacbacacacacb$,
$acbcbabcbacbcbabcb$, $acbcacbcbacbcacbcb$
SF: 2^0,2^1,2^3,2^6,2^9,2^{13},2^{17},2^{21},2^{25}
Gr: 1,4,10,22,46,94,184,338,600,1022,1682

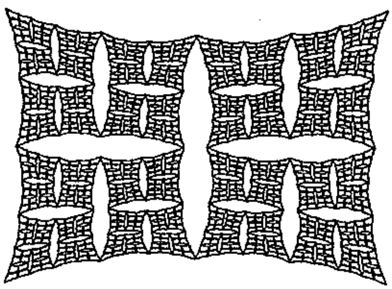


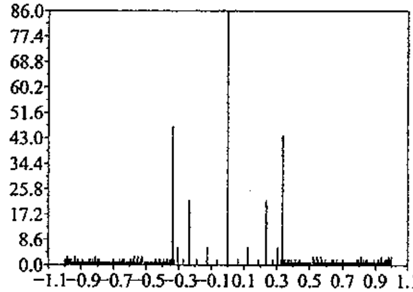
Automaton number 2205

$a = \sigma(c, c)$ Group: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = \sigma(b, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*



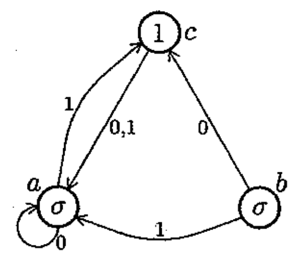
Rels: a^2, c^2, $acac$, $acbcab$, $acbacb$, $abcabc$, acb^2cab^2, acb^2acb^2, ab^2cab^2c, acb^3cab^3, acb^3acb^3, ab^3cab^3c, acb^4cab^4, acb^4acb^4, ab^4cab^4c
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48},2^{91}
Gr: 1,5,16,40,88,184,376,746,1458
Limit space:

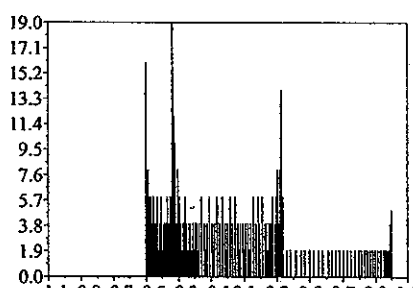




Automaton number 2212

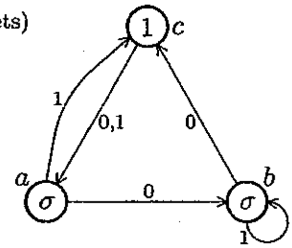
$a = \sigma(a, c)$ Group: *Klein bottle group*, virtually $\mathbb{Z}^2$
$b = \sigma(c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: ca^2, cb^2
SF: 2^0,2^1,2^2,2^4,2^6,2^8,2^{10},2^{12},2^{14}
Gr: 1,7,19,37,61,91,127,169,217,271,331

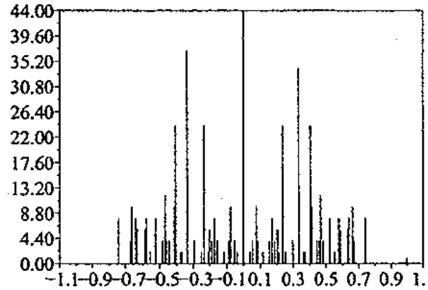




Automaton number 2240

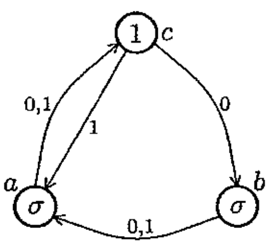
$a = \sigma(b, c)$ Group: F_3 — free of rank 3 (Aleshin-Vorobets)
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *no*
Rels:
SF: 2^0,2^1,2^2,2^4,2^7,2^{10},2^{14},2^{21},2^{34}
Gr: 1,7,37,187,937,4687

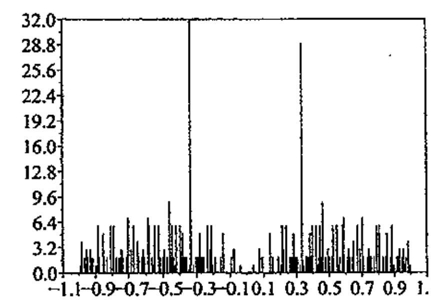




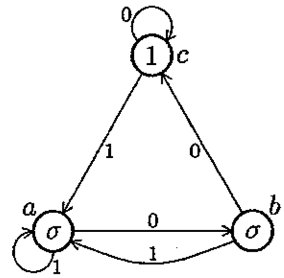
Automaton number 2277

$a = \sigma(c, c)$ Group: $C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$
$b = \sigma(a, a)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, *acbacb*, *acbcbacbcb*, *bacacbacac*, *acbcbcbacbcbcb*, *bacacacbacacac*, *acbcbcbcbacbcbcbcb*, *bacacacacbacacacac*, *acbcbcbcbcbacbcbcbcbcb*, *bacacacacacbacacacacac*, *acbcbcbcbcbcbacbcbcbcbcbcb*
SF: 2^0,2^1,2^2,2^4,2^5,2^6,2^7,2^8,2^9
Gr: 1,4,10,19,31,46,64,85,109,136,166
Limit space: 2-dimensional sphere S_2

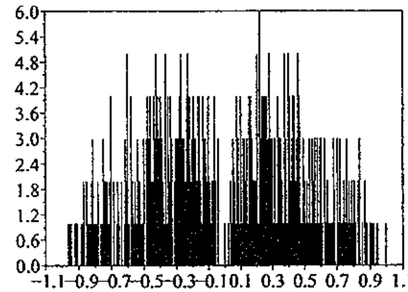




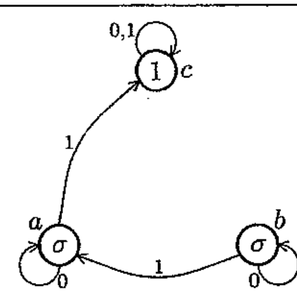
Automaton number 2369



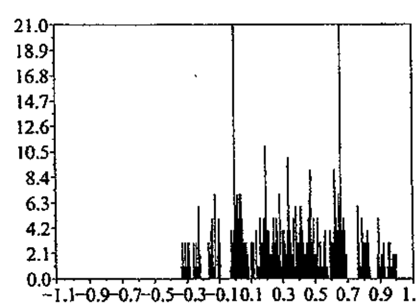
$a = \sigma(b, a)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $a^{-1}ba^{-1}b$, $b^{-1}cb^{-1}c$, $b^{-1}ca^{-1}ba^{-1}c$,
$a^{-2}b^2a^{-1}b^{-1}ab$, $a^{-2}bab^{-2}ab$, $a^{-1}ca^{-1}ca^{-1}ca^{-1}c$,
$b^{-1}ca^{-1}cb^{-1}ca^{-1}c$, $a^{-1}bab^{-1}a^{-1}b^2a^{-1}$,
$a^{-2}bac^{-1}bc^{-1}b^{-1}ab$, $a^{-1}bc^{-1}ac^{-1}ac^{-1}ac^{-1}b$,
$b^{-1}ca^{-2}b^2a^{-1}b^{-1}ac$, $b^{-1}ca^{-2}bab^{-2}ac$,
$b^{-1}cab^{-2}aba^{-2}c$, $b^{-1}cab^{-1}a^{-1}b^2a^{-2}c$,
$a^{-1}bab^{-1}c^{-1}bc^{-1}aba^{-1}$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,602,2514



Automaton number 2851

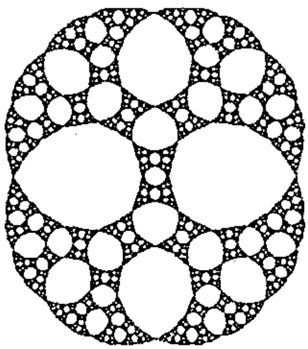


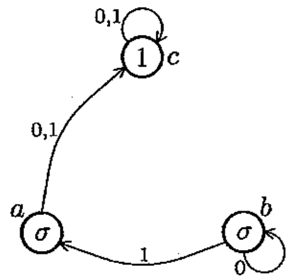
$a = \sigma(a, c)$ Group: Isomorphic to G_{929}
$b = \sigma(b, a)$ Contracting: *no*
$c = (c, c)$ Self-replicating: *yes*
Rels: c, $a^{-4}bab^{-1}a^2b^{-1}ab$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174},2^{345}
Gr: 1,5,17,53,161,485,1445

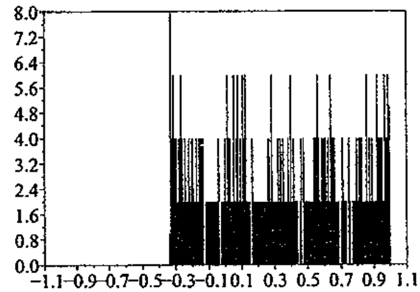


Automaton number 2853

$a = \sigma(c, c)$ Group: $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = \sigma(b, a)$ Contracting: *yes*
$c = (c, c)$ Self-replicating: *yes*
Rels: c, a^2, $ab^{-1}ab^{-2}ab^{-1}abab^2ab$
SF: 2^0,2^1,2^2,2^3,2^5,2^8,2^{14},2^{25},2^{47}
Gr: 1,4,10,22,46,94,190,375,731,1422,2752,5246,9908
Limit space:







7. Proofs of some facts about the selected groups

We start this section with a few useful observations, which simplify computations and were used often in the classification process.

First, we need to mention the classification of the groups generated by 2-state automata over a 2-letter alphabet. The following theorem is proved in [GNS00].

Theorem 8. *There are, up to isomorphism,* 6 *different groups generated by* 2*-state automata over a* 2*-letter alphabet (automata of complexity* $(2, 2)$*). Namely, trivial group,* C_2*,* $C_2 \times C_2$*, infinite cyclic group* $\mathbb{Z}$*, infinite dihedral group* D_∞ *and the lamplighter group* $\mathbb{Z} \wr C_2$*.*

The following proposition allows sometimes to see directly from the automaton structure that the corresponding group is not a torsion group.

Proposition 1. *Let* G *be a group generated by an automaton* A *over the* 2*-letter alphabet* $X = \{0, 1\}$ *that has the following property. The set of states of* A *splits into two nonempty parts* P *and* Q *such that*

(*i*) *one of the parts contains all active states and the other contains all inactive states;*

(ii) for each state in part P, both arrows go to states in the same part (either both to P pr both to Q);
(iii) for each state in part Q, one arrow goes to part P and the other to part Q

Then any element of the group that can be written as a product of odd number of active generators and odd number of inactive generators in any order, has infinite order. In particular, G is not a torsion group.

Proof. Let g be such an element. Let us prove by induction on n that, for each $n \geq 0$, there exists a vertex v fixed by g^{2^n} such that the section of g^{2^n} at v has the same form (i.e. is a product of odd number of active generators and odd number of inactive generators).

For $n = 0$ this is true. Suppose it is true for $n = k$ and let h be a section of g^{n^k} at some vertex $v \in X^*$ satisfying the conditions of the assumption. Since h is a product of odd number of active states and odd number of inactive states we can write $h = \sigma(h_1, h_2)$. Since v is fixed under g^{2^k} we have that $v0$ is fixed under $(g^{2^k})^2 = g^{2^{k+1}}$ and $g^{2^{k+1}}|_{v0} = h_2h_1$. The element h_2h_1 is product (in some order) of the first level sections of the generators (and/or their inverses) used to express h. By assumption, among these generators, there are odd number of active and odd number of inactive. The generators from part P, by condition (ii), will produce even number of active and even number of inactive generators, while the generators from part Q, by condition (iii), will give odd number of generators from both categories, which proves the induction step. Thus $g^{2^n} \neq 1$ for all n. □

There is an algorithm which determines whether a given element of a self-similar group generated by a finite automaton over the 2-letter alphabet $X = \{0, 1\}$ acts level transitively on the tree.

The abelianization of $\operatorname{Aut} X^*$ is isomorphic to the infinite Cartesian product $\prod_{i=0}^{\infty} C_2$. The canonical isomorphism sends $g \in G$ to $(c_i \mod 2)_{i=0}^{\infty}$, where c_i is the number of vertices $v \in X^i$, such that g_v acts nontrivially on the first level (i.e. c_i is the number of active sections of g at level i)

The abelianization group $\prod_{i=0}^{\infty} C_2$ can be endowed with the structure of a ring of formal power series $C_2[[t]]$ by $(a_i)_{i=0}^{\infty} \mapsto \sum_{i=0}^{\infty} a_i t^i$, where $a_i \in C_2$.

Algorithm 1 (Element transitivity). Let G be a self-similar group generated by a finite automaton over the 2-letter alphabet $X = \{0, 1\}$, and let $g \in G$ be given as a product of states and their inverses. Denote by $\bar{g}$ the image of g in $C_2[[t]]$. The element g acts level transitively on X^* if and only if $\bar{g} = (1, 1, 1, \ldots)$.

Suppose $g = \sigma^i(g_0, g_1)$, $i \in 0, 1$. Then

$$\overline{g} = i + t \cdot (\overline{g_0} + \overline{g_1}).$$

We can produce similar equations for the sections g_0, g_1 and so on. Since G is generated by a finite automaton the number of different sections of g is finite. Therefore we get finite linear system of equations over the variables $\{g_v, v \in X^*\}$, whose solution will express $\overline{g}$ as a rational function $P(t)/Q(t)$, where P, Q are polynomials with degrees not higher than $k = |\{g_v, v \in X^*\}|$.

Expanding this rational function as a power series will produce a preperiodic sequence of coefficients from C_2 with period and preperiod no longer than 2^k. In particular, g acts level transitively if and only if all c_i, $i = 1, \ldots, 2^{k+1}$ are equal to 1.

We often need to show that a given group of tree automorphisms is level transitive. Here is a very convenient necessary and sufficient condition for this in the case of a binary tree.

Proposition 2 (Group transitivity). *A self-similar group of binary tree automorphisms is level transitive if and only if it is infinite.*

Proof. Let G be an infinite self-similar group acting on a binary tree.

Level transitivity clearly implies that G is infinite.

For the converse, let us first prove that all level stabilizers $\mathrm{St}_G(n)$ are different. For this it suffices to show that for every $n \geq 1$ $\mathrm{St}_G(n-1) \setminus \mathrm{St}_G(n) \neq \varnothing$. Since all stabilizers have finite indices in G and G is infinite we get that all of them are infinite.

Let $g \in \mathrm{St}_G(n-1)$ be an arbitrary nontrivial element and $v = x_1 \ldots x_k$ be a word of shortest length (one of them) such that $g(v) \neq v$ (in other words, $g \in \mathrm{St}_G(k-1)$ and such k is maximal). Clearly $k \geq n$ and we can consider the section $h = g_{x_1 x_2 \ldots x_{k-n}}$ which is an element of G because of self-similarity. The fact that $g \in \mathrm{St}_G(k-1)$ implies $h \in \mathrm{St}_G(n-1)$. On other hand $x_1 \ldots x_{k-n} x_{k-n+1} \ldots x_k = v \neq g(v) = g(x_1 \ldots x_{k-n}) h(x_{k-n+1} \ldots x_k) = x_1 \ldots x_{k-n} h(x_{k-n+1} \ldots x_k)$. Therefore $h(x_{k-n+1} \ldots x_k) \neq x_{k-n+1} \ldots x_k$, thus $h \notin \mathrm{St}_G(n)$ and we found the desired element.

Now let us prove transitivity by induction on the level. The section of any nontrivial element at the vertex where it acts nontrivially gives transitivity on the first level.

Suppose G acts transitively on level n. Let $h \in \mathrm{St}_G(n) \setminus \mathrm{St}_G(n+1)$ be an arbitrary element and let $w = x_1 \ldots x_n \in X^n$ be one of the words such that $h(wx) = w\overline{x}$, where $\overline{x} = 1 - x$. For $u = y_1 \ldots y_{n+1} \in X^{n+1}$, let us find an element $g \in G$ such that $g(w0) = u$. This will prove the induction step.

By inductive assumption there exists $f \in G$ such that $f(w) = y_1 \dots y_n$. Suppose $f(w0) = y_1 \dots y_n \tilde{y}_{n+1}$. Then, if $\tilde{y}_{n+1} = y_{n+1}$ we are done, otherwise $\tilde{y}_{n+1} = \overline{y_{n+1}}$ and, for $g = fh$, we obtain $g(w0) = f(h(w0)) = f(w1) = y_1 \dots y_n \overline{\overline{y_{n+1}}} = u$. □

Note, that the last proof works also for self-similar subgroups of the infinitely iterated permutational wreath product $\wr_{i \geq 1} C_d$ (the subgroup of $\mathrm{Aut}(\mathcal{T})$ consisting of those automorphisms of the d-ary tree for which the activity at every vertex is a power of some fixed cycle of length d. Also, certain generalizations of this method could be used in more complex situations (see, for example, the construction of $G_{2240} \cong C_2 * C_2 * C_2$ in [Nek05], or proof of transitivity of Sushchansky groups on a subtree in [BS]).

We provide below additional information and proofs about the groups listed in Section 6.

739: $C_2 \ltimes (C_2 \wr \mathbb{Z})$. For G_{739}, we have $a = \sigma(a, a)$, $b = (b, a)$ and $c = (a, a)$.

All generators have order 2. The elements $u = acba = (1, ba)$ and $v = bc = (ba, 1)$ generate $\mathbb{Z}^2$ because $ba = \sigma(1, ba)$ is the adding machine and has infinite order. Also we have $ac = \sigma$ and $\langle u, v\rangle$ is normal in $H = \langle u, v, \sigma\rangle$ because $u^\sigma = v$ and $v^\sigma = u$. In other words, $H \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z}) = C_2 \wr \mathbb{Z}$.

Furthermore, $G_{739} = \langle H, a\rangle$ and H is normal in G_{739} because $u^a = v^{-1}$, $v^a = u^{-1}$ and $\sigma^a = \sigma$. Thus $G_{739} = C_2 \ltimes (C_2 \wr \mathbb{Z})$, where the action of C_2 on H is specified above.

744: For G_{744}, we have $a = \sigma(c, b)$, $b = (b, a)$ and $c = (a, a)$.

Since $(a^{-1}c)^2 = (c^{-1}ab^{-1}a, b^{-1}ac^{-1}a)$ and $c^{-1}ab^{-1}a = ((c^{-1}ab^{-1}a)^{-1}, a^{-1}c)$ we see that $(a^{-1}c)^2$ fixes the vertex 01 and its section at this vertex is equal to $a^{-1}c$. Hence, $a^{-1}c$ has infinite order.

Furthermore, the element $c^{-1}ab^{-1}a$ has infinite order, fixes the vertex 00 and its section at this vertex is equal to $c^{-1}ab^{-1}a$. Therefore G_{744} is not contracting (all powers of $c^{-1}ab^{-1}a$ would have to belong to the nucleus).

748: $D_4 \times C_2$. For G_{748}, we have $a = \sigma(a, a)$, $b = (c, a)$ and $c = (a, a)$.

It follows from the relations $a^2 = b^2 = c^2 = acac = bcbc = abababab = 1$ that G_{748} is a homomorphic image of $D_4 \times C_2$. Since $a \neq 1$, $b \neq 1$ and $(ab)^2 \neq 1$, it follows that $\langle a, b\rangle = D_4$. One can verify directly that c is not equal to any of the four elements in $\langle a, b\rangle$ that stabilize level 1 (namely 1, b, aba and $abab$). Thus $G_{748} = D_4 \times C_2$.

753: For G_{753}, we have $a = \sigma(c, b)$, $b = (c, a)$ and $c = (a, a)$.

Since $ab^{-1} = \sigma(1, ba^{-1})$, this element is conjugate to the adding machine.

For a word w in $w \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}^*$, let $|w|_a$, $|w|_b$ and $|w|_c$ denote the sum of the exponents of a, b and c in w. Let w represents the element $g \in G$. If $|w|_a$ and $|w|_b$ are odd, then g acts transitively on the first level, and $g^2|_0$ is represented by a word w_0, which is the product (in some order) of all first level sections of all generators appearing in w. Hence, $|w_0|_a = |w|_b + 2|w|_c$ and $|w_0|_b = |w|_a$ are odd again. Therefore, similarly to Proposition 1, any such element has infinite order.

In particular c^2ba has infinite order. Since $a^4 = (caca, a^4, acac, a^4)$ and $caca = (baca, c^2ba, bac^2, caba)$, the element a^4 has infinite order (and so does a). Since a^4 fixes the vertex 01 and its section at that vertex is equal to a^4, the group G_{753} is not contracting.

771: $\mathbb{Z}^2$. For G_{771}, we have $a = \sigma(c, b)$, $b = (b, b)$ and $c = (a, a)$.

Since G_{771} is finitely generated, abelian, and self-replicating (easy to check), it follows from [NS04] that it is free abelian. There are two options: either it has rank 1 or rank 2 (since $b = 1$). Let us prove that the rank is 2. For this it is sufficient to show that $c^n \neq a^m$ in G. Assume on the contrary that $c^n = a^m$ for some integer n and m and choose such integers with minimal $|n| + |m|$. Since c^n stabilizes level 1, m must be even and we have $(a^n, a^n) = c^n = a^m = (c^{m/2}, c^{m/2})$. But then $a^n = c^{m/2}$ and by the minimality assumption m must be 0, implying $c^n = 1$. The last equality can only be true for $n = 0$ since G_{771} is torsion free (free abelian) and $c \neq 1$. Thus $G_{771} \cong \mathbb{Z}^2$.

775: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. For G_{775}, we have $a = \sigma(a, a)$, $b = (c, b)$, $c = (a, a)$.

We have $a^2 = b^2 = c^2 = 1$, $ac = ca = \sigma(1, 1)$ and $ba = \sigma(ba, ca)$. Hence, for the subgroup $H = \langle ba, ca \rangle \leq G$, we have $H \cong G_{2853} \cong IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. On the other hand H is normal in G since $(ba)^a = ab = (ba)^{-1}$ and $(ca)^a = ac = ca$. Thus $G \cong C_2 \ltimes H$, where C_2 is generated by a and the action of a on H is given above. It is proved below (see G_{783}), that $G_{775} \cong G_{783}$. Therefore G_{775} also contains a torsion free subgroup of index 4.

783$\cong G_{775}$: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. For G_{783}, we have $a = \sigma(c, c)$, $b = (c, b)$ and $c = (a, a)$.

All generators have order 2 and a commutes with c. Conjugating this group by the automorphism $\gamma = (c\gamma, \gamma)$ yields an isomorphic group generated by the 4-state automaton defined by the recursive relations $a' = \sigma(1, 1)$, $b' = (c', b')$ and $c' = (a', a')$. On the other hand, we obtain the same automaton after conjugating G_{775} by $\mu = (a\mu, \mu)$ (here a denotes the generator of G_{775}).

It can be proved that the subgroup $H = \langle ba, cabc\rangle$ is torsion free and not metabelian. Furthermore, $G_{783} = \langle a, c\rangle \ltimes H \cong (C_2 \times C_2) \ltimes H$. The group G_{783} is regular weakly branch group over H''.

Since $bca = \sigma(bca, a)$, $G_{783} = \langle acb, a, c\rangle \cong G_{2205}$.

803$\cong G_{771}$: $\mathbb{Z}^2$. For G_{803}, we have $a = \sigma(b, a)$, $b = (c, c)$, $c = (a, a)$.

Since G_{771} is finitely generated, abelian, and self-replicating, it follows from [NS04] that it is free abelian. Consider the $\frac{1}{2}$-endomorphism ϕ : $\mathrm{Stab}_{G_{803}}(1) \to G_{803}$ associated to the vetrex 0, given by $\phi(g) = h$ for $g \in \mathrm{Stab}_{G_{803}}(1)$, provided $g = (h, *)$. Consider also the linear map $A : \mathbb{C}^3 \to \mathbb{C}^3$ induced by ϕ. It has the following matrix representation with respect to the basis corresponding to the triple $\{a, b, c\}$:

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial $\chi_A(\lambda) = -\lambda^3 + \frac{1}{2}\lambda^2 + \frac{1}{2}$ has three distinct complex roots $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{4} - \frac{1}{4}i\sqrt{7}$ and $\lambda_3 = -\frac{1}{4} + \frac{1}{4}i\sqrt{7}$. Choose an eigenvector v_i associated to the eigenvalue λ_i, $i = 1, 2, 3$. In particular, we may choose $v_1 = (2, 1, 1)$, which shows that $a^2bc = 1$ in G_{803}. In order to show that $a^{2m}c^n \neq 1$ (except when $m = n = 0$) we will prove that the vector $v = (2m, 0, n)$ is eventually pushed out from the domain corresponding to the first level stabilizer, i.e. from the set $D = \{(2i, j, k), i, j, k \in \mathbb{Z}\}$, by iterations of the action of A.

Consider the expansion of v in the basis $\{v_1, v_2, v_3\}$: $v = a_1v_1 + a_2v_2 + a_3v_3$. Since $m \neq 0$ or $n \neq 0$, v is not a scalar multiple of v_1. We have $A^t(v) = a_1v_1 + \lambda_2^t a_2v_2 + \lambda_3^t a_3v_3 \to a_1v_1$, as $t \to \infty$, since $|\lambda_2| = |\lambda_3| < 1$. We can choose a neighborhood of a_1v_1 that does not contain points from D, except maybe a_1v_1. Eventually $A^t(v)$ will be in this neighborhood and, since $A^t(v) \neq a_1v_1$ for all t, $A^t(v)$ will be outside of D. This implies that the word $a^{2m}c^n$ represents a nontrivial element in G_{803}. Thus $G_{803} = \langle a, c\rangle \cong \mathbb{Z}^2$.

846: $C_2 * C_2 * C_2$. This is a result of Muntyan. See the proof in [Nek05]. In particular, G_{846} contains a self-similar free group of rank 2 generated by a 6-state automaton. The automaton 846 is sometimes called Bellaterra automaton.

852: Basilica group $\mathcal{B} = IMG(z^2 - 1)$. First studied in [GŻ02a], where it is proved that $\mathcal{B}$ is not in the class SG of sub-exponentially amenable groups, does not contain a free subgroup of rank 2, and the nontrivial generators a and b generate a free subsemigroup. Spectral properties are considered in [GŻ02b]. It is proved in [BV05] that $\mathcal{B}$ is amenable, providing the first example of an amenable group not in the class SG.

857: For G_{857}, we have $a = \sigma(b, a)$, $b = (c, b)$ and $c = (b, a)$.

Let us prove that b has infinite order. For any $w \in X^*$ $b(w0^\infty) = b(w0)b_{w0}(0^\infty)$. Since b_{w0} equals either b or c and $b(0^\infty) = c(0^\infty) = 0^\infty$, we have $b(w0^\infty) = b(w0)0^\infty$. Therefore all elements in the forward orbit of 010^∞ under the action of b end in 0^∞. The length of the non-zero prefix of any infinite word ending in 0^∞ cannot decrease under the action of b. Indeed, for any $w \in X^*$ $b(w10^\infty) = b(w)b_w(10^\infty)$. The section b_w is one of the three generators, for which we have $a(10^\infty) = 010^\infty$, $b(10^\infty) = 10^\infty$ and $c(10^\infty) = 110^\infty$.

On the other hand, the length of the non-zero prefix along the orbit cannot stabilize, because in this case the orbit must be finite and we must have $b^k(010^\infty) = 010^\infty$, for some $k \geq 1$. But this is impossible since $b(010^\infty) = 0110^\infty$ and thus the length of the non-zero prefix of $b^k(010^\infty)$ must be at least 3. Thus the orbit is infinite and b has infinite order.

Since $b = (c, b)$, G_{857} is not contracting.

858: For G_{858}, we have $a = \sigma(c, a)$, $b = (c, b)$ and $c = (b, a)$.

The element $ab^{-1} = \sigma(1, ab^{-1})$ is the adding machine.

Using the same approach as for G_{857} one can show that c has infinite order. Namely the length of the non-zero prefix of the forward orbit of 10^∞ under c is nondecreasing, which then implies that this orbit is not finite.

Since $b = (c, b)$, G_{858} is not contracting.

870: Baumslag-Solitar group $BS(1, 3)$. For G_{870}, we have $a = \sigma(c, b)$, $b = (a, c)$, and $c = (b, a)$.

The automaton satisfies the conditions of Proposition 1. Thus, ab has infinite order in G, which implies that $bc = (ab, ca)$, $a^2 = (bc, cb)$ also have infinite order. Hence, we can claim the same for a and $b = (a, c)$.

Furthermore, the element $\mu = b^{-1}a = \sigma(1, a^{-1}b) = \sigma(1, \mu^{-1})$ also has infinite order (it is conjugate of the adding machine). Since $a^{-1}c = \sigma(1, c^{-1}a) = \mu$ we have $c = ab^{-1}a$ and $G = \langle a, b\rangle = \langle \mu, b\rangle$. Let us check that $b^{-1}\mu b = \mu^3$. Since $b^{-1}\mu b = \sigma(c^{-1}a, a^{-2}bc)$ and $\mu^3 = \sigma(\mu^{-1}, \mu^{-2})$ all we need to check is that $a^{-2}bc = a^{-1}ba^{-1}b$, i.e. $a^{-1}bcb^{-1}ab^{-1} = 1$. The last is correct since $a^{-1}bcb^{-1}ab^{-1} = (1, b^{-1}aba^{-1}bc^{-1})$ and $b^{-1}aba^{-1}bc^{-1}$ is a conjugate of the inverse of $a^{-1}bcb^{-1}ab^{-1}$. Thus, since b and μ have infinite order, $G_{870} \cong BS(1, 3)$.

See [BŠ06] for realizations of $BS(1, 3)$ and other Baumslag-Solitar groups by automata.

878: $C_2 \ltimes IMG(1 - \frac{1}{z^2})$. For G_{878}, we have $a = \sigma(b, b)$, $b = (b, c)$ and $c = (b, a)$.

Denote $x = bc$ and $y = ca$. All generators have order 2, and therefore the subgroup $H = \langle x, y\rangle$ is a normal subgroup of index 2 in G_{878}. Moreover $G_{878} \cong C_2 \ltimes H$, where C_2 is generated by c and the action of C_2 on H is given by $x^c = x^{-1}$ and $y^c = y^{-1}$. We have $x = (1, ca) = (1, y)$ and $y = \sigma(ab, 1) = \sigma(y^{-1}x^{-1}, 1)$. Exchanging the letters 0 and 1 leads to an isomorphic copy of H defined by $x = (y, 1)$ and $y = \sigma(1, y^{-1}x^{-1})$, which is the iterated monodromy group $IMG(1 - \frac{1}{z^2})$, according to [BN06]. Thus, $G_{878} \cong C_2 \ltimes IMG(1 - \frac{1}{z^2})$.

929. See G_{2851}.

942: For G_{942}, we have $a = \sigma(c, b)$, $b = (c, b)$ and $c = (c, a)$.

It is known [GŻ01] that the group $L = \langle a', b'\rangle$ defined by

$$\begin{aligned} a' &= \sigma(a', b') \\ b' &= \quad (a', b') \end{aligned}$$

is the lamplighter group $\mathbb{Z} \wr C_2$ (compare to (8)). Consider the subtree Y^* of X^* consisting of all words over the alphabet $Y = \{01, 11\}$. The element a swaps the letters of Y and b fixes them. Since $a_{01} = b_{01} = a$, $a_{11} = b_{11} = b$, the tree Y^* is invariant under the action of $H = \langle a, b\rangle$ and the action of H on Y^* coincides with the action of the lamplighter group $L = \langle a', b'\rangle$ on X^* (with the identification $0 \leftrightarrow 01$, $1 \leftrightarrow 11$). Therefore the map $\phi : H \longrightarrow L$ given by $a \mapsto a'$, $b \mapsto b'$ extends to a homomorphism. We claim that this homomorphism has trivial kernel. Indeed, let $w = w(a, b)$ be a group word representing an element of the kernel of ϕ. Since the word $w(a', b')$ represents the identity in L the total exponent of a in w must be even and the total exponent ϵ of both a and b in w must be 0. But in that case the element $g = w(a, b)$ fixes the top two levels of the tree X^* and has decomposition

$$g = (c^\epsilon, *, c^\epsilon, *),$$

where the $*$'s denote words over a and b representing the identity in H (these words correspond to the first level sections of $w(a', b')$ in L). Therefore $g = 1$ and the kernel of ϕ is trivial.

Thus, the lamplighter group is a subgroup of G_{942}.

968. For G_{968}, we have $a = \sigma(b, b)$, $b = (c, c)$ and $c = (c, a)$.

This group contains $\mathbb{Z}^5$ as a subgroup of index 16. It is contracting with nucleus consisting of 73 elements, whose self-similar closure consists of 77 elements. All generators have order 2.

Let $x=(ac)^2$, $y=bcba$, and consider the subgroup $K=\langle x,y\rangle$. Direct computations show that x and y commute ($xy=\sigma(bcbacacb,ba)$ and $yx=\sigma(cacabc,ba)$). Conjugating by $\gamma=(b\gamma,a\gamma)$ leads to the self-similar copy K' of K generated by $x'=((y')^{-1},(y')^{-1})$ and $y'=\sigma(x',y')$, where $x'=x^\gamma$ and $y'=y^\gamma$. Since $(y')^2=(x'y',x'y')$, the virtual endomorphism of K' is given by the matrix

$$A=\begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues $\lambda=\frac{1}{4}\pm\frac{1}{4}\sqrt{7}i$ of this matrix are not algebraic integers, hence, according to [NS04], the group K' is free abelian of rank 2, and so is K.

Since all generators have order 2, the subgroup $H=\langle ab,bc\rangle$ has index 2 in G_{968}. The stabilizer $\mathrm{St}_H(2)$ of the second level has index 8 in H. Moreover, the quotient group is isomorphic to the dihedral group D_4 (since ab acts on the second level by permuting $00\leftrightarrow 10$ and $01\leftrightarrow 11$, while bc acts by permuting $10\leftrightarrow 11$). The stabilizer $\mathrm{St}_H(2)$, conjugated by the element $g=(b,c,b,1)$, is generated by

$$\begin{array}{lll}
g_1=\left((bc)^2\right)^g & =(bcbc)^g= & (1,\ 1,\ y,\ y^{-1}\ ),\\
g_2=\left((bc)^2\right)^{bag} & =(acbcba)^g= & (y,\ y,\ 1,\ 1\ ),\\
g_3=\left((ab)^2\right)^{bcg} & =(cbabac)^g= & (1,\ x,\ x,\ 1\ ),\\
g_4=\left((ab)^2\right)^g & =(abab)^g= & (1,\ x,\ 1,\ x^{-1}\ ),\\
g_5=\left((ab)^2\right)^{(bc)^{ba}g} & =(abcbabacba)^g= & (x,\ 1,\ 1,\ x^{-1}\ ),\\
g_6=\left((ab)^2\right)^{bc(bc)^{ba}g} & =(abcacbabacacba)^g= & (x,\ 1,\ x,\ 1\ ).
\end{array}$$

Therefore, all g_i commute and $g_6=g_5g_3g_4^{-1}$. If $\prod_{i=1}^5 g_i^{n_i}=1$, then all sections must be trivial, hence, $x^{n_5}y^{n_2}=x^{n_3+n_4}y^{n_2}=x^{n_3}y^{n_1}=x^{n_4+n_5}y^{n_1}=1$. But K is free abelian, whence $n_i=0$, $i=1,\dots,5$. Thus, $\mathrm{St}_H(2)$ is a free abelian group of rank 5.

2205$=G_{775}$. $C_2\ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. For G_{2205}, we have $a=\sigma(c,c)$, $b=\sigma(b,a)$ and $c=(a,a)$. See G_{783} for an isomorphism.

2212: Klein bottle group, $\langle a,b\mid a^2=b^2\rangle$, contains $\mathbb{Z}^2$ as subgroup of index 2. For G_{2212}, we have $a=\sigma(a,c)$, $b=\sigma(c,a)$ and $c=(a,a)$.

Since $ac=\sigma(a^2,ca)$ and $ca=\sigma(a^2,ac)$, the generators a and c commute. Further, $a^2c=(ca^2,aca)=(a^2c,a^2c)$, which shows that $c=a^{-2}$, and therefore $a=\sigma(a,a^{-2})$ and $b=\sigma(a^{-2},a)$. Since $a^2=(a^{-1},a^{-1})$, the element a has infinite order and so does $x=ab^{-1}=(a^3,a^{-3})$. Finally, since $x^a=b^{-1}a=(a^{-3},a^3)=x^{-1}$, we have $G_{2212}=\langle x,a\mid x^a=x^{-1}\rangle$ and

G_{2212} is the Klein bottle group. Going back to the generating set consisting of a and b, we get the presentation $G_{2212} = \langle a, b \mid a^2 = b^2 \rangle$.

2240: Free group of rank 3. The automaton generating this group first appeared in [Ale83]. It is proved in [VV05] that G_{2240} is a free group of rank 3 with basis $\{a, b, c\}$. This is the smallest example of a free nonabelian group among all automata over a 2-letter alphabet (see Theorem 4).

2277: $C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$. For G_{2277}, we have $a = \sigma(c, c)$, $b = \sigma(a, a)$, $c = (b, a)$.

All generators have order 2. Let $x = cb$ and $y = ab$ and let $H = \langle x, y \rangle$. Then $x = \sigma(1, y^{-1})$ and $y = (xy^{-1}, xy^{-1})$. It is easy to check that x and y commute and that H is self-replicating. The matrix of the associated virtual endomorphism is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix}.$$

Since the eigenvalues $-\frac{1}{2} \pm \frac{1}{2}i$ are not algebraic integers, according to [NS04] H is free abelian of rank 2.

The subgroup H is normal of index 2 in G_{2277} because the generators of G_{2277} are of order 2. Thus $G_{2277} = \langle H, b \rangle = C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$, where the action of $C_2 = \langle b \rangle$ on H is by inversion of the generators.

2369. For G_{2369} we have $a = \sigma(b, a)$, $b = \sigma(c, a)$ and $c = (c, a)$.

For any vertex $v \in X^*$, we have $a_{v1} = a$, $a_{v10} = b$ and $a_{v10^{n+2}} = c$, for $n \geq 0$. Therefore, for any vertex $w \in X^*$, $a(w10^\infty) = a(w1)110^\infty$ and the forward orbit of 10^∞ under a is infinite, because the length of the non-zero prefix grows by 2 with each application of a. Thus a has infinite order.

Since $a^2 = (ab, ba)$, the element ab also has infinite order. Furthermore, $ab = (ac, ba)$ and $ba = (ab, ca)$. Thus, G_{2369} is not contracting.

2851=G_{929}. For G_{2851} we have $a = \sigma(a, 1)$, $b = \sigma(b, a)$, $c = (c, c) = 1$.

The element a is conjugate of the adding machine (in fact it is its inverse). Since $ba^{-1} = (a, ba^{-1})$, the order of ba^{-1} is infinite and G_{2851} is not contracting.

The group G_{2851} is a regular weakly branch group over G' since it is self-replicating and $[a^2, b] = ([a, b], 1)$.

The subsemigroup $\langle a, b \rangle$ is free. Indeed, let w be a nonempty word in $\{a, b\}^*$. If $w = 1$ in G_{2851}, then w contains both a and b, because they both have infinite order. Suppose the length of w is minimal among all nonempty words over $\{a, b\}$ representing the identity element in G_{2851}. Then one of the projections of w will be shorter than w, nonempty, and will represent the identity in G_{2851}, which contradicts the minimality assumption. Thus $w \neq 1$ in G_{2851}, for any nonempty word in $\{a, b\}^*$.

Now let w and v be two words in $\{a,b\}^*$ with minimal sum $|w|+|v|$ such that $w=v$ in G_{2851}. Suppose w ends in a and v ends in b. Then

(1) if w ends in a^2 then w_0 is a word that is shorter than w ending in a, while v_0 is a word not longer than v ending in b. Since $w_0=v_0$ in G_{2851} and $|w_0|+|v_0|<|w|+|v|$, we have a contradiction.
(2) if w ends in ba then w_1 is a word shorter than w ending in b, while v_1 is a word not longer than v ending in a. Since $w_1=v_1$ in G_{2851} and $|w_1|+|v_1|<|w|+|v|$, we have a contradiction.
(3) if $w=a$ then $v_1=1$ in G and v_1 is a nonempty word, which is impossible, as already proved above.

Thus G has exponential growth. On the other hand, the orbital Schreier graph $\Gamma(G,000\dots)$ has intermediate growth (see [BH05, BCSN]).

The group G_{2851} coincides with G_{929} as subgroup of $\mathrm{Aut}(X^*)$. Indeed, $G_{2851}=\langle a^{-1}=\sigma(1,a^{-1}), b^{-1}a=(b^{-1}a,a^{-1})\rangle = G_{929}$. Therefore all properties proved for G_{2851} above hold also for G_{929}.

2853: $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. For G_{2853}, we have $a=\sigma(c,c)$, $b=\sigma(b,a)$ and $c=(c,c)=1$.

It is proved in [BN06] that $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$ is generated by $\alpha=\sigma(1,\beta)$ and $\beta=(\alpha^{-1}\beta^{-1},\alpha)$. We have then $\beta\alpha=\sigma(\alpha,\alpha^{-1})$. Conjugate the right hand side of the wreath recursion by $(1,\alpha)$ to obtain a copy of $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$ given by $\beta=(\alpha^{-1}\beta^{-1},\alpha)$, $\beta\alpha=\sigma$ and $\alpha=\sigma(\alpha^{-1},\beta\alpha)$ (this is equivalent to conjugating by $\gamma=(\gamma,\alpha\gamma)$ in $\mathrm{Aut}(X^*)$).

This shows that G_{2853} is isomorphic to $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$ via the isomorphism $a\mapsto\beta\alpha$ and $b\mapsto\alpha$. Moreover, they are conjugate by the element $\delta=(\delta_1,\delta_1)$, where $\delta_1=\sigma(\delta,\delta)$ (this is the automorphism of the tree changing all letters which stand on even places).

Consequently, the limit space of G_{2853} is the Julia set of the rational map $z\mapsto\left(\frac{z-1}{z+1}\right)^2$.

The group G_{2853} is contained in G_{775} as a subgroup of index 2 (see G_{775}). It contains the torsion free subgroup H mentioned in the discussion of G_{775} as subgroup of index 2 and is a weakly branch group over H''. All Schreier graphs on the boundary of the tree have polynomial growth of degree 2. Diameters of Schreier graphs on the levels grow as $\sqrt{2}^n$ (see [BN] for details).

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